Good-for-Games Automata: State of the art and perspectives

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Dagstuhl Seminar
Unambiguity in Automata Theory

Games

Two Players: Eve Adam

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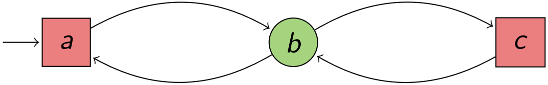
Arena: finite graph $G = (V, E)$, with $V = V_{\bigcirc} \uplus V_{\square}$.



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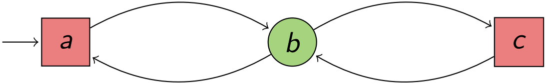

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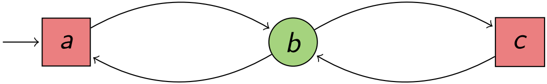

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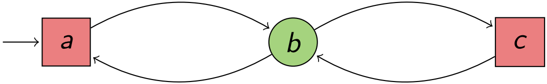

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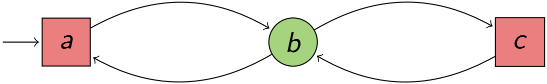

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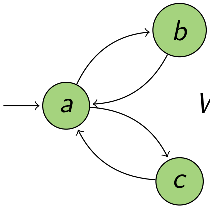
Eve wins a play π if $\pi \in W$

ω-regular games

Winning condition W: an ω-regular language.



$$W = (a^* b a^* c)^\omega$$

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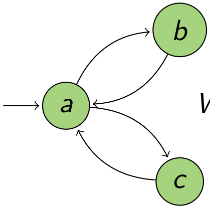

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Particular case: Parity games

W is a parity condition: each vertex has a color in $\mathbb{N}$, the maximal color appearing infinitely often must be even.

Büchi=Parity $[1, 2]$ CoBüchi=Parity $[0, 1]$.

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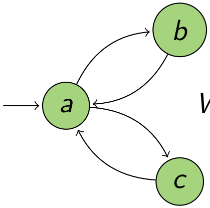

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Theorem (Positional Determinacy [Emerson, Jutla '91])

In a parity game, Eve or Adam has a ***positional*** *winning strategy.*

Solving an ω-regular game

Input: G game with ω-regular winning condition $W \subseteq V^\omega$.
Question: Who wins G ? How ?

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Solution:

1. Build Det Parity automaton $\mathcal{A}_{\mathrm{Det}}$ for W,
2. Solve the parity game $G' = \mathcal{A}_{\mathrm{Det}} \circ G$.

Theorem

G' has same winner as G.
σ_{pos} in G' gives σ in G with memory $\mathcal{A}_{\mathrm{Det}}$.
$\rightarrow$ ω-regular games are finite-memory determined.

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Automatically build a program from a specification L
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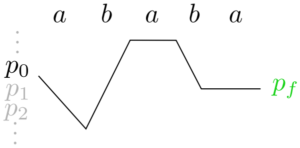
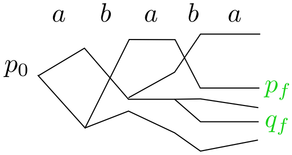
Problem: Determinization is expensive. Maybe too strong ?

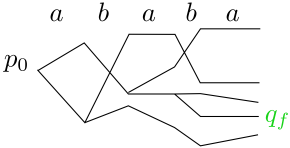
Good-for-Games Automata



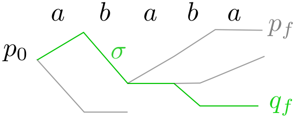
Deterministic



Non-deterministic



Unambiguous



Good-for-Games

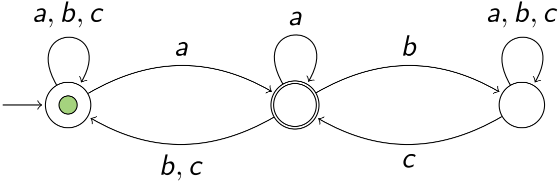
$\sigma : A^* \to Q$

Definition of GFG via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:
Adam plays letters:

Eve: resolves non-deterministic choices for transitions



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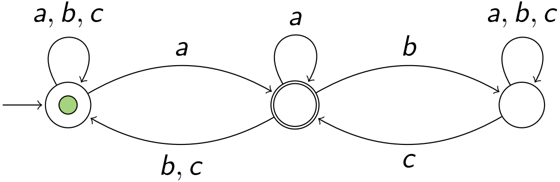

