Positive first-order logic on words and graphs

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Highlights of Logic, Games and Automata
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First-Order Logic (FO)

Signature: Predicate symbols $(P_1, \ldots, P_n)$ with arities $k_1, \ldots, k_n$.

Syntax of FO:

$$\varphi, \psi := P_i(x_1, \ldots, x_{k_i}) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \neg\varphi \mid \exists x.\varphi \mid \forall x.\varphi$$

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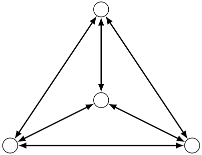
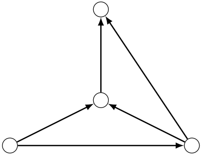
Example: For directed graphs, signature = one binary predicate E.

Graph class	Cliques	No node points to everyone
Formula	$\varphi = \forall x.\forall y.E(x,y)$	$\psi = \neg\exists x.\forall y.E(x,y)$
Example graph	 Model of φ	 Model of ψ

Positive versus Monotone

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For graph classes: monotone $=$ closed under adding edges.

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Motivation: Logics with fixed points.
Fixed points can only be applied to monotone φ.
Hard to recognize $\rightarrow$ replace by positive φ, syntactic condition.

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Theorem (Lyndon 1959)

If φ is monotone then φ is equivalent to a positive formula.

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- [This work]
  EF games on words, elementary

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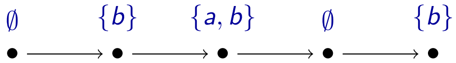

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$\to$ Words on alphabet $\mathcal{P}(\{a, b, \dots\})$:

$$\overset{\emptyset}{\bullet} \longrightarrow \overset{\{b\}}{\bullet} \longrightarrow \overset{\{a,b\}}{\bullet} \longrightarrow \overset{\emptyset}{\bullet} \longrightarrow \overset{\{b\}}{\bullet}$$

FO^+: $\neg a$ forbidden

L Monotone: $u\alpha v \in L$ and $\alpha \subseteq \beta \Rightarrow u\beta v \in L$

Our results

Finite Model Theory:

Lyndon's theorem fails on

- Finite words
- Finite graphs
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Finite Model Theory:

Lyndon's theorem fails on

- **Finite words:** $(ABC)^*$
- Finite graphs
- Finite structures

Regular Language Theory:

Monotone FO languages	$\neq$	**Positive FO languages**
Algebraic characterization		Logical characterization
Decidable membership		Undecidable membership

Ongoing work

With Quentin Moreau (internship):

- Link with LTL
- 2-variable fragment

With Thomas Colcombet:
Exploring the consequences of this in other frameworks:

- regular cost functions,
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- ...

Slogan:
FO variants without negation will often display this behaviour.

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Thanks for your attention !