Stamina: Stabilisation Monoids in Automata theory

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Quantitative Automata

Kinds of quantitative automata:

- Weighted automata [Schützenberger 1961]
- Probabilistic automata [Rabin 1963]
- Timed automata [Alur, Dill 1994]
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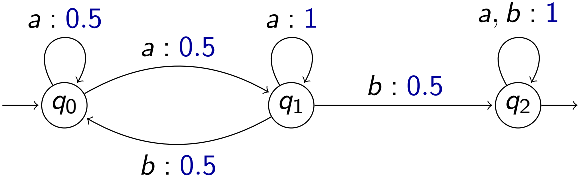
This tool: Analyse asymptotic behaviours of quantitative automata.

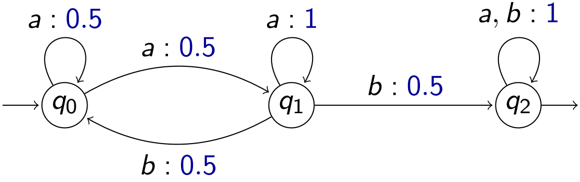
Probabilistic Automata



Probabilistic automaton: computes function $P_{\mathcal{A}} : A^* \to [0, 1]$, mapping words to probability of acceptance.

Example: $P_{\mathcal{A}}(ab) = 0.25$

Value 1 Problem



Value 1 problem: is there a sequence of words $(u_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} P_{\mathcal{A}}(u_n) = 1 \ ?$$

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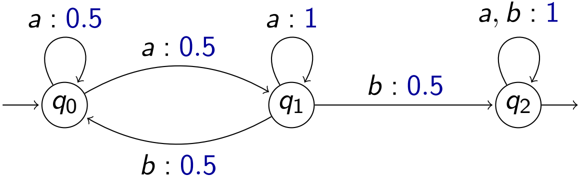

Value 1 problem: is there a sequence of words $(u_n)_{n\in\mathbb{N}}$ such that

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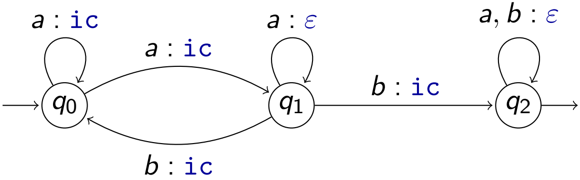
Yes: $P_{\mathcal{A}}((a^n b)^n) \xrightarrow{n\to\infty} 1.$

Cost automata



Cost automaton: Non-deterministic automaton equipped with finitely many counters

- Operations on counters: increment (ic), reset (r), wait (ε).

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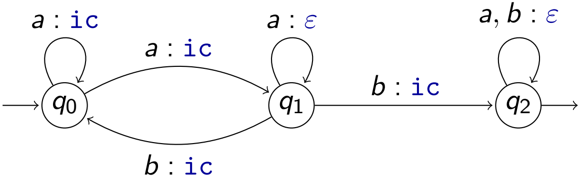

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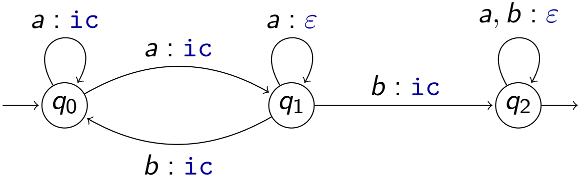

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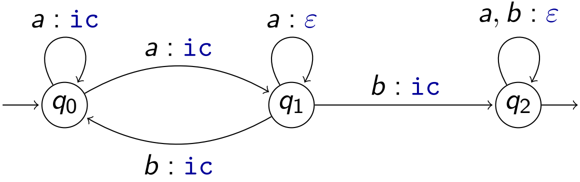
- Operations on counters: increment (ic), reset (r), wait (ε).
- Value of a run: highest value taken by a counter.
- Semantics $[\![\mathcal{A}]\!] : L(\mathcal{A}) \to \mathbb{N}$, mapping to each accepted word u the minimal value of a run on u. **Example:** $[\![\mathcal{A}]\!](ab) = 2$.

Limitedness Problem



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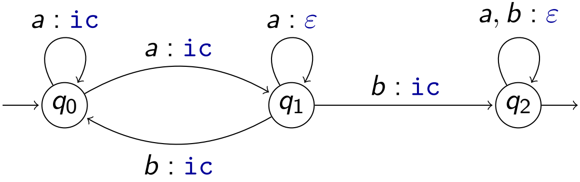

Limitedness problem: is there a sequence of words $(u_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} [\![\mathcal{A}]\!](u_n) = \infty \ ?$$

No. For all $u \in L(\mathcal{A})$, we have $[\![\mathcal{A}]\!](u) = 2$.

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Monoid with unary "stabilisation" operation ($\sharp$) describing asymptotic behaviour.

