The Mathematics of Juggling

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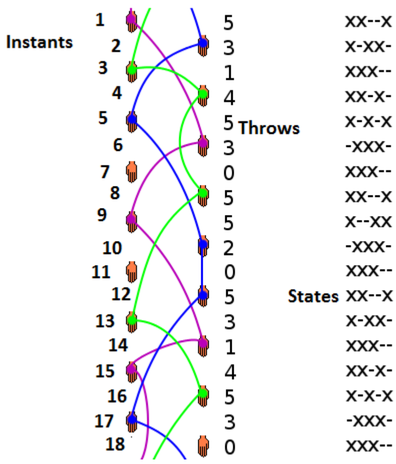
Juggling is an ancient art, and its origins seem to go back to the origin of mankind, as witnessed for instance by drawings of juggling women found in an Egyptian tomb dating from 1994-1781 BC. It just requires a few number of rocks, and a little practice, so it is not surprising that humans started to play with it very soon in the history.

However, it is only very recently that people started to get interested into the mathematical aspects of it. The first appearances of mathematical notations for juggling patterns originated in the 80's independently in Santa Cruz, Caltech University and Cambridge University. I will give here here a brief description of how mathematics can help jugglers structure their patterns, and also how this interaction of different fields can give back to mathematicians.

The first step is a rough modeling of the juggling activity, in order to be able to say something rigorous about it. We will consider (for starters) that time is discretized into a sequence of instants: $1, 2, 3, \dots$, and that jugglers have two hands, which can each contain at most one object (ball) at a given instant. The timing alternates between the two hands, so one hand can catch and/or throw a ball at even instants, and the other hand at odd instants.

Now it gets more interesting: there are different ways to throw a ball, and we will modelize them only by the time (number of instants) the ball stays in the air. It means that a throw of type t at instant i lands at instant $i+t$. Notice that this means all even throws land in the same hand that threw them, and odd throws switch hands. We will note 0 for empty throw, i.e. a hand does a throw 0 at instant i if the hand is empty at instant i.

Here is an exemple of a sequence of throws, with time going down. Each ball is identified with a color, and additional information (states) is also represented for further description. Notice that here the juggler is using three balls.



This particular sequence will be described only by the sequence of throws performed, i.e. 531453055205314530. This notation is called *siteswap*. Jugglers are usually interested into simpler sequences than the one presented here. Of special interest are *periodic* sequences, which can be repeated ad infinitum. For instance the first pattern that most people learn with 3 balls is called *cascade*, and corresponds to the sequence $33333\dots$.

To avoid repeating useless information, we usually describe a periodic sequence by what happens during one period, so the cascade corresponds just to the pattern 3.

To the mathematician minds, this can raise a lot of questions. Are all patterns valid? How the number of balls reflects on the pattern? How many patterns exist if we fix some constraints (like number of balls and maximal throw)?

Here are a few basic properties of siteswap patterns:

Definition: Let $t_0 \dots t_{k-1}$ be a siteswap pattern with $k \geq 1$. It is valid if and only if $\sigma : [0, k-1] \to [0, k-1]$ defined by $\sigma(i) = i + t_i \mod k$ is a permutation of $[0, k-1]$. All patterns with $k = 1$ are valid, and pattern n corresponds to the cascade with n balls.

This definition just translates the fact that it is forbidden for two throws to land at the same time. So all throws of the pattern must land at different times.

Lemma: In a valid siteswap pattern for n balls, the average of the throws is n.

Proof is left as exercise for readers who like a challenge...

This can be used as a first sanity check for a pattern: if the average is not an integer, then the pattern cannot be valid.

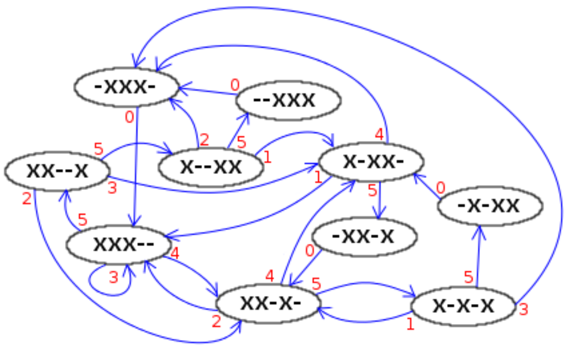
Examples:

- 521 has an average of 8/3 so it is not a valid pattern.
- 321 has an average of 2, so it could describe a pattern with 2 balls. However, the first condition is not respected: $\sigma(0) = \sigma(1) = \sigma(2) = 0$, so all throws would land at the same time.
- 441, 531, 55500 are valid patterns with 3 balls. 6451, 7333, 71 are valid patterns with 4 balls.

An interesting nontrivial result from Benoit Guerville is the following:

Reorganisation theorem: Any sequence with integer average can be reorganised in a valid pattern.

I will now explain the "states" part of the above figure. At every given moment, we can look at the "height" of each ball, i.e. when it will land back. Since two balls can never land at the same time, they are all at different height. We will note from left to right, X if there is a ball and $-$ if there is not. In particular, the leftmost symbol corresponds to height 0, meaning the current active hand. We stop the description of the state to an arbitrary maximal height (5 in the example), so description of states are finite, and we limit all throws to this maximal height. This means that if a state starts with $-$, the current hand is empty, and the next throw is a 0. But if there is a X, the ball can be throw at any position marked with $-$, which corresponds to an empty slot. The maximal throw is always allowed, corresponding to a virtual $-$ at the last position. Then all the symbols are shifted to the left, to account for a passing time unit. Using this states, we always know at any time what are the allowed throws, and this can be summed up by a state-transitions diagram (or automaton) like the following (here for 3 balls and maximal height 5):



So the allowed sequences are exactly the labels of paths in this automaton, and periodic patterns correspond to cycles. This could be used to define exotic juggling patterns, and a number of performers (for instance the company Gandini Juggling) use Siteswap extensively to design routines for their shows.

Notice that for the mathematical results, we never used the fact that there are two hands, and everything is still valid with more juggling "sites" (feet, head, several people,...) On the mathematical side, people still continue to explore Siteswaps, and it was generalized in a number of ways: synchronous time (different hands can throw at the same time), multiplexes (one hand can throw several balls simultaneously),. . . .

It also led to more abstract research, many papers were (and still are) published on the subject, mostly dealing with the combinatorics of siteswap. Interestingly, some ideas developed for studying juggling were then used in some other mathematical areas. Lots of papers, and sometimes even PhD theses (for instance "Combinatorial aspects of Juggling" by Anthony Mays) relate concepts from siteswap to cutting-edge research in mathematics.

So there is no such things as useless maths: interconnectedness often catches on the most innocent subjects!

Images adapted from Wikipedia.