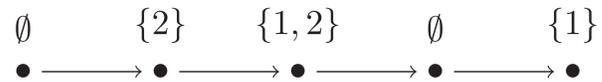


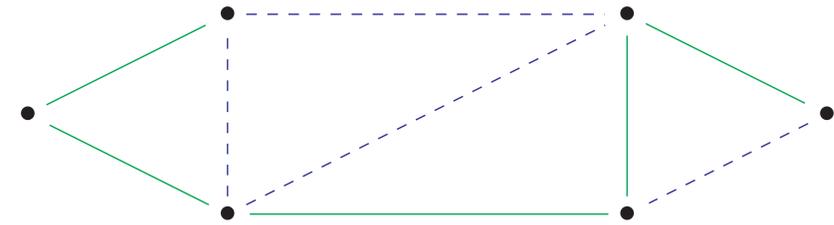


WORDS

Alphabet: $A = 2^\Sigma$



ARBITRARY STRUCTURES



UPWARD-CLOSED LANGUAGES

L is upward-closed if:

$$\left. \begin{array}{l} uav \in L \\ a \subseteq b \end{array} \right\} \implies ubv \in L.$$

Examples:

$L_0 = A^*\{1, 2\}A^*$: Yes

$L_1 = A^*\{1\}A^*$: No

UPWARD-CLOSED CLASSES OF STRUCTURES

Fix any relational signature R_1, \dots, R_k .

A class \mathcal{C} of structures is upward-closed if:

$$\left. \begin{array}{l} (S, R_1, \dots, R_k) \in \mathcal{C} \\ \forall i, R_i \subseteq R'_i \end{array} \right\} \implies (S, R'_1, \dots, R'_k) \in \mathcal{C}$$

\mathcal{C} upward-closed $\iff \mathcal{C}$ closed under surjective morphisms.

POSITIVE FO (FO^+)

Grammar for FO^+ , a ranges over Σ :

$$\varphi := x \leq y \mid x < y \mid a^\uparrow(x) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x. \varphi(x) \mid \forall x. \varphi(x)$$

- no negation

- $a^\uparrow(x)$: label of x contains a .

only defines upward-closed languages.

LYNDON'S THEOREM

[Lyndon 1959]

FO -definable and upward-closed $\iff \text{FO}^+$ -definable

Also true if monotonicity is with respect to a subset of the signature.

SYNTAX VERSUS SEMANTIC

L definable in $\text{FO}^+ \implies L$ FO -definable and upward-closed.

Theorem [This work]: The converse does **not** hold:

$\exists L$ upward-closed and FO -definable but not FO^+ -definable.

FAILURE ON FINITE STRUCTURES

Lyndon's theorem fails on finite structures:

- **Very hard:** on signature $(4, 3, 3, 3, 3, 2, 1, \underline{1})$ [Ajtai Gurevich 1987] (lattices, probabilities, number theory, topology)
- **Hard:** on signature $(2, \underline{2})$ [Stolboushkin 1995] (Ehrenfeucht-Fraïssé games on grids)
- **Easier:** on signature $(2, \underline{1}, \underline{1}, \underline{1})$ [This work] (E-F games on words)

Numbers are arities, underlined in green for positive predicates.

FUTURE RESEARCH

- Decidability of FO^+ -definability ?
- Algebraic characterization of FO^+ -definable languages ?
- Generalization to quantitative settings ?