Directed Minors for Minimal Automata

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Abstract

We study the following problem, that we call $\mathcal{C}$-recognizability, where $\mathcal{C}$ is a minor-closed class of undirected graphs: given a regular language L, is there a deterministic automaton for L whose underyling graph is in $\mathcal{C}$? We call such a language $\mathcal{C}$-recognizable. We aim at characterizing $\mathcal{C}$-recognizable languages via the underlying graph structure of their minimal automata. For this, we introduce a new minor relation for directed graphs, and show that the class of graphs of minimal automata of $\mathcal{C}$-recognizable languages is preserved under taking directed minors. We study the particular case where $\mathcal{C}$ is the class of planar graphs, and show that open problems from undirected graph theory can be reduced to planar recognizability for regular languages.

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Introduction

Regular languages form a robust and well-studied class of languages: they are recognized by finite deterministic automata (DFA), as well as various formalisms such as Monadic Second-Order logic, finite monoids, regular expressions, nondetermistic automata (NFA).

These robusts features are partly due to a canonical object that can be associated with each regular language: its minimal DFA. This object allows to efficiently test properties such as inclusion of regular languages, and can also be used as a measure of complexity, via its number of states. Almost all natural questions on natural languages can be answered by computing minimal DFAs, and check their properties.

Usually, this minimal DFA is considered "optimal", in the sense that the number of states is the most commonly accepted measure for the complexity of a DFA. However, there can be contexts where the crucial parameter is not the number of states, but rather another property related to the structure of the automaton, for instance tree-width, size of strongly connected components, or topological considerations such as planarity.

In this paper, we will be interested in the graph-theoretical properties of all DFAs recognizing a given language. The question we address is: given a minor-closed class $\mathcal{C}$ of undirected graphs, and a language L, is there a DFA for L whose underlying graph is in $\mathcal{C}$? For a fixed class $\mathcal{C}$, we call this problem $\mathcal{C}$-recognizability, and we aim at showing its decidability. We can also consider that $\mathcal{C}$ is part of the input, given by its finite list of forbidden minors, in which case we call the problem General Recognizability.

Contrary to most properties of regular languages, it does not suffice here to compute the minimal DFA, and check whether it verifies the wanted property, i.e. whether its underlying graph belongs to $\mathcal{C}$. Indeed, it can happen that L is recognized by a DFA whose graph is in $\mathcal{C}$, but that it is not the case of the minimal DFA.

We propose to study this problem by introducing a notion of directed minors, designed to reflect graph properties of the set of DFAs recognizing a language, while looking only at the minimal DFA for this language.

Our notion of directed minors is strictly richer than most alternatives from the literature (see Related Work section). Moreover, it preserves several good properties of the undirected minor relation, and interacts well with the notion of DFA. Therefore we hope it could serve as the good notion of directed minor relation in several contexts.

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Our approach is inspired by the success of the graph minor theorem [5], giving decidability of membership for any class of undirected graphs closed under the minor relation.

The emblematic example is the class of planar graphs, which is characterized by the two forbidden minors K_5 and $K_{3,3}$ [17]. Therefore, we will detail the special case where $\mathcal{C}$ is the class of planar graphs, called Planar Recognizability, in order to show the kind of behaviour that can occur in our framework.

Planar graphs have attracted considerable attention and continue to do so. Their study yielded many deep theorems, among which the Kuratowski and Wagner theorems [15, 17] and the four color theorem [1]. Moreover, open problems are still studied by the community, and we will be particulary interested in the existence of planar cover and planar emulation, linked to Negami's conjecture [7], and show how it connects to Planar Recognizability.

Related work

The famous minor theorem was obtained by Robertson and Seymour in a serie of papers culminating with [16].

A notion of directed minors, called butterfly minors, was used in [9, 10], to study directed tree-width. In our formalism, butterfly minors corresponds to allowing only the edge in- and out-contraction operations. The result from [9] concerning grid minors for planar digraphs was extended to all graphs in [12]. The same notion of butterflies minor is used as well to study the k-disjoint paths problem in [11].

TODO: Cite "Directed Graph Minors and Serial-Parallel Width" and what it cites as examples of directed minors.

In [13], a notion of directed minors specifically defined for tournaments is introduced, and it is shown that tournaments form a well quasi-order under this notion. Minors are obtained by contracting strongly connected components to single vertices.

A more general notion of minor was considered in [14], using both cycle contraction and in- and out-edge contraction. It is used to characterize particular classes of directed graphs. The authors aim at a directed graph minor theorem, that would induce decidability of membership for every class of directed graph closed under their minor relation.

Planar automata were investigated in [2], where it is shown that some regular languages are not recognized by planar DFA, but all regular languages are recognized by a planar NFA.

TODO: cite also Inherently Nonplanar Automata Book and Chandra

In the undirected framework, decidability of related problems called planar cover and planar emulation are given by a nonconstructive blackbox application of the Robertson-Seymour theorem [6]. Finding explicit algorithms, lists of forbidden minors, as well as topological characterizations remains open [7].

In [3], the more general notion of language genus was introduced. A language has genus g if it is recognized by a DFA $\mathcal{A}$ whose underlying graph can be embedded in a surface of genus g. Languages recognized by a planar DFA corresponds to languages of genus 0. It was shown in [3] that the genus classification induces a strict hierarchy among regular languages.

Contributions

We start by showing that labels of transitions can be ignored in the simulation relation of DFAs, thereby making the problem purely graph-theoretical.

We introduce a new notion of minor for directed graphs, well-behaved with respect to DFA minimization. We allow operations that restrict the power of undirected edge contraction:

edge in- and out-contractions and cycle contractions, and an additional operation called amalgamation, that allows to merge distant vertices.

We show that the resulting minor relation is strictly richer than the one from [10, 14], and therefore has a better chance of being a well-order.

We show that for any minor-closed class $\mathcal{C}$ of undirected graphs, the class of graphs of minimal automata of $\mathcal{C}$-recognizable languages is closed under this directed minor relation. We actually show a stronger result: we can consider that $\mathcal{C}$ is a class of directed graphs, closed only under directed minors in the sense of [14].