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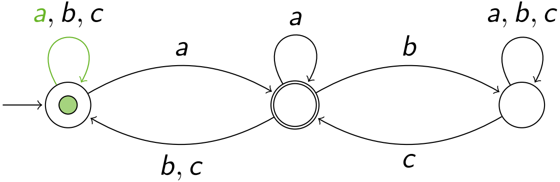

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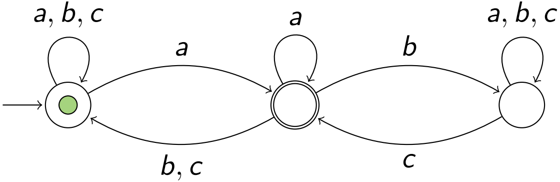

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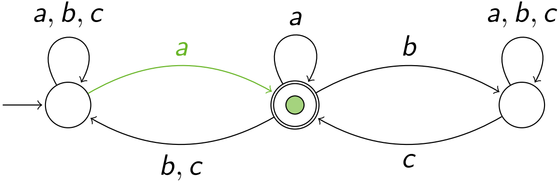

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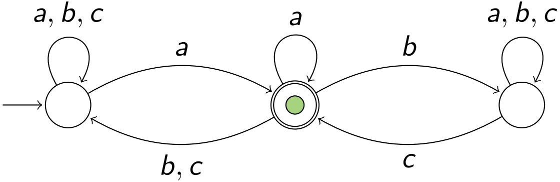

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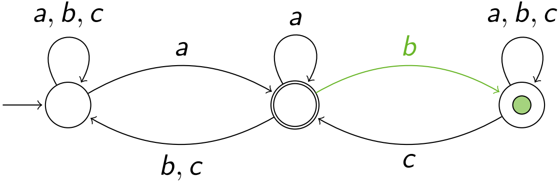

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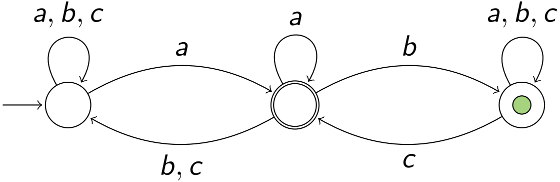

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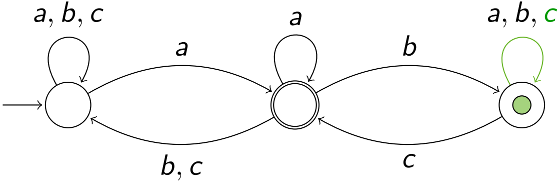

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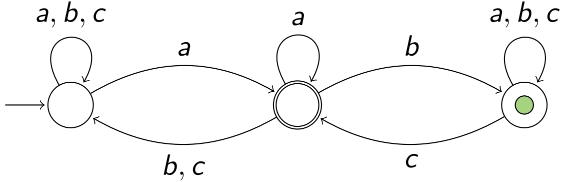

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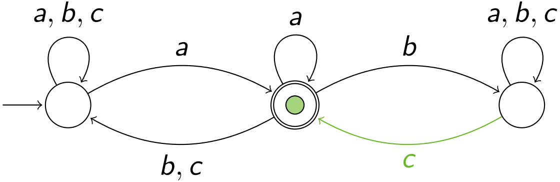

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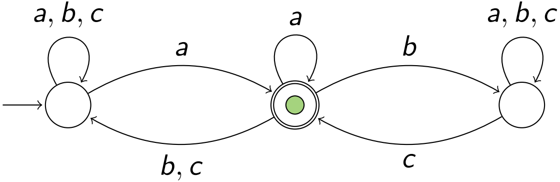

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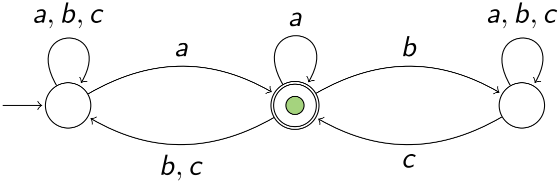

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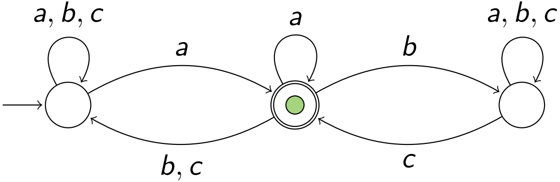

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Not a parity game! Only ω-regular.

