Positive first-order logic on words

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FO^+ **and the powerset alphabet**

A special language

Background: Lyndon's theorem

Undecidability result

The FO^+ logic, words as structures

FO^+ Logic: a ranges over Σ, no $\neg$

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Example: On $\Sigma = \{a, b\}$:

$$\exists x, y.(x \leq y) \wedge a(x) \wedge b(y) \;\rightsquigarrow\; A^*\{a\}A^*\{b\}A^* \cup A^*\{a, b\}A^*$$

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Question [Colcombet]: FO & monotone $\overset{?}{\Rightarrow}$ FO^+

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Our first result

There is L monotone, FO-definable but not FO^+-definable.

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Language $L = (a^\uparrow b^\uparrow c^\uparrow)^* \bigcup A^* \begin{pmatrix} a \\ b \\ c \end{pmatrix} A^*$.

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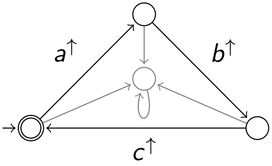
Lemma: L is FO-definable.

Proof: 

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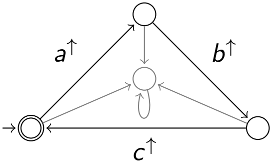
Proof: 

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To prove L is not FO^+-definable: Ehrenfeucht-Fraïssé games.

Ehrenfeucht-Fraïssé games for FO

Definition (EF games)

Played on two words u, v. At each round i:

- Spoiler places token i in u or v.
- Duplicator must answer token i in the other word such that
  - the letter on token i is the same in u and v.
  - the tokens are in the same order in u and v.

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Theorem (Ehrenfeucht,Fraïssé, 1950-1961)

L not FO-definable $\Leftrightarrow$ For all n, there are $u \in L$, $v \notin L$ s.t. $u \equiv_n v$.

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Example

Proving $(aa)^*$ is not FO-definable:

$$\begin{array}{lll} u = a^{2k} & \in (aa)^* : & a\ a\ a\ a\ a\ a\ a\ a\ a\ a \\ v = a^{2k-1} & \notin (aa)^* : & a\ a\ a\ a\ a\ a\ a\ a\ a \end{array}$$

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Definition (EF^+ games)

New rule:
Letters in u just have to be included in corresponding ones in v.

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Application: Proving L is not FO^+-definable

$$
\begin{array}{lcccccccccc}
u \in L: & a & b & c & a & b & c & a & b & c \\
v \notin L: & \binom{a}{b} & \binom{b}{c} & \binom{c}{a} & \binom{a}{b} & \binom{b}{c} & \binom{c}{a} & \binom{a}{b} & \binom{b}{c} &
\end{array}
$$

Background: Lyndon's theorem

First-order logic on arbitrary structures, signature $(P_1, \ldots, P_k)$.

Theorem (Lyndon 1959)

Let $\varphi \in \mathrm{FO}$, stable under making predicates true on more tuples. Then φ is equivalent to a negation-free formula.

Example: If a language of graphs is FO-definable and closed under adding edges, then it is FO-definable without $\neg$.

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- *[This work]*
  EF games on words, elementary thanks to L

Can we decide FO^+-definability?

Theorem

Given L regular on an ordered alphabet, we can decide

- *whether L is monotone (e.g. automata inclusion)*
- *whether L is FO-definable [Schützenberger, McNaughton, Papert]*

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FO^+-definability is undecidable for regular languages.

Reduction from *Turing Machine Mortality*:

A deterministic TM M is *mortal* if there a uniform bound n on the runs of M from **any** configuration.

Undecidable [Hooper 1966].

Undecidability proof sketch

Given a TM M, we build a regular language L such that

$$M \text{ mortal } \Leftrightarrow L \text{ is } \mathrm{FO}^+\text{-definable.}$$

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Building L:
Inspired from $(a^{\uparrow} b^{\uparrow} c^{\uparrow})^{*}$, but:

- $a, b, c \rightsquigarrow$ Words from C_1, C_2, C_3 encoding configs of M.
- All transitions of M follow the cycle: $\begin{array}{ccc} & C_2 & \\ \nearrow & & \searrow \\ C_1 & \longleftarrow & C_3 \end{array}$
- $\binom{a}{b}, \binom{b}{c}, \binom{c}{a} \rightsquigarrow \binom{u_1}{u_2}$, exists iff $u_1 \xrightarrow{M} u_2$.

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⚠ $u \in L \not\Rightarrow u$ encodes a run of M.

The reduction

If M not mortal:
Let $u_1, u_2, \ldots, u_n$ a long run of M, and play Duplicator in :

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Play Spoiler in the abstracted game (here $n = 5$):

$$\begin{array}{lcccccccccc} u: & 2 & 3 & 2 & 4 & 3 & 5 & 4 & 3 & 4 & 4 \\ v: & \binom{2}{1} & \binom{3}{2} & \binom{2}{1} & \binom{4}{3} & \binom{3}{2} & \binom{5}{4} & \binom{4}{3} & \binom{5}{4} & \binom{5}{4} & \end{array}$$

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Spoiler always wins in $2n$ rounds $\rightarrow$ L is FO^+-definable.

Ongoing work

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Exploring the consequences of this in other frameworks:

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Thanks for your attention !