This is stronger than the undirected case, as all operations of directed minors from [14] can be seen as minor operations of the underlying undirected graph. Moreover, this allows to specify constraints related to the directed graph structure, for instance bounding the size of strongly connected components.

We hope this is a step towards proving the computability of the $\mathcal{C}$-recognizability problem.

We then study the particular case where $\mathcal{C}$ is the class of planar graphs. We show that in this case, the answer depends only on the structure of each strongly connected component of the minimal automaton. We give examples of minimal forbidden directed minors for Planar Recognizability, and show that they all must have size at least 7. This is to be compared with forbidden minors for planar graphs, of size 5 and 6.

We connect the problem of Planar Recognizability to conjectures in the theory of undirected graphs, where finding an algorithm for the existence of a planar emulator, as well as characterizing the class of graphs having a planar directed emulator, are famous open problems still attracting attention [4].

This gives an indication that Planar Recognizability, and more generally $\mathcal{C}$-recognizability, is likely to be a difficult challenge, as it subsumes long-standing open problems in graph theory. Nevertheless, we hope that this new approach using a rich directed minor relation will prove to be useful in the study of such problems.

1 Automata and graphs

1.1 Definitions: Automata and Graphs

A deterministic automaton (DFA) $\mathcal{A}$ is a tuple $(Q, \Sigma, p_0, F, \delta)$, where Q is a finite set of states, Σ is a finite alphabet, $p_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of final states, and $\delta : Q \times \Sigma \to Q$ is the transition function.

The run of $\mathcal{A}$ on a word $w = a_1 \dots a_n \in \Sigma^*$ is the sequence of states $p_0, p_1, \dots, p_n$ with $p_i = \delta(p_{i-1}, a_i)$ for all $i \in [1, n]$. We say that the run is accepting if $p_n \in F$.

The language $L(\mathcal{A})$ of $\mathcal{A}$ is the set of words $w \in \Sigma^*$ such that the run of $\mathcal{A}$ on w is accepting.

Let $\mathcal{A} = (Q_\mathcal{A}, \Sigma, p_0, F_\mathcal{A}, \delta_\mathcal{A})$ and $\mathcal{B} = (Q_\mathcal{B}, \Sigma, q_0, F_\mathcal{B}, \delta_\mathcal{B})$ be two DFAs on the same alphabet. An *automaton morphism* from $\mathcal{A}$ to $\mathcal{B}$ is a map $f : Q_\mathcal{A} \to Q_\mathcal{B}$ with the following properties:

(1) $f(p_0) = q_0$;
(2) $f^{-1}(F_\mathcal{B}) = F_\mathcal{A}$;
(3) For every $(p, a) \in Q_\mathcal{A} \times \Sigma$, we have $f(\delta(p, a)) = \delta(f(p), a)$.

We say that $\mathcal{B}$ is a quotient of $\mathcal{A}$ if there is a surjective automaton morphism $\mathcal{A} \to \mathcal{B}$. It is straightforward to show that in this case, $L(\mathcal{A}) = L(\mathcal{B})$.

▶ **Fact 1.** *For any regular language L, there is a unique DFA $\mathcal{A}_L$ recognizing L with minimal number of states. Moreover, given any DFA $\mathcal{A}$ recognizing L, $\mathcal{A}_L$ is a quotient of $\mathcal{A}$.*

A directed graph (or digraph) G is a pair (V,E) where V is a finite set of vertices and $E \subseteq V \times V$ is the set of directed edges. We say that G is undirected if E is symmetric. If $G=(V,E)$ is a digraph, its undirected support G_u is the undirected graph obtained by taking the symmetric closure of E, i.e. forgetting the direction of edges.

Given a DFA $\mathcal{A}$, we can forget the label of transitions, which states are initial or final, as well as multiplicity of edges, and obtain a digraph $G(\mathcal{A})$. More formally, if $\mathcal{A}=(Q,A,p_0,F,\delta)$, then $G(\mathcal{A})=(V,E)$ with $V=Q$ and $E=\{(p,q) \mid \exists a \in A, \delta(p,a)=q\}$. We note $G_u(\mathcal{A})$ the undirected support of $G(\mathcal{A})$, forgetting the direction of edges.

1.2 Directed emulators and $\mathcal{C}$-recognizability

▶ **Definition 1** (Directed emulators, amalgamation)**.** *Let $G=(V,E)$ be a digraph. We say that a digraph $G'=(V',E')$ is a* directed emulator *of G if there is a surjective map $\pi : V' \to V$ such that for all $(x,y) \in E$ and all $x' \in \pi^{-1}(x)$, there is $y' \in \pi^{-1}(y)$ such that $(x',y') \in E'$. Such a map π will be called a* directed emulator map*. We say that a digraph H is an amalgamation of a digraph G if G is a directed emulator of H.*

Our long-term goal is the decidability of the following problem, called $\mathcal{C}$-recognizability :

▶ **Definition 2** ($\mathcal{C}$-recognizability)**.** *Given a class $\mathcal{C}$ of undirected graphs and a regular language L, is there a DFA $\mathcal{A}$ such that $L(\mathcal{A})=L$ and $G_u(\mathcal{A}) \in \mathcal{C}$.*

We will be interested in several variants of the question: $\mathcal{C}$ can be a parameter of the problem or part of the input, and can also be a class of directed graphs. We will also restrict our attention to particular classes $\mathcal{C}$: minor-closed classes, and as a running example we will detail the case where $\mathcal{C}$ is the class of planar graphs in Section 4

We recall the classical notion of graph minor:

▶ **Definition 3** (Graph Minor)**.** *Let H,G be undirected graphs, we say that H is a minor of G if H can be obtained from G by a sequence of edge-contractions (merging two neighbours), edge deletions, vertices deletions.*

The next Lemma connects the notions of automata morphism and amalgamation:

▶ **Lemma 4.**

1. *If $\mathcal{B}$ is a quotient of $\mathcal{A}$, then $G(\mathcal{B})$ is an amalgamation of $G(\mathcal{A})$.*
2. *If $G(\mathcal{B})$ is an amalgamation of G, then there is a DFA $\mathcal{A}$ such that $G(\mathcal{A})=G$ and $\mathcal{B}$ is a quotient of $\mathcal{A}$*

