First-order logic and regular languages

Overview and recent results

Denis Kuperberg

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First-Order Logic (FO)

Signature: Predicate symbols $(P_1, \ldots, P_n)$ with arities $k_1, \ldots, k_n$.

Syntax of FO:

$$\varphi, \psi := P_i(x_1, \ldots, x_{k_i}) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \neg\varphi \mid \exists x.\varphi \mid \forall x.\varphi$$

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Example: For directed graphs, signature = one binary predicate E.

Graph class	Cliques	No node points to everyone
Formula	$\varphi = \forall x.\forall y.E(x, y)$	$\psi = \neg\exists x.\forall y.E(x, y)$
Example graph	Model of φ	Model of ψ

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Motivation: Logics with fixed points.
Fixed points can only be applied to monotone φ.
Hard to recognize $\rightarrow$ replace by positive φ, syntactic condition.

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- ▶ [This work]
  EF games on words, elementary

Our results

Lyndon's theorem fails on

- ▶ Finite words

- ▶ Finite graphs

- ▶ Finite structures (elementary proof), several versions:

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- **Finite words**
- Finite graphs
- Finite structures (elementary proof), several versions:

Regular Language Theory:

Monotone FO languages	$\neq$	**Positive FO languages**
Algebraic characterization		Logical characterization
Decidable membership		Undecidable membership

Regular Languages

Fixed alphabet A. Language = set of words on A.

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$$e := \emptyset \mid \varepsilon \mid a \in A \mid e + e \mid e \cdot e \mid e^*$$
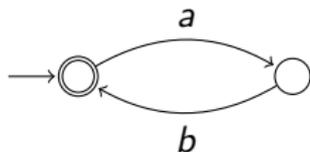
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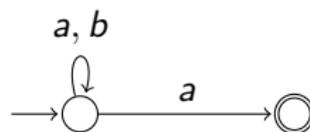Fixed alphabet A. Language = set of words on A.

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Deterministic finite automaton
(DFA)

Nondeterministic finite automaton
(NFA)

Properties and applications

Regular languages are closed under:
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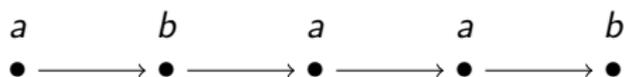
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Applications:

▶ Proving decidability

▶ Text processing (grep,...)

▶ Model checking (LTL,...)
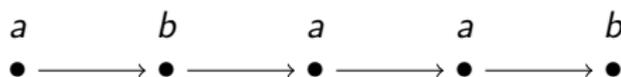
▶ Compilation

▶ ...

FO on words, the usual way

Words on alphabet $A = \{a, b[, \dots]\}$: signature $(\leq, a, b[, \dots])$

$$a \quad b \quad a \quad a \quad b$$
$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

- ▶ $x \leq y$ means position x is before position y.
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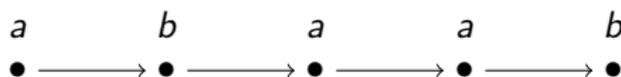\end{array}
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Examples of formulas:

▶ $\exists x.a(x)$: words containing a. Language $A^* a A^*$.

▶ $\exists x, y.(x \leq y \wedge a(x) \wedge b(y))$. Language $A^* a A^* b A^*$.

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- ► $\exists x, y.(x \leq y \land a(x) \land b(y))$. Language $A^*aA^*bA^*$.

> **Theorem**
>
> First-order languages form a strict subclass of regular languages.

Example: $(aa)^*$ is not FO-definable. (Proof later)

Monadic second-order logic (MSO)

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Theorem (Büchi 1960, Elgot 1961, Trakhtenbrot 1962)

MSO captures exactly regular languages.

FO-definable languages

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> ### Theorem (Schützenberger 1965, McNaughton+Papert 1971)
>
> Language $L \subseteq A^*$ is FO-definable iff definable by:
>
> - ▶ Star-free expression (no $\cdot^*$, but complement allowed)
> - ▶ aperiodic monoid (no group)
> - ▶ Linear Temporal Logic (LTL) formula
> - ▶ counter-free automaton ...

Understanding FO-definable languages

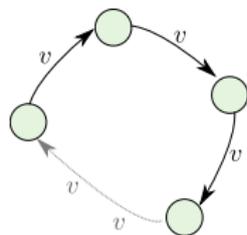**Intuition**: FO languages are "Aperiodic": cannot count modulo

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$\Leftrightarrow$ _Counter-free automaton:_ No cycle of the form:
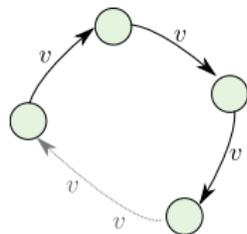
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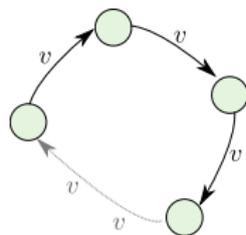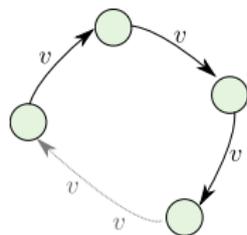**Corollary:** FO-definability is decidable for regular languages.