Why "Good-for-games" ?

Theorem (Henzinger, Piterman '06)

Let $\mathcal{A}$ a parity GFG automaton, and G game with winning condition $L(\mathcal{A})$.
Then $\mathcal{A} \circ G$ (where Eve controls $\mathcal{A}$) is a parity game with same winner as G.

Proof: Eve can drive $\mathcal{A}$ according to σ_{GFG}.

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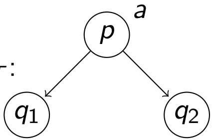
Remark: Synthesis is ExpTime-complete for nondet specification, and 2ExpTime-complete for LTL specification.

Good-for-Trees

$\mathcal{A}$ automaton on infinite words $\mapsto$ $\mathcal{A}_T$ automaton on infinite trees

If $p \xrightarrow{a} q_1$ and $p \xrightarrow{a} q_2$ in $\mathcal{A}$, then put in $\mathcal{A}_T$:



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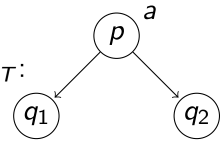

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$\mathcal{A}$ is **Good-for-Trees** (GFT) if

$$L(\mathcal{A}_T) = \{t \mid \text{all branches of } t \text{ are in } L(\mathcal{A})\}$$

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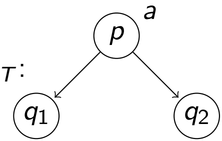

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If rank of the trees $\geq$ size of the alphabet, then

$$\mathrm{GFG} = \mathrm{GFT}$$

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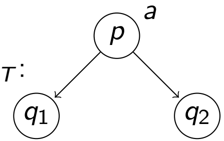

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Hypothesis is necessary !

Link with determinism

Fact

Every deterministic automaton is GFG.

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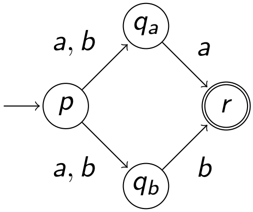
Some (unamb.) non-GFG automaton:

$$L = (a + b)(a + b)$$



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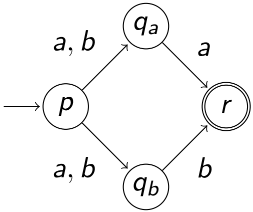

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Determinizable by Pruning (DBP):

Determinizable by removing some transitions.

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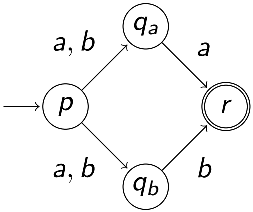

Definition

Determinizable by Pruning (DBP):

Determinizable by removing some transitions.

Fact

DBP = *"GFG with a positional strategy".*

$\rightarrow$ *Every DBP automaton is GFG.*

Some GFG automata

Theorem

On finite words, $\mathrm{DBP} = \mathrm{GFG}$*.*

Proof: σ_{GFG}: always go to state accepting the maximal language.

Some GFG automata

Theorem

On finite words, DBP $=$ GFG.

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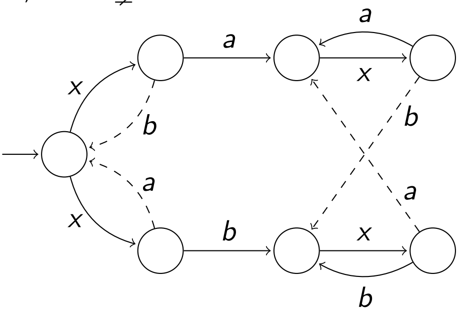
Theorem ([Boker, K., Kupferman, Skrzypczak '13])

On infinite words, DBP $\subsetneq$ GFG.



A GFG coBüchi automaton for $(xa + xb)^*[(xa)^\omega + (xb)^\omega]$.

Algorithmic properties of GFG automata

Theorem (Easy inclusion checking)

If $\mathcal{A}$ is nondet and $\mathcal{B}$ is GFG, we can decide whether $L(\mathcal{A}) \subseteq L(\mathcal{B})$ in $\textsc{PTime}$.