Proof. (1) Let $\pi : \mathcal{A} \to \mathcal{B}$ be the surjective automaton morphism witnessing that $\mathcal{B}$ is a quotient of $\mathcal{A}$. We want to show that π is a directed emulator map $G(\mathcal{A}) \to G(\mathcal{B})$. Let (x,y) be an edge in $G(\mathcal{B})$. This means that there is a transition $x \xrightarrow{a} y$ in $\mathcal{B}$, for some letter $a \in \Sigma$. Let $x' \in \pi^{-1}(x)$, and $y' = \delta_{\mathcal{A}}(x',a)$. By definition of automata morphism, $\pi(y')=y$. The existence of the edge (x',y') in $G(\mathcal{A})$ shows that π is indeed a directed emulator map $G(\mathcal{A}) \to G(\mathcal{B})$.

(2) Let $G=(V,E)$ and $\mathcal{B}=(Q,\Sigma,p_0,F_{\mathcal{B}},\delta_{\mathcal{B}})$. Let π be the directed emulator map $G \to G(\mathcal{B})$. We build $\mathcal{A}=(V,\Sigma,q_0,F_{\mathcal{A}},\delta_{\mathcal{A}})$ based on G in the following way. We take q_0 arbitrarily in $\pi^{-1}(\{p_0\})$ (non-empty because π is surjective). We define $F_{\mathcal{A}}$ as $\pi^{-1}(F_{\mathcal{B}})$. Finally, if $v \in V$ and $a \in \Sigma$, by the definition of directed emulator map and since $(\pi(v),\delta_{\mathcal{B}}(\pi(v),a))$ is an edge in $G(\mathcal{A})$, there is $v' \in V$ such that $\pi(v') = \delta_{\mathcal{A}}(\pi(v),a)$ and $(v,v') \in E$. We set $\delta_{\mathcal{A}}(v,a) = v'$. It is straightforward to verify that π is an automaton morphism $\mathcal{A} \to \mathcal{B}$. ◀

▶ Remark 5. It is not true that $G(\mathcal{B})$ is an amalgamation of $G(\mathcal{A})$ if and only if $\mathcal{B}$ is a quotient of $\mathcal{A}$. Consider for instance that $\mathcal{A}$ and $\mathcal{B}$ can be identical up to a permutation of the letters, in which case they can recognize different languages, but have the same underlying directed graph.

In the following, we fix a class $\mathcal{C}$ of directed graphs. We call $\mathcal{CL}$ the class of languages recognized by a DFA $\mathcal{A}$ with $G(\mathcal{A}) \in \mathcal{C}$.

We aim at deciding membership in $\mathcal{CL}$ for an input language L by looking solely at the minimal DFA $\mathcal{A}_L$, and more precisely at its graph $G(\mathcal{A}_L)$. Therefore, we define

$$\mathcal{C}_{min} = \{H \mid \exists G \in \mathcal{C},\ H \text{ is an amalgamation of } G\}.$$

The following Lemma follows directly from Fact 1 together with Lemma 4.

▶ **Lemma 6.** *Let L be a regular language, we have $L \in \mathcal{CL}$ if and only if $G(\mathcal{A}_L) \in \mathcal{C}_{min}$.*

Proof.

$$\begin{aligned} L \in \mathcal{CL} &\iff \exists \text{ DFA } \mathcal{A} \text{ such that } L(\mathcal{A}) = L \text{ and } G(\mathcal{A}) \in \mathcal{C} \\ &\overset{\text{Fact 1}}{\iff} \mathcal{A}_L \text{ is the quotient of a DFA } \mathcal{A} \text{ with } G(\mathcal{A}) \in \mathcal{C} \\ &\overset{\text{Lem. 4}}{\iff} G(\mathcal{A}_L) \text{ is an amalgamation of a graph in } \mathcal{C} \\ &\iff G(\mathcal{A}_L) \in \mathcal{C}_{min}. \end{aligned}$$

◀

Therefore, deciding $\mathcal{C}$-recognizability can be reduced to deciding membership in $\mathcal{C}_{min}$.

When $\mathcal{C}$ is defined by a set of directed or undirected forbidden minors, we aim at characterizing $\mathcal{C}_{min}$ via its forbidden minors, with respect to a new notion of directed minors.

2 Directed minors

We keep notations $\mathcal{C}, \mathcal{CL}, \mathcal{C}_{min}$ from the previous section.

▶ **Definition 7.** *We say that $\mathcal{C}$ is closed under undirected minors if there is a minor-closed class C_u of undirected graphs, such that $\mathcal{C} = \{G \mid G_u \in C_u\}$.*

From now on, we will assume that the class $\mathcal{C}$ is closed under undirected minors, and investigate the impact of this assumption on $\mathcal{C}_{min}$.

Our goal is to define a directed minor relation, such that $\mathcal{C}_{min}$ is closed under taking directed minors. We start with the following lemma, which is actually true for any class $\mathcal{C}$.

▶ **Lemma 8.** *The class $\mathcal{C}_{min}$ is closed under amalgamations.*

Proof. It suffices to show that the composition of two directed emulator maps is a directed emulator map. Let $\pi_1 : G \to K$ and $\pi_2 : K \to H$ be directed emulator maps, on digraphs $G = (V_G, E_G)$, $K = (V_K, E_K)$ and $H = (V_H, E_H)$.

We want to show that $\pi = \pi_2 \circ \pi_1$ is a directed emulator map $G \to K$. First, notice that π is surjective, since π_1 and π_2 are.

Let $(x, y) \in E_H$ and $x'' \in \pi^{-1}(x)$. There is $x' \in \pi_2^{-1}(x)$ such that $x \in \pi_1^{-1}(x')$. Since π_2 is a directed emulator map, there is $y' \in \pi_2^{-1}(y)$ such that $(x', y') \in E_K$. Since π_1 is a directed emulator map, there is $y'' \in \pi_2^{-1}(y')$ such that $(x'', y'') \in E_G$. We have $y'' \in \pi^{-1}(y)$, so this achieves the proof that π is a directed emulator map.

