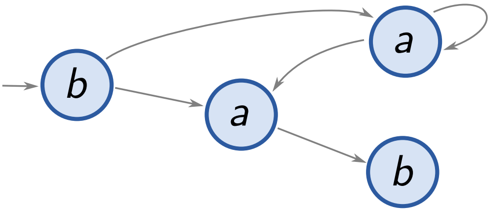
Tree Algebras and Bisimulation-Invariant MSO on Finite Graphs

Thomas Colcombet, Amina Doumane, Denis Kuperberg

CNRS, LIP, ENS Lyon

GdT Plume, 23/06/25

Transition systems



Specifying properties

MSO formulas:

$$\varphi, \psi := a(x) \mid E(x,y) \mid \exists x.\varphi \mid \exists X.\varphi \mid x \in X \mid \varphi \vee \psi \mid \neg\varphi$$

Example: $\varphi(r)$ for "$\exists$ ∞ path from r":
$\exists X.$
$r \in X \wedge$
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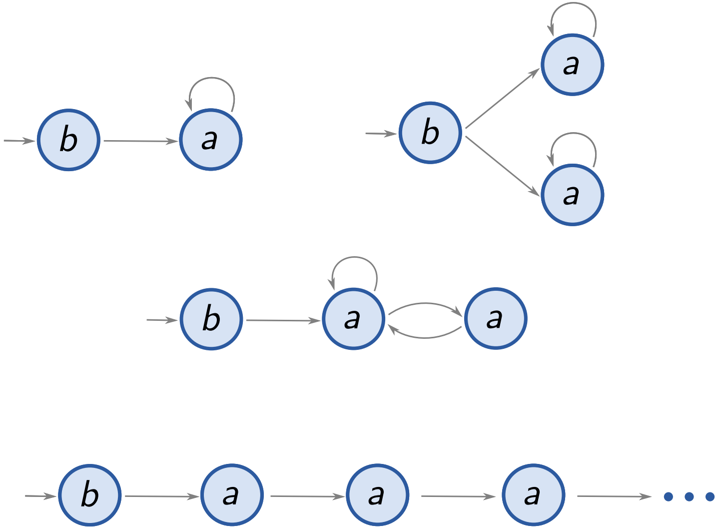
μ-calculus formulas:

$$\varphi, \psi := a \mid \diamond\varphi \mid \Box\varphi \mid \mu X.\varphi \mid \nu X.\varphi \mid \varphi \vee \psi \mid \neg\varphi$$

Example: ψ for "$\exists$ ∞ path from the current vertex": $\nu X. \diamond X$
Fact: μ-calculus is bisimulation-invariant.

Bisimulation



Starting point

Theorem (Janin and Walukiewicz 1996)

For properties of systems, the following are equivalent:

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μ-calculus $\rightarrow$ bisim-inv MSO : Easy

Bisim-inv MSO $\rightarrow$ μ-calculus : Hard

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Correctness:
Infinite trees suffice to define bisim-inv properties of systems.

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Main Contribution

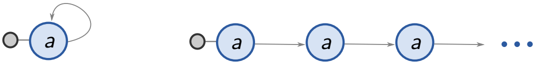
For properties of **finite** systems, the following are equivalent:

1. Being MSO-definable and bisimulation-invariant.
2. Being μ-calculus-definable.

Example of the difference

MSO formula φ for "$\exists$ cycle":

- φ is not bisim-invariant on all systems.



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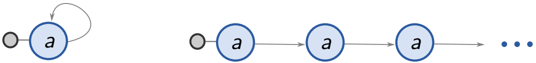

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$\Longrightarrow$ using Janin-Walukiewicz does not work for finite systems.

Ranked systems

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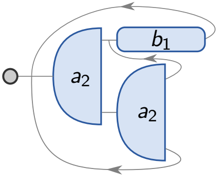

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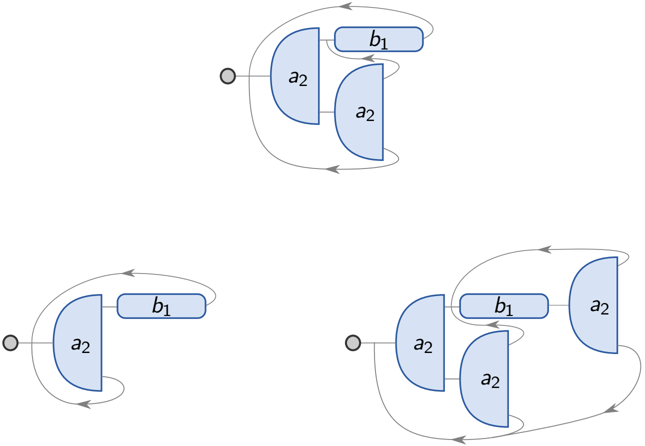
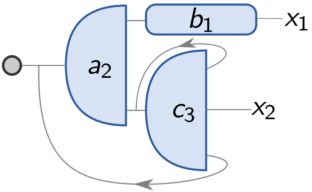


