Positive first-order logic on words

Denis Kuperberg

CNRS, LIP, ENS Lyon

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This work was presented at LICS 2021

Background: Lyndon's theorem

First-order logic on arbitrary structures, signature $(P_1, \ldots, P_k)$.

Theorem (Lyndon 1959)

Let $\varphi \in \mathrm{FO}$, stable under making predicates true on more tuples. Then φ is equivalent to a negation-free formula.

Example: If a language of graphs is FO-definable and closed under adding edges, then it is FO-definable without $\neg$.

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  lattices, probabilities, number theory, topology, very hard
- *[Stolboushkin 1995]*
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- *[This work]*
  EF games on words, easy: language $(ABC)^*$

The FO+ logic, words as structures

FO^+ Logic: a ranges over Σ, no $\neg$

$$\varphi, \psi := a(x) \mid x \leq y \mid x < y \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \forall x.\varphi$$

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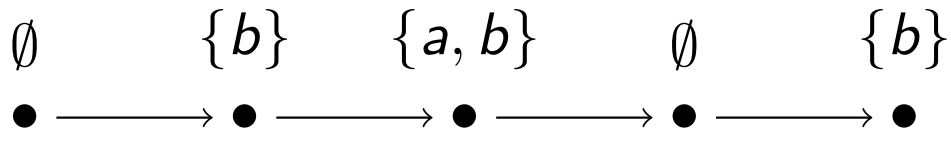
Word on alphabet 2^Σ:



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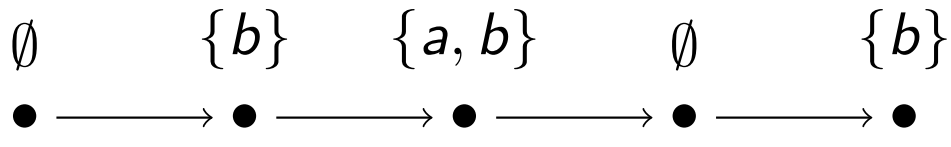

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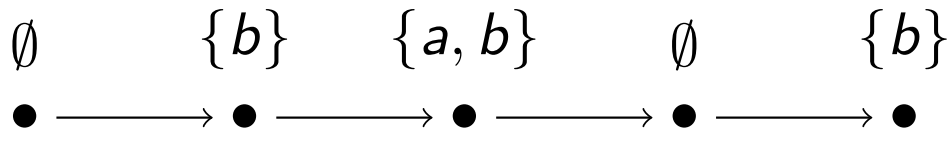

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There is FO-definable language on 2^Σ, closed under adding true Σ-predicates, but undefinable in FO^+.

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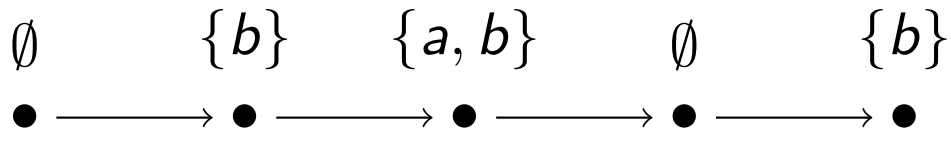

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Result 2: Undecidability

For regular languages on 2^Σ, FO^+-definability is undecidable.

Ongoing work

With Thomas Colcombet:
Exploring the consequences of this in other frameworks:

- regular cost functions,
- logics on linear orders,
- ...

Slogan:
FO variants without negation will often display this behaviour.

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Thanks for your attention !