◀

The notions of in-contraction, out-contraction and cycle-contraction are introduced in [14].

▶ **Definition 9** (Edge in-contraction). *Let $G = (V, E)$ be a digraph and let $e = (u, v) \in V$. The* in-contraction *along e of G, is the digraph $G_e = (V', E')$ where $V' = V \setminus \{v\}$ and $E' = (E \cup \{(u, x) \mid (v, x) \in E\}) \setminus \{(v, x), (x, v) \mid x \in V\}$*

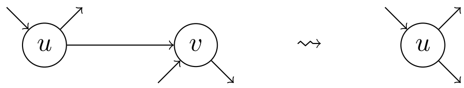


Figure 1 Edge in-contraction

Notice that this definition forgets edges with target v.

▶ **Definition 10** (Edge out-contraction). *Let $G = (V, E)$ be a digraph and let $e = (u, v) \in V$. The* out-contraction *along e of G, is the digraph $G_e = (V', E')$ where $V' = V \setminus \{u\}$ and $E' = (E \cup \{(x, v) \mid (x, u) \in E\}) \setminus \{(u, x), (x, u) \mid x \in V\}$*

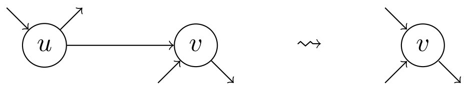


Figure 2 Edge out-contraction

Notice that this definition forgets edges outgoing from u.

▶ **Definition 11** (Cycle contraction). *Let $G = (V, E)$ be a digraph and let $C = \{v_0, v_2, \ldots, v_{r-1}\} \subseteq V$ be a directed cycle in G i.e., $(v_i, v_{i+1}) \in E$ for $0 \leq i < r - 1$ and $(v_{r-1}, v_0) \in E$. We note $E_{\overline{C}}$ the edges avoiding vertices from C. The C-contraction $G' = (V', E')$ of G is defined by $V' = (V - C) \cup \{w\}$ where w is a new vertex, and*

$$E' = E_{\overline{C}} \cup \{(x, w) \mid (x, v_i) \in E \text{ for some } 0 \leq i \leq r - 1 \text{ and } x \notin C\}$$
$$\cup \{(w, x) \mid (v_i, x) \in E \text{ for some } 0 \leq i \leq r - 1 \text{ and } x \notin C\}.$$

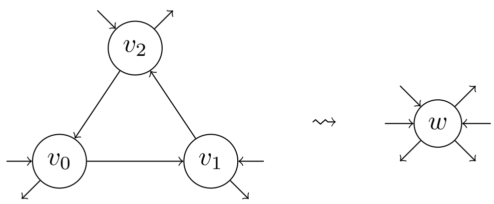


Figure 3 Cycle contraction

In the following, we assume that $\mathcal{C}$ is closed under undirected minors.

▶ **Theorem 12.** *The class $\mathcal{C}_{min}$ is closed under edge out-contractions.*

Proof. Let $G = (V, E) \in \mathcal{C}_{min}$: there is a directed emulator map $\pi : G_1 \to G$ where $G_1 = (V_1, E_1) \in \mathcal{C}$. Let $e = (u, v) \in E$ and $G_e = (V_e, E_e)$ the out-contraction of G along e. We recall that $V_e = V \setminus \{u\}$. We need to prove that $G_e \in \mathcal{C}_{min}$. Let $S = E_1 \cap (\pi^{-1}(u) \times \pi^{-1}(v))$ be the subset of all edges in E_1 connecting a preimage of u to a preimage of v. Let $G' = (V', E')$ be the result of performing a sequence of out-contractions from G_1 with respect to all edges from S, in an arbitrary order. Notice that in this process some edges from S can disappear before being treated, by out-contracting other edges from S. However, for each $u' \in \pi^{-1}(u)$, there is $v' \in \pi{-}1(v)$ such that an out-contraction along the edge (u', v') is performed in the process. In this process, all vertices in $\pi^{-1}(u)$ are removed, while their in-edges are redirected to vertices in $\pi{-}1(v)$. Since out-contraction is a particular case of the undirected minor relation on the underlying undirected graph, and $\mathcal{C}$ is closed under undirected minors, we have that G' is in $\mathcal{C}$.

We claim that G' is a directed emulator of G_e.

We build $\pi' : V' \to V_e$ by restricting π to $V' \subseteq V_1$. This is well-defined, since there is no $x' \in V'$ such that $\pi(x') = u$.

We show that π' is a directed emulator map $G' \to G_e$. Let $(x, y) \in E_e$ and $x' \in \pi'^{-1}(x)$.

- if $y \neq v$, then $(x, y) \in E$ and there is $y' \in \pi'^{-1}(y)$ such that $(x', y') \in E'$. Notice that if $x = v$, we use the fact that all edges outgoing from v in G_e correspond to edges outgoing from v in G, by definition of out-contraction.
- if $y = v$ then either $(x, v) \in E$ (i.e. was among the edges of G before the contractions) and we conclude as before, or $(x, u) \in E$. Then there is $u' \in \pi^{-1}(u)$ such that $(x', u') \in E_1$. Moreover, there is $v' \in \pi^{-1}(v)$ such that an out-contraction along (u', v') happened in the building of G'. Therefore, there is an edge $(x', v') \in E'$ with $\pi'(v') = v$.