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Example: Star-free expression for $(ab)^*$: $a\overline{\emptyset}b \cap \overline{\overline{\emptyset}(aa + bb)\overline{\emptyset}}$.

FO on words, the "unconstrained" way

For now, a word is a structure $(X, \leq, a, b, \dots)$ where

- $\leq$ is a total order
- $a, b, \dots$ form a partition of X.

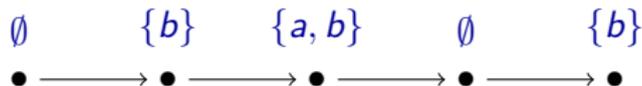# FO on words, the "unconstrained" way

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We will note $\Sigma = \{a, b, \dots\}$, and $A = \mathcal{P}(\Sigma)$ the alphabet.

- ▶ Useful e.g. in verification (LTL,...):
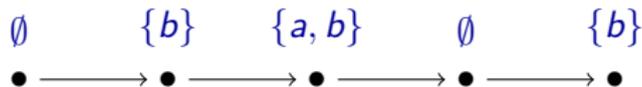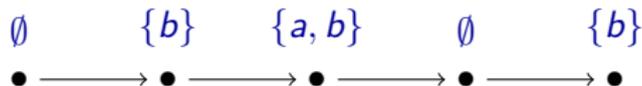 independent signals can be true or false simultaneously.

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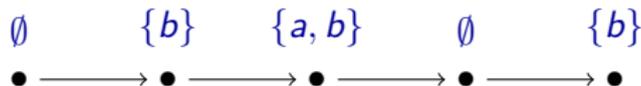- FO languages on alphabet A are the same (Preds=Σ or A).

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  $\rightarrow$ Negation necessary for full FO.

The FO^+ logic: positive formulas

<u>FO^+ Logic</u>: a ranges over Σ, no $\neg$

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Example: On $\Sigma = \{a, b\}$:

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Question [Colcombet]: FO & monotone $\overset{?}{\Rightarrow} \mathrm{FO}^+$

A counter-example language

Our first result

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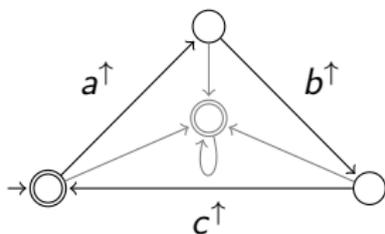There is L monotone, FO-definable but not FO^+-definable.

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Language $\quad L = (a^\uparrow b^\uparrow c^\uparrow)^* \ \bigcup \ A^* \left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right) A^*$.

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Language $\quad L = (a^\uparrow b^\uparrow c^\uparrow)^* \ \bigcup \ A^* \begin{pmatrix} a \\ b \\ c \end{pmatrix} A^*$. Monotone

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There is L monotone, FO-definable but not FO^+-definable.

Alphabet $A = \{\emptyset, a, b, c, \binom{a}{b}, \binom{b}{c}, \binom{c}{a}, \left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)\}$. Let $a^\uparrow = \{a, \binom{a}{b}, \binom{c}{a}\}$.

Language $\quad L = (a^\uparrow b^\uparrow c^\uparrow)^* \ \bigcup \ A^* \left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right) A^*$. Monotone

Lemma: L is FO-definable.

Proof:

is counter-free. (no cycle labelled $v^{\geq 2}$)

A counter-example language

> **Our first result**
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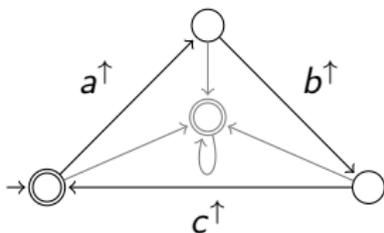
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To prove L is not FO^+-definable: Ehrenfeucht-Fraïssé games.

Ehrenfeucht-Fraïssé games for FO

Definition (EF games)

Played on two words u, v. At each round i:

- Spoiler places token i in u or v.
- Duplicator must answer token i in the other word such that
  - the letter on token i is the same in u and v.
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Theorem (Ehrenfeucht,Fraïssé, 1950-1961)

L not FO-definable $\Leftrightarrow$ For all n, there are $u \in L$, $v \notin L$ s.t. $u \equiv_n v$.