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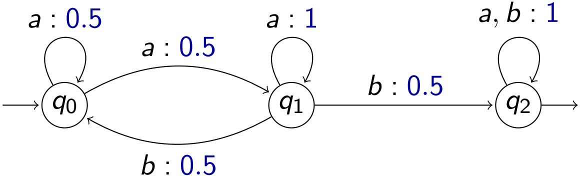
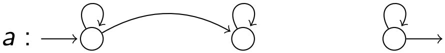
- "Most correct" algorithm for Value 1 problem [Fijalkow 2016]

Example



Elements of the stabilisation monoid:



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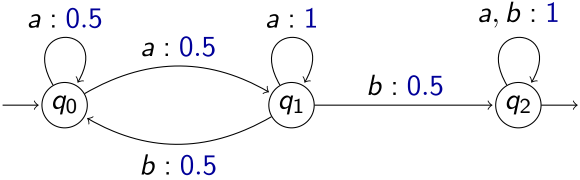

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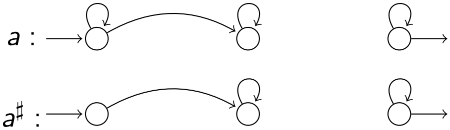

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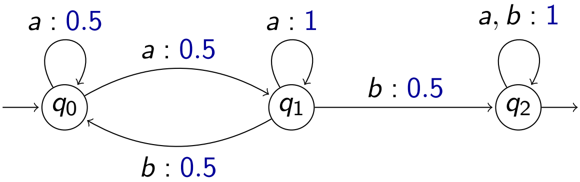

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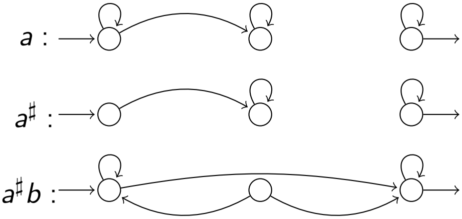

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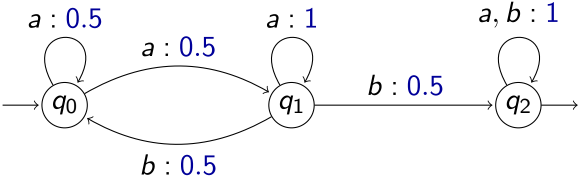

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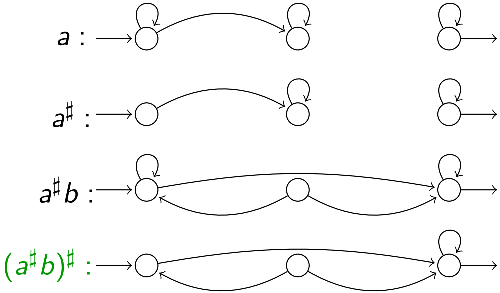

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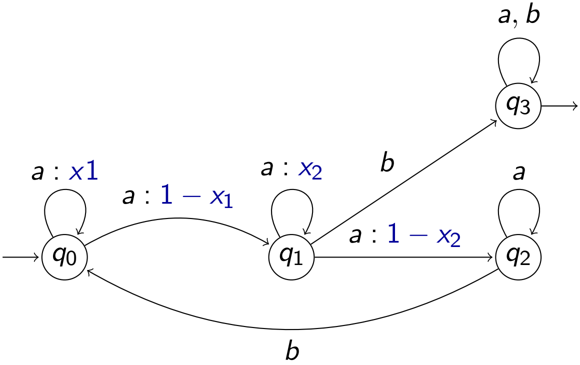

Idea of the algorithm

- Start with matrices of letters,
- Saturation of the stabilisation monoid by product and stabilisation,
- Stop if witness is found, answer **Value 1/Unlimited**.
- If no witness:
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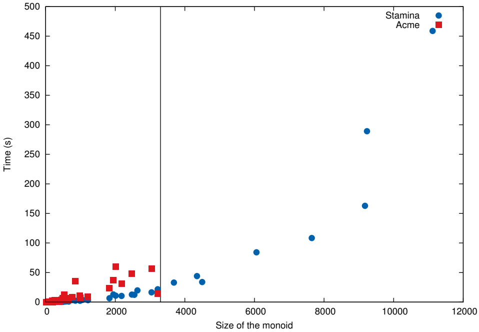
Optimizations

- basic CPU operations on vectors
- shared vectors between matrices
- rewrite rules like $A^\sharp B \rightarrow AB$ to avoid computations

Performances

ACME [F.,K. 2014]: previous version in OCaml for Value 1 and Limitedness problems.
Comparison on random automata of size 10, according to monoid size.



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L defined by $(a^*b)^*(a^*c)^*$

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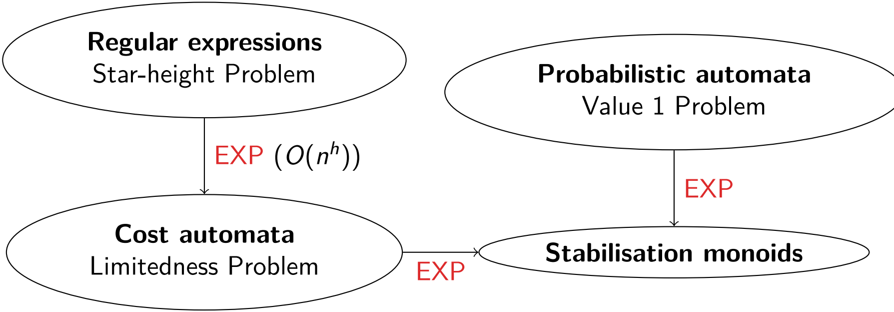
- Implemented here for the first time.

Big picture


Regular expressions
Star-height Problem
EXP ($O(n^h)$)
Probabilistic automata
Value 1 Problem
EXP
Cost automata
Limitedness Problem
EXP
Stabilisation monoids

More on the star-height algorithm

Loop Complexity (LC) of a NFA: minimal star-height of an expression derived from this NFA.
From [Eggan 1963]:

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Heuristics for Star-height problem:

- Loop complexity of input automaton
  - computable in **PTIME**
  - gives a regular expression e of low star-height
  - optimal in many cases [Cohen 1970]
- Test probable unlimited witnesses based on e
- Minimisation of an intermediary automaton [Colcombet, Löding 2008].