We have showed that G_e has a directed emulator in $\mathcal{C}$, thereby proving that $G_e \in \mathcal{C}_{min}$. ◀

▶ Remark 13. TODO: adapt to in-contractions, only full minors for now The class $\mathcal{C}_{min}$ is in general *not* closed under in-contraction. We take $\mathcal{C}$ to be the class of planar languages for this counter-example. Consider the DFA A on the alphabet $\Sigma = \mathbb{Z}/7\mathbb{Z}$ defined as follows. The set of states is $Q = (\mathbb{Z}/7\mathbb{Z} \times \{0, 1\}) \cup \{p_0, \top, \bot\}$. The initial state is p_0 and the unique final state is $\top$. The transitions are defined by $p_0 \xrightarrow{j} (j, 0)$, $(i, 0) \xrightarrow{j} (i + j, 1)$, $(i, 1) \xrightarrow{j} \bot$ if $i \neq j$ and $(i, 1) \xrightarrow{i} \top$ for all $i, j \in \mathbb{Z}/7\mathbb{Z}$. The language computed by $\mathcal{A}$ is $L = \{a_0 a_1 a_2 \mid a_2 = a_0 + a_1 \mod 7\}$ and $\mathcal{A}$ is the minimal DFA computing L. Note that $G_u(\mathcal{A})$ is not planar since it contains $K_{3,3}$ as subgraph. However, $G(\mathcal{A})$ has a planar directed emulator obtained by unfolding $G(\mathcal{A})$ into a tree (of depth 4). On the other hand, contracting the edges $((i, 0), (i, 1))$ for $i = 0, \ldots, 6$ in G yields a digraph G' containing the full directed graph on 7 vertices as a subgraph. It is shown in the next Section (Lemma 22) that such a G' is not in $\mathcal{C}_{min}$.

▶ **Theorem 14.** *The class $\mathcal{C}_{min}$ is closed under cycle contractions.*

Proof. Let $G_0 = (V_0, E_0) \in \mathcal{C}_{min}$: there is a directed emulator $G_1 = (V_1, E_1)$ of G_0 with $G_{1u} \in \mathcal{C}$, witnessed by a directed emulator map $\pi : G_1 \to G_0$. Consider a cycle $C = (v_0, \ldots, v_{r-1})$ in G_0. We wish to prove that the digraph $G = (V, E)$ obtained by contracting C in G_0 lies in $\mathcal{C}_{min}$. Let $w \in V$ be the new vertex replacing the cycle C. The lift $\tilde{C} = \pi^{-1}(C)$ of the cycle C induces a subgraph $(\tilde{C}, E_{\tilde{C}})$ of G_1.

▶ **Lemma 15.** *Each weakly connected component of $\tilde{C}$ is composed of a cycle of length a multiple of r, containing antecedents for every node of C, together with a set of finite paths leading to this cycle.*

Proof. Let $i \in \{1, \ldots, r\}$, and $v'_i \in \pi^{-1}(v_i)$. Since π is a directed emulator map, there is an edge $v'_i \to v'_{i+1}$ with $v'_{i+1} \in \pi^{-1}(v_{i+1})$, where $i + 1$ is modulo r. We can continue this process, until the same vertex of G_1 is visited twice. This means we built a lasso of the form $v'_i \to v'_{i+1} \ldots v'_j \to v'_{j+1} \to \cdots \to v'_j$. The cycle around v'_j can correspond to several times the cycle C, if a vertex $v'''_j \in \pi^{-1}(\pi(v_j)) \setminus \{v_j\}$ is reached along the way.

Each starting point in $\tilde{C}$ eventually reaches such a cycle, which achieves the proof of the lemma. ◀

For each such weakly connected component (WCC) P in $\tilde{C}$, we create a vertice w_P, that will serve as the contraction of W.

Let $N(E_{\tilde{V}})$ denote the set of all edges in E_1 such that at least one endpoint lies in $\tilde{C}$. Let $G' = (V', E')$ the digraph defined as follows: $V' = (V_1 - \tilde{C}) \cup \{w_P | P \text{ WCC of } \tilde{C}\}$;

$$\begin{aligned} E' = E_1 - N(E_{\tilde{C}}) &\cup \{(x', w_P) \mid (x', v') \in E_1 \text{ for some } v' \in P, P \text{ WCC of } \tilde{C}\} \\ &\cup \{(w_P, x') \mid (v', x') \in E_1 \text{ for some } v' \in P, P \text{ WCC of } \tilde{C}\}. \end{aligned}$$

Observe that G'_u is obtained from G_{1u} by a sequence of usual undirected edge contractions (merging of adjacent vertices). Since $\mathcal{C}$ is preserved under edge contraction (a particular case of the minor relation on the underlying undirected graph), G'_u is in $\mathcal{C}$. here out-contraction suffices, no full edge contration needed Secondly, we claim that $G' = (V', E')$ is a directed emulator of G. We define $\pi' : V' \to V$ by $\pi'(v') = \pi(v')$ if $v' \in V_1 - \tilde{C}$, and $\pi'(w_P) = w$.

Let $(x, y) \in E$ and $x' \in \pi'^{-1}(x)$.

• If $x, y \neq w$ then $(x, y) \in E_0$ and there is $y' \in \pi'^{-1}(y)$ such that $(x', y') \in E'$, using the fact that π is a directed emulator map.

• If $x \neq w$ and $y = w$, then (x, v_j) is a directed edge of C for some $1 \leq j \leq r$. In particular, there is some $v'_j \in \pi^{-1}(v_j) \subseteq V_1$ such that $(x', v'_j) \in E_1$. It follows that $(x', w') \in E'$.

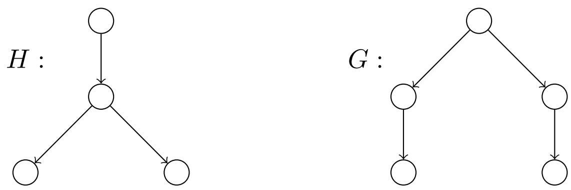
• If $x = w$ and $y \neq w$, then we must have $x' = w_P$ for some WCC P of $\tilde{C}$. There is an edge from some $v_j \in C$ to y in G_0. Let v'_j be a node in $P \cap \pi^{-1}(v_j)$, which is nonempty by Lemma 15. Since π is a directed emulator map, there is an edge $v'_j \to y'$ in G_1, with $\pi(y') = y$. Finally, we have an edge $w_P \to y'$ in G'.