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Example

Proving $(aa)^*$ is not FO-definable:

$$u = a^{2k} \quad \in (aa)^* : \quad a\,a\,a\,a\,a\,a\,a\,a\,a\,a$$
$$v = a^{2k-1} \quad \notin (aa)^* : \quad a\,a\,a\,a\,a\,a\,a\,a\,a$$

Proving FO^+-undefinability

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New rule:

Letters in u just have to be included in corresponding ones in v.

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Theorem (Correctness of EF^+ games)
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[Stolboushkin 1995+this work]

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Application: Proving L is not FO^+-definable

$$u \in L: \quad a \quad b \quad c \quad a \quad b \quad c \quad a \quad b \quad c$$
$$v \notin L: \quad \binom{a}{b} \binom{b}{c} \binom{c}{a} \binom{a}{b} \binom{b}{c} \binom{c}{a} \binom{a}{b} \binom{b}{c}$$

From finite words to finite structures.

Goal: Lift L to finite structures.

Theorem (Ajtai & Gurevich 1987, Stolboushkin 95, this work)

There is a class of finite structures monotone, FO-definable but not FO^+-definable.

Several variants: are all predicates monotone ?

From finite words to finite structures: variants

For now: signature $(\leq, a, b, c)$ assuming $\leq$ is a total order.

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Alphabet encoded by one binary predicate A.

$$a(x) \equiv A(0, x) \qquad b(x) \equiv A(1, x) \qquad c(x) \equiv A(2, x)$$

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▶ Axiomatize that $\leq$ is total assuming $\not\leq$ is its complement.

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- ▶ Axiomatize that $\leq$ is total assuming $\not\leq$ is its complement.

$a, b, c, \leq, \not\leq$ are monotone.

From finite words to finite graphs

Finite graphs: signature = one binary predicate E.

> ### Theorem (this work)
>
> There exists a class of finite graphs closed under edge addition, FO-definable but not FO$^+$-definable.

From finite words to finite graphs: the reduction

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Encode words into (directed) graphs, here $ab\binom{a}{b}c$:
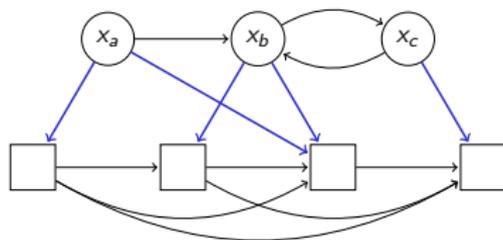
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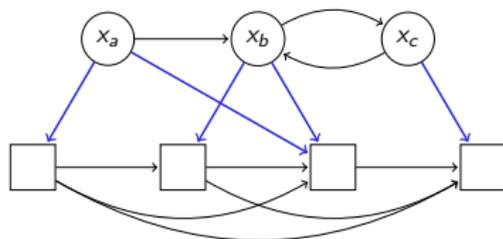
$\rightarrow$ formula ψ_L for graphs encoding words of $L = (a^{\uparrow}b^{\uparrow}c^{\uparrow})^* \cup (A^* \left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right) A^*)$.

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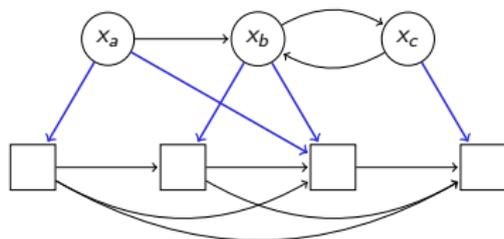Encode words into (directed) graphs, here $ab\binom{a}{b}c$:

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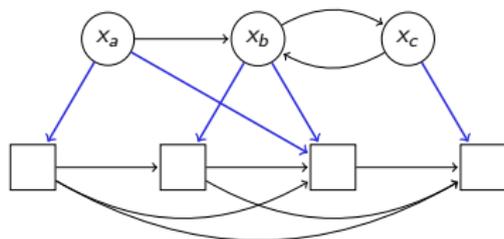
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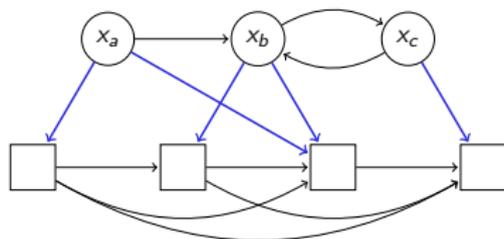Encode words into (directed) graphs, here $ab\binom{a}{b}c$:

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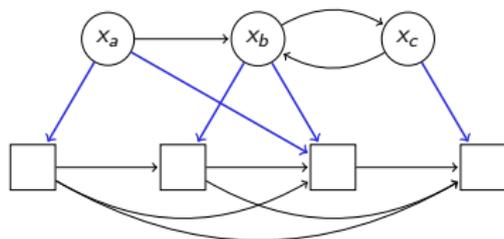
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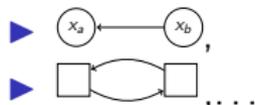   - ,

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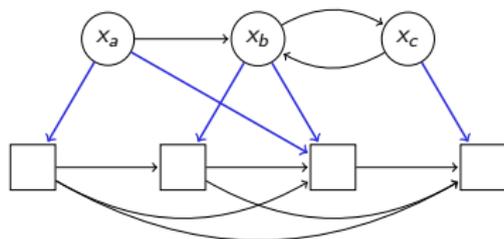
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    - $x_a \longleftarrow x_b$,
    - $\square \longleftrightarrow \square$,...