The free algebra of systems

Systems have open ports and arities:



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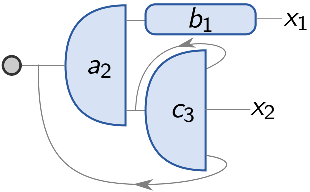
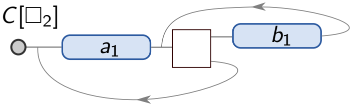
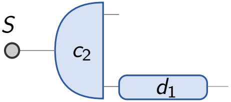
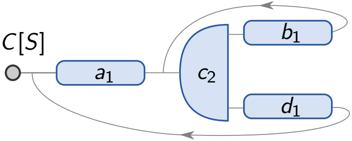


Operation: Plug into context.







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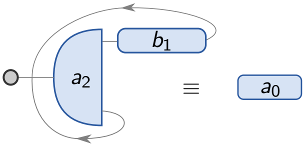
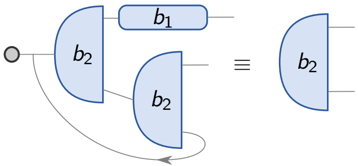


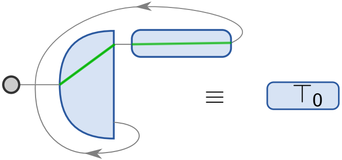


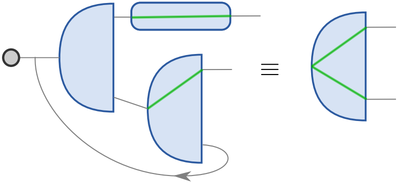
Then $L = \{\text{Systems evaluating to } a_0\}$, via h :Systems$\rightarrow \mathcal{A}$.

Another example of algebra

Language $L = \{\exists \text{ cycle containing } a\}$.

Then $A_n = \{\top_n\} \cup \mathcal{P}(\{1, \ldots, n\})$





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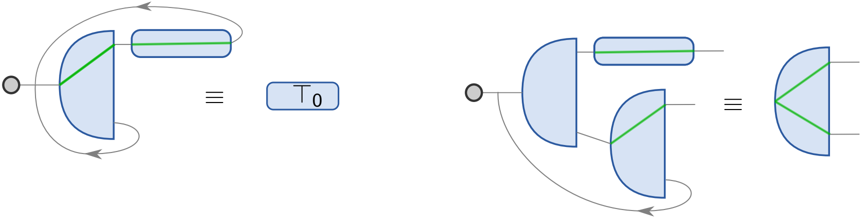

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$\mathcal{A}$ is rankwise-finite and unfold-invariant.
Intuition: Enough for regularity.

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Main Contribution 2

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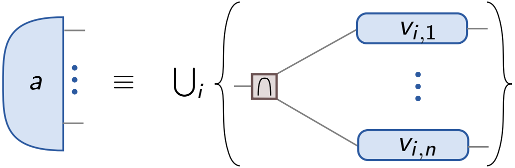
Add operators such as intersection $\cap$ to the algebra.

Key Lemma

$\forall a \in A_n$, $\exists (v_{i,j})$ from A_1 such that:



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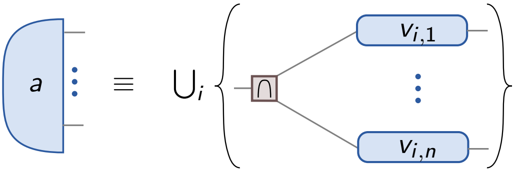
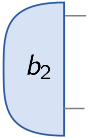


Consequences

- A_1 actually contains all the information about A_n.
- Algebras can be turned into automata.

Key Lemma Example

Algebra: $A_n = \{a_n, b_n\}$, for "$\exists a$".



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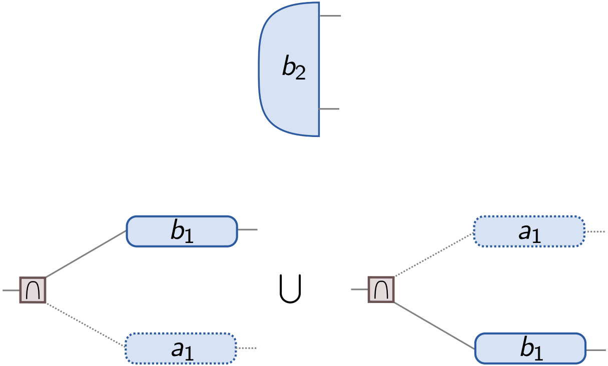

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- Bisim-invariant automata $\rightarrow$ μ-calculus as in [Janin-Walukiewicz 1996].

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Thanks for your attention!