In conclusion, we have showed that G has a directed emulator in $\mathcal{C}$. This proves that $G \in \mathcal{C}_{min}$. ◀

▶ **Definition 16.** *Let G, H be two digraphs. We say that H is a* directed minor *of G if H can be obtained from G by a succession of operations from this list: edge deletion, vertex deletion, amalgamation, edge out-contraction, cycle contraction.*

Compared to the notion of directed minors from [14], we added amalgamation. The example below shows that adding this operation makes our relation strictly richer than the one from [14].

▶ **Example 17.** Here H is a directed minor of G, but the amalgamation operation is necessary.



The results from this section imply the following theorem:

▶ **Theorem 18.** *The class $\mathcal{C}_{min}$ is closed under directed minors.*

Proof. It only remains to show that $\mathcal{C}_{min}$ is closed under edge and vertice deletions, which is a direct consequence of the fact that it is the case for $\mathcal{C}$. ◀

As a consequence, the class $\mathcal{C}_{min}$ can be characterized by a (possibly infinite) set of forbidden directed minors. If moreover this set can be chosen finite, and since the directed minor relation between two given graphs is decidable, the membership in $\mathcal{C}_{min}$ would be decidable. Therefore, a path to proving the decidability of $\mathcal{C}$-recognizability (i.e. membership in $\mathcal{CL}$) of regular languages via their minimal automata would be to show that the directed minor relation is a well-quasi-order.

We believe this notion of directed minors can serve as an analog of the graph minor relation in the framework of regular languages.

We can further generalize all the results from this section by taking $\mathcal{C}$ to be a class of directed graphs closed under directed minors in the sense of [14] i.e. under in-contractions, out-contractions and cycle-contractions (and as usual, edge and vertice deletion). Indeed, we only used these properties in the above proofs. Notice that if $\mathcal{C}$ is additionnaly preserved under amalgamation, the $\mathcal{C}$-recognazbility problem becomes easy, as we would have $\mathcal{C} = \mathcal{C}_{min}$.

Considering that $\mathcal{C}$ is closed under directed minors (and hence is a class of directed graphs) is more general, because any minor-closed class $\mathcal{C}$ is also closed under directed minors, when undirected graphs are viewed as particular cases of directed graphs. This allows us to study richer constraints, since $\mathcal{C}$ can relate to the digraph structure of the DFA, for instance bounding the size of strongly connected components.

2.1 From minimal $\mathcal{C}$ forbidden minors to minimal $\mathcal{C}_{min}$ forbidden minors

▶ **Theorem 19.** *Let M be a minimal forbidden minor of $\mathcal{C}$. Let M' be an orientation of M such that every vertex has in-degree at most 1. Then M' is a minimal forbidden minor of $\mathcal{C}_{min}$.*

Proof. Utilise le lemme que dans M' il y a un sommet qui peut atteindre tout le monde.

On peut reconstruire M dans n'importe quel expansion de M'.

Minimalité du fait que tout mineur strict de M' est un mineur strict de M, et donc dans $\mathcal{C}$. ◀

2.2 Oriented minor is not a well quasi-order

We have an infinite sequence $(C_p)_{p \text{ prime}}$ of graphs that are all independent.

C_p is the graph with $2p$ vertices, arranged in a cycle, where orientation of edges alternate, and target vertices also form a directed cycle.

3 Simple graph classes

Each class correspond to a machine having a special bounded memory structure.

3.1 DAGs: R-trivial languages

This class is actually also closed under amalgamations, so $\mathcal{C} = \mathcal{C}_{min}$. Forbidden minor: C_2.

3.2 Paths

Memory structure: counter with increment and decrement

Theorem:

Forbidden directed minors for $\mathcal{C}$: 3-cycle (C_3), transitive graph T_3: $1 \leq 2 \leq 3$, and all 3-stars: vertex with 3 distincts neighbours.

Forbidden directed minors for $\mathcal{C}_{min}$: 3-cycle (C_3), out-degree 3 (O_3), out-degree 2+ in-degree 1 ($IO_{1,2}$)

3.3 Directed Paths

Memory structure: counter with increment

Intersection of Paths and R-trivial.

$\mathcal{C} = \mathcal{C}_{min}$, and forbidden are C_2, and out-degree 2.

3.4 Pathwidth 1

Memory structure: counter with increment and decrement, and can be locked

Forbidden minors for $\mathcal{C}$: C_3, T_3, and 3-spiders $S_{3,2}$ of 7 vertices with any orientation: central vertex connected to 3 lines of length 2. Might be others because it is false that oriented minors come from non-oriented minors in general. Counter-example: K_5+1 intermediary vertex.

Forbidden minors for $\mathcal{C}_{min}$: C_3 and 3-spiders $S_{3,2}$ with the 3 possibles versions for choice of source vertex deciding orientation of edges. Probably more...

3.5 Cycles (and paths)

Memory structure: counter (modulo k) with increment and decrement

Conjecture: Forbidden directed minors: O_3, $IO_{1,2}$

3.6 Trees

Memory structure: stack (modulo k) with push and pop.

Conjecture: Forbidden directed minors: C_3

3.7 Pseudo-trees

Memory structure: stack (modulo k) with push and pop + counter modulo k modifiable only when the stack is empty.

3.8 Vertex set feedback of size at most 1

Memory structure: stack (modulo k) with push and pop + counter modulo k modifiable only when the stack is empty, and k can be changed to $\{k_1, k_2, \dots\}$ when the counter is at 0.

4 The case of planar languages

We now turn to a particular instance of the $\mathcal{C}$-recognizability, where $\mathcal{C}$ is the class of planar graphs. Given a regular language L, we want to decide whether there exists a planar DFA for L.

We will call a language planar if it is recognized by a planar DFA, and we will note $\mathcal{P}_{min}$ instead of $\mathcal{C}_{min}$, i.e. the class of digraphs having a planar directed emulator.

4.1 A family of examples

We denote by G_k (resp. $G_k^{i_1,\ldots,i_r}$) the digraph associated to Z_k (resp. $Z_k^{i_1,\ldots,i_r}$).

