Peano Arithmetic and $\bar{\bar{\mu}}$MALL (an abstract)

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We examine some of the proof theory of arithmetic using three proof systems.

1. A linearized version of arithmetic, named $\bar{\bar{\mu}}$MALL, is the multiplicative-additive fragment of linear logic plus logical connectives to treat first-order term structures: equality, inequality, first-order quantifier, and the least and greatest fixed point operators. This proof system was studied in [1, 2] (where it was named μMALL$^=$) and it is known to satisfy cut-elimination and is, therefore, consistent. It also has a focused proof systems that is known to be complete [1].
2. The proof system $\bar{\bar{\mu}}$LKp is an extension of $\bar{\bar{\mu}}$MALL in which contraction and weakening are permitted. This proof system is a polarized version of classical logic with fixed points. We have shown that $\bar{\bar{\mu}}$LKp is consistent by embedding it into second-order linear logic [6].
3. $\bar{\bar{\mu}}$LKp$^+$ is a further extension in which the cut rule is permitted. We also show that $\bar{\bar{\mu}}$LKp$^+$ contains Peano arithmetic and that in a couple of different situations, $\bar{\bar{\mu}}$LKp is conservative over $\bar{\bar{\mu}}$MALL. Whether or not $\bar{\bar{\mu}}$LKp$^+$ and $\bar{\bar{\mu}}$LKp prove the same theorems is currently open.

Finally, we show that if we can prove (in $\bar{\bar{\mu}}$LKp$^+$) that a given relation encodes a function, then a simple proof-search-based algorithm using unification and non-deterministic search, can compute that function. Since we are interested in using $\bar{\bar{\mu}}$MALL, $\bar{\bar{\mu}}$LKp, and $\bar{\bar{\mu}}$LKp$^+$ to study *arithmetic*, we use first-order structures to encode natural numbers and fixed points to encode (partially recursive) relations among numbers. This focus is in contrast to uses of the propositional subset of $\bar{\bar{\mu}}$MALL as a typing systems (see, for example, [5]). We use invariants to reason about fixed points instead of employing other methods, such as infinitary proof systems (e.g., [3]) and cyclic proof systems (e.g., [4, 8]).

This abstract is based on a paper presented at Linearity & TLLA 2022 [7].

References

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