Proof: Simulation game.

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Theorem (Hard complementation)

If $\mathcal{A}$ is GFG for L and $\mathcal{B}$ is GFG for $\overline{L}$, then we can build in $\mathrm{PT{\scriptsize IME}}$ a deterministic automaton for L based on $\mathcal{A} \times \mathcal{B}$.

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If $\mathcal{A}$ is GFG for L and $\mathcal{B}$ is GFG for $\overline{L}$, then we can build in $\mathrm{PT{\small IME}}$ a deterministic automaton for L based on $\mathcal{A} \times \mathcal{B}$.

Proof: Pos. Strategy in Letter game of $U = \mathcal{A} \times \mathcal{B}$ accepting all words.

How much memory is needed in the GFG strategy ?

Lemma

If $\mathcal{A}$ is GFG and $\mathcal{D}$ is a det. automaton for $L(\mathcal{A})$, then there is a strategy σ_{GFG} with memory $\mathcal{D}$.

Proof: Solve the letter game the classical way.

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Important Remark:

GFG automata can be used in algorithms without knowing σ_{GFG}.
The strategy σ_{GFG} “hides” the determinism.

State-blow-up for determinization

Finite words:
GFG=DBP, Determinization in PTIME [Löding].

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CoBüchi (Parity $[0, 1]$):
GFG automata can be exponentially more succinct than deterministic ones [K., Skrzypczak '15]

Exponential succinctness

To define L_n, letters $\{a, b, \sharp\}$ act on $\{0, 1, \ldots, n-1\}$:

$$a : \; +1 \mod n \qquad b : \; 0 \leftrightarrow 1 \qquad \sharp : \text{“cuts” } 0$$

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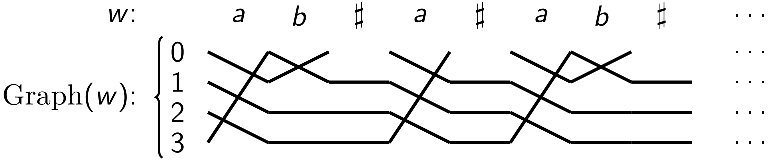

$L_n = \{w \mid \mathrm{Graph}(w) \text{ contains an } \infty \text{ path}\}.$

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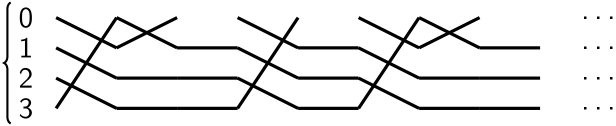
$\mathrm{Graph}(w)$:



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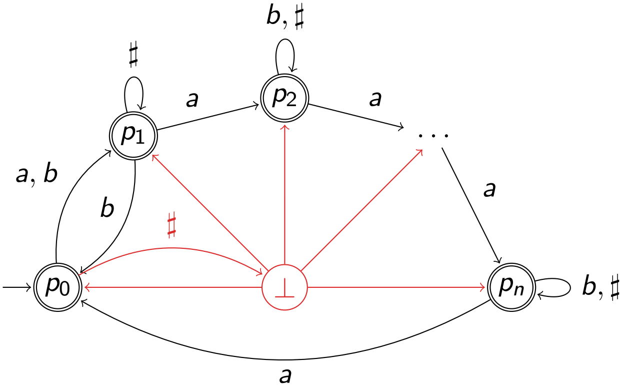
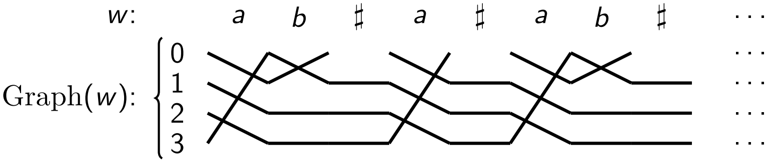
Lemma: Any Det Parity automaton for L_n needs $\dfrac{2^{\frac{n}{2}}}{n+1}$ states.

Small CoBüchi GFG for L_n





More general frameworks

Co-invention: History-deterministic for **cost functions** [Colcombet '09]

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Building GFG automata

Some efforts:

- Incremental powerset construction [K', Majumdar '18]
- Fragment of CoBüchi languages [Iosti, K. '19]
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Less ambitious goal: recognizing GFG automata.