For each $k \geq 1$, we define the regular language on alphabet $\mathbb{Z}/k\mathbb{Z}$:

$$Z_k := \{a_1 a_2 \ldots a_n \mid \sum_{i=1}^{n} a_i \equiv 0 \mod k\}.$$

It will be convenient to denote $Z_k^{a_1,\ldots,a_r}$ the regular language obtained from Z_k by restriction to the subalphabet $\{a_1,\ldots,a_r\} \subseteq \mathbb{Z}/k\mathbb{Z}$.

▶ **Example 20.** The minimal DFA for the language $Z_5^{0,1,2}$ has K_5 as underlying undirected graph, therefore this automaton is not planar.

However, Figure 4 shows a planar DFA with six states recognizing the same language.

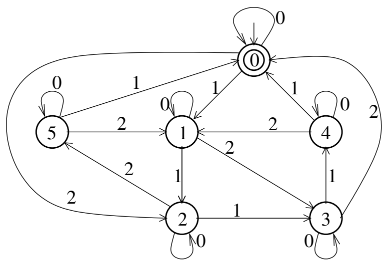


Figure 4 A planar DFA for Z_5

In the previous example, adding just an extra state suffices to produce a planar equivalent automaton.

The following lemma shows that even the language Z_6 with 6 letters, whose minimal automaton is the complete directed graph on 6 vertices, is still planar. In this case, each state needs to be duplicated.

▶ **Lemma 21.** *Z_6 is planar.*

Proof. The result follows from the existence of a planar cover for the complete graph K_6 (see e.g. [8] for a cover with 12 states). We do not give here the definition of planar covers, see for instance [7], but in this context it suffices to know that it is a particular case of undirected planar emulators. In Section 4.3, we recall the definition of undirected emulators and explicit an exact connection between planar emulators and planar regular languages. ◀

On the other hand, techniques from [2] allow to show that $Z_7^{1,2,3}$ and $Z_8^{1,5}$ are not planar. Indeed, Euler's formula for planar graph imply that if the minimal degree to distinct vertices is at least 3, or at least 2 in a bipartite graph, then the digraph cannot be in $\mathcal{P}_{min}$. TODO: Clarify, this should exlcude K_4, problem

Therefore, we have:

▶ **Lemma 22.** *If a digraph G has $G(Z_7^{1,2,3})$ or $G(Z_8^{1,5})$ as a directed minor, then $G \notin \mathcal{P}_{min}$.*

4.2 A general decomposition result

▶ **Theorem 23.** *Let L be a regular language. Then L is planar if and only if the strongly connected components (SCCs) $C_1, \ldots, C_n$ of $G(\mathcal{A}_L)$ are all in $\mathcal{P}_{min}$.*

Proof. If $G(\mathcal{A}_L)$ are all in $\mathcal{P}_{min}$, then it is clear that all its SCC are in $\mathcal{P}_{min}$ as well, since we know from Section 2 that $\mathcal{P}_{min}$ is closed under taking subgraphs.

We now assume that each SCC $C_i \in \{C_1, \dots, C_n\}$ of $G(\mathcal{A}_L) = (V, E)$ has a planar directed emulator G_i, with directed emulator map $\pi_i : G_i \to C_i$. We show that this is enough to build a planar DFA for L.

Let C_1 be the SCC containing the initial state p_0 of $\mathcal{A}_L$. We can assume without loss of generality that G_1 is minimal in the topological order of G, meaning it cannot be reached by another SCC.

We define an intermediary graph G by taking the union of the components G_i, and adding all edges $\{(p,q) | (p',q') \in E(L), p \in \pi_i^{-1}(p'), q \in p_j^{-1}(q')$ for some $i \neq j\}$. These new edges are called transient edges. It is clear that G is a directed emulator of $G(\mathcal{A}_L)$, however it is not planar in general.

We will now turn G into a planar directed emulator of $G(\mathcal{A}_L)$, by making copies of its SCC to organize them in a tree structure.

Let G_i be a SCC of G, and S_i be the set of paths reaching G_i from G_1. To each path s we associate the subpath $f(s) = e_1 \dots e_k$ of transient edges from s. Notice that for any such path s, the length k of $f(s)$ is at most n. Let $T_i = f(S_i)$, the set of transient subpaths reaching G_i. for each $t \in T_i$, we build a copy G_t of G_i.

Notice that only one copy of G_1, namely G_ϵ, is built this way.

We build the graph G' by taking the union of all the G_t (for all initial components G_i), and by connecting them in the intuitive way: $G_t \xrightarrow{e} G_{te}$.

It is straightforward to verify that this graph G' is still a directed emulator of $G(\mathcal{A}_L)$, witnessed by a planar emulator map π, defined by aggregating all the maps π_i on every copy for all i. Moreover, it is planar, since it consists in planar components arranged in a tree, and connected via single transitions.

It remains to show that we can build a planar DFA $\mathcal{A}$ using G' as underlying structure. let $q_0 \in G_1$ such that $\pi_1(q_0) = p_0$, we choose q_0 as initial state of $\mathcal{A}$. The accepting set of $\mathcal{A}$ is $\pi{-}1(F)$.

Finally, let $p \xrightarrow{a} q$ be a transition in $\mathcal{A}_{\min}$. If $p = q$ then for all $p' \in \pi^{-1}(p)$ we add a transition $p' \xrightarrow{a} p'$ in $\mathcal{A}$. Notice that this does not change the planarity of the graph. If $p \neq q$, then there is an edge $p \to q$ in $G(L)$. This means that for any $p' \in \pi^{-1}(p)$, there is an edge $p' \to q'$, with $\pi(q') = q$. We can therefore add an edge $p' \xrightarrow{a} q'$ in $\mathcal{A}$, without modifying the underlying graph G'. This achieves the description of the planar DFA $\mathcal{A}$ recognizing L. ◀

Together with Lemma 21, we obtain the following corollary:

▶ **Corollary 24.** *If all SCCs of $\mathcal{A}_L$ have size at most 6 then L is planar.*

It is interesting to compare forbidden minors for $\mathcal{P}_{min}$ to the classical case of planar graphs. It is well-known that a graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor. Here however, forbidden directed minors for $\mathcal{P}_0$ must be of size at least 7, by Lemma 21.