Final Formula: $\exists x_a, x_b, x_c.(\psi^- \wedge (\psi_L \vee \psi^+))$

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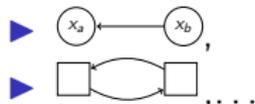
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Left as exercise: Same with undirected graphs.

Back to regular languages

Theorem
Given L regular on an ordered alphabet, it is decidable *whether*

- ▶ *L is monotone (e.g. automata inclusion)*
- ▶ *L is FO-definable [Schützenberger, McNaughton, Papert]*

Can we decide whether L is FO^+-definable ?

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Reduction from *Turing Machine Mortality*:
A deterministic TM M is *mortal* if there a uniform bound n on the runs of M from **any** configuration.
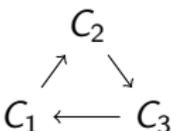
Undecidable [Hooper 1966].

Undecidability proof sketch

Given a TM M, we build a regular language L such that

$$M \text{ mortal } \Leftrightarrow L \text{ is } \mathrm{FO}^+\text{-definable.}$$

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$$M \text{ mortal} \iff L \text{ is } \mathrm{FO}^+\text{-definable}.$$

Building L:
Inspired from $(a^\uparrow b^\uparrow c^\uparrow)^*$, but:

▶ $a, b, c \rightsquigarrow$ Words from languages C_1, C_2, C_3 encoding configs of M.

▶ All transitions of M follow the cycle:

$$C_1 \xleftarrow{\quad} C_3 \quad \begin{array}{c} C_2 \\ \nearrow \quad \searrow \end{array}$$

▶ $\binom{a}{b}, \binom{b}{c}, \binom{c}{a} \rightsquigarrow \binom{u_1}{u_2}$, exists iff $u_1 \xrightarrow{M} u_2$.
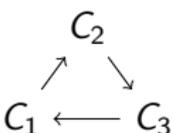
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\begin{array}{ccc}
 & C_2 & \\
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$$L := (C_1^\uparrow \cdot C_2^\uparrow \cdot C_3^\uparrow)^*$$

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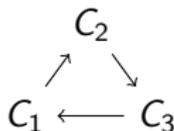
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⚠ $u \in L \not\Rightarrow u$ encodes a run of M.

The reduction

If M not mortal:

Let $u_1, u_2, \ldots, u_n$ a long run of M, and play Duplicator in :

$$
\begin{array}{llllll}
u \in L : & u_1 & u_2 & u_3 & \ldots & u_{n-1} & u_n \\
v \notin L : & \binom{u_1}{u_2} & \binom{u_2}{u_3} & \binom{u_3}{u_4} & \ldots & \binom{u_{n-1}}{u_n}
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Abstract u_i by the length of the run of M starting in it (at most n).

Play Spoiler in the abstracted game (here $n = 5$):

$$
\begin{array}{llllllllll}
u : & 2 & 3 & 2 & 4 & 3 & 5 & 4 & 3 & 4 & 4 \\
v : & & \binom{2}{1} & \binom{3}{2} & \binom{2}{1} & \binom{4}{3} & \binom{3}{2} & \binom{5}{4} & \binom{4}{3} & \binom{5}{4} & \binom{5}{4}
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$$

Spoiler always wins in $2n$ rounds $\rightarrow$ L is FO^+-definable.

Follow-up and perspectives

Other frameworks:
Counter-example and undecidability apply to other logics, e.g.:

$FO + \exists^{\mathrm{bounded}} X$ on words, $FO + \exists^{\infty} X$ on linear orders, ...

Slogan:
FO variants without negation will often display this behaviour.

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Follow-up [Iosti, K., Moreau 2025]: LTL^+, fragments of FO/LTL
Open question:

▶ Monotone $\mathrm{FO}_2 \stackrel{?}{=}$ Positive FO_2 ?

▶ Decidability of FO_2^+-definability ?

Follow-up and perspectives

Other frameworks:
Counter-example and undecidability apply to other logics, e.g.:

$FO+\exists^{bounded}X$ on words, $FO+\exists^{\infty}X$ on linear orders, ...

Slogan:
FO variants without negation will often display this behaviour.

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Thanks for your attention !