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- Compute a deterministic parity automaton for $L(\mathcal{A})$,
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So GFGness is decidable, but can we do better than $\textsc{ExpTime}$?

Abstracting the Letter game: finite words

Theorem (Löding)

The GFGness problem is in $\mathrm{PT{\scriptsize IME}}$ *for finite words automata.*

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The game G_1:

Adam plays letters:

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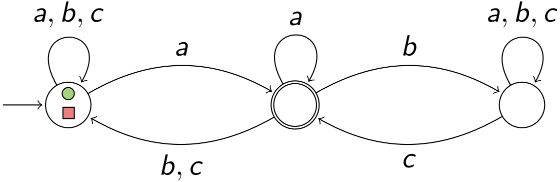

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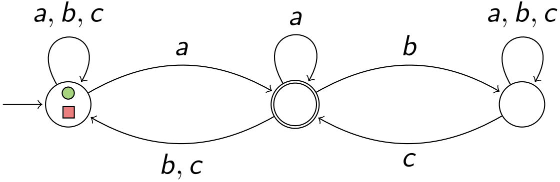

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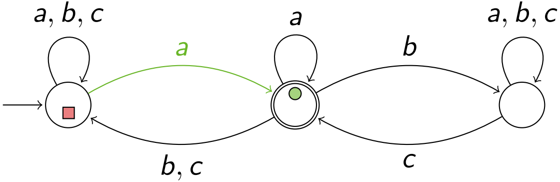

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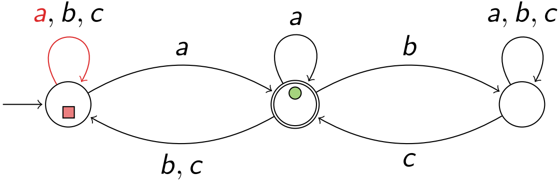

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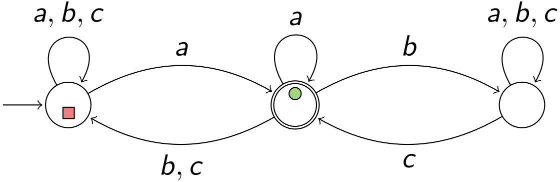

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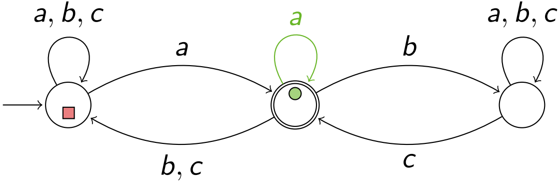

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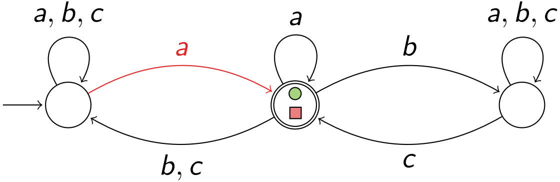

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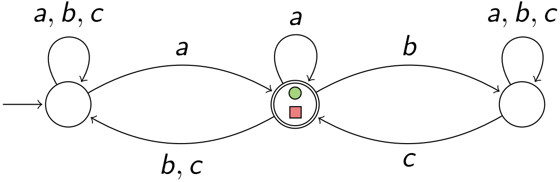

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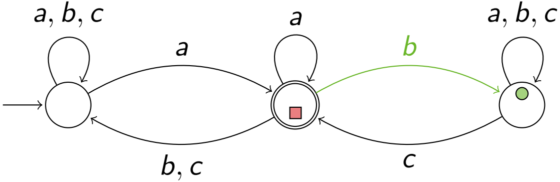

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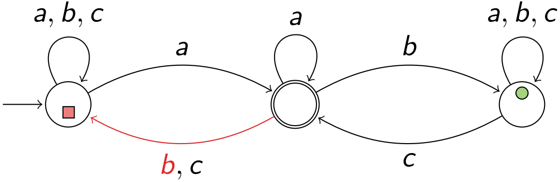

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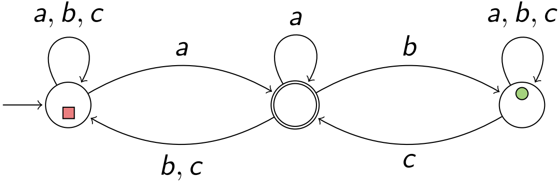

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