We show in the next section that this problem generalizes the (famously difficult) problem of existence of planar emulators in the undirected case.

4.3 Link with undirected emulation

We recall here definitions from undirected graph theory.

▶ **Definition 25.** *Let $G = (V, E), G' = (V', E')$ be undirected graphs. We say G is an* emulator *of G' if there is a surjective mapping $\pi : V \to V'$, such that for each $v \in V$, π maps surjectively the neighbours of v to the neighbours of $\pi(v)$.*

Because the class of undirected graphs having planar emulators is closed under the classical minor relation [6], we have the following:

▶ **Theorem 26** ([6]). *It is decidable in $O(n^3)$ whether a graph has a planar emulator, where n is the number of vertices of G.*

However, no explicit algorithm is known for this problem. Indeed, finding a full set of forbidden minors for this class of graphs is an open problem.

We will call *Planar Emulation* the above problem in the undirected case, and *Planar Recognizability* the problem of deciding whether a regular language is planar.

▶ **Theorem 27.** *Planar Emulation polynomially reduces to Planar Recognizability.*

The rest of this section is dedicated to proving Theorem 27.

We assume the existence of an algorithm deciding Planar Recognizability, and we describe an algorithm for Planar Emulation. Remark that it suffices to decide Planar Emulation on connected graphs, since the algorithm can be called on each component in the case of general disconnected graphs.

Let $G = (V, E)$ be a connected undirected graph, for which we want to decide Planar Emulation. We build an alphabet $\Sigma = \{a_e \mid e \in E\} \cup \{b_e \mid e \in E\}$ of size $2|E|$.

We turn G into a DFA $\mathcal{A}$ by turning each undirected edge $e = \{x, y\}$ into a pair of transitions $x \xrightarrow{a_e} y$ an $y \xrightarrow{b_e} x$. We complete the automaton with a sink $\perp$, and for each $p \in V$ and $a \in \Sigma$ such that a does not label any outgoing edge of p, we add a transition $p \xrightarrow{a} \perp$. We choose any $p_0 \in V$ as initial state, and $\perp$ is the only non-accepting state. This completes the description of the DFA $\mathcal{A}$.

▶ **Lemma 28.** *$\mathcal{A}$ is the minimal DFA of $L(\mathcal{A})$.*

Proof. For each letter a, there is a unique state p such that the single-letter word a is accepted from p. Therefore, no two states accept the same language, and $\mathcal{A}$ is minimal. ◀

▶ **Lemma 29.** *$L(\mathcal{A})$ is planar if and only if G has a planar emulator.*

Proof. Assume $L(\mathcal{A})$ is planar, and let $\mathcal{B}$ be a planar DFA accepting $L(\mathcal{A})$. This means there is an automaton morphism $f : \mathcal{B} \to \mathcal{A}$. Let $H = G_u(\mathcal{B}) = (V_H, E_H)$ be the underlying graph of $\mathcal{B}$. The function f induces a surjective function $f_H : V_H \to V$. Let $q \in V_H$ and $p = f_H(q) \in V$. Let $p' \in V$ be a neighbour of p, this means there is a transition $p \xrightarrow{a} p'$ in $\mathcal{A}$ for some letter $a \in \Sigma$. Additionally, let b be a letter accepted from p'. The word ab is accepted from p so it must be accepted from q in $\mathcal{B}$. Therefore, there is a transition $q \xrightarrow{a} q'$ in $\mathcal{B}$ such that b is accepted from q'. This means that q' is a neighbour of q and $f_H(q') = q$. So any neighbour of p is the image of a neighbour of q. Let q' be a neighbour of q, this means either there is a transition $q' \xrightarrow{a} q$ or a transition $q \xrightarrow{a} q'$. In both cases, $f_H(q)$ and $f_H(q')$ are neighbours in G, so f_H maps surjectively neighbours of q in H to neighbours of $f_H(q)$ in G. This shows that H is a planar emulator of G.

Conversely, let $H = (V_H, E_H)$ be a planar emulator of G witnessed by a mapping $f : V_H \to V$, we want to show that $L(\mathcal{A})$ is planar. We design a DFA $\mathcal{B}$ based on H:

- The initial state is a $q_0 \in f^{-1}(p_0)$
- We add a sink state $\perp_q$ next to each state q of V_H.

- Let $q \in V_H$ and $a \in \Sigma$. Let $p = f(q)$. If there is a transition $p \xrightarrow{a} p'$ in $\mathcal{A}$ with $p' \neq \bot$, then we choose q' a neighbour of q with $f(q') = p'$, and we add a transition $q \xrightarrow{a} q'$ in $\mathcal{B}$. Otherwise, if $p \xrightarrow{a} \bot$ in $\mathcal{A}$, then we add a transition $q \xrightarrow{a} \bot_q$ in $\mathcal{B}$.
- All states of V_H are accepting, while the $\bot_q$ are rejecting.

It is straightforward to verify that $\mathcal{B}$ is planar and recognizes $L(\mathcal{A})$. ◀

Lemmas 28 and 29 put together show that deciding Planar Emulation for G amounts to deciding whether $L(\mathcal{A})$ is planar.

This means that finding an algorithm for Planar Recognizability would in particular provide an algorithm for Planar Emulation. This would give an algorithm answering open problems in graph theory, namely whether particular graphs have planar emulators [7].

Conclusion

We introduced a notion of minors for directed graphs, generalizing existing alternatives, in particular the ones from [10, 14]. We showed that if $\mathcal{C}$ is a class closed under directed minors in the sense of [14] (which is less restrictive than undirected minors, or our notion), then the class of minimal automata having a DFA in $\mathcal{C}$ is closed under our notion of directed minors. This paves the way to show decidability of $\mathcal{C}$-recognizability, since our notion of directed minors could form a well-order even if the one from [14] does not.

The decidability of General Emulation where forbidden minors for $\mathcal{C}$ is part of the input poses a more difficult but very interesting challenge. Indeed, coming up with an algorithm for General Emulation would mean that we understand a systematic (and computable) link between the forbidden minors of $\mathcal{C}$ and the ones of $\mathcal{C}_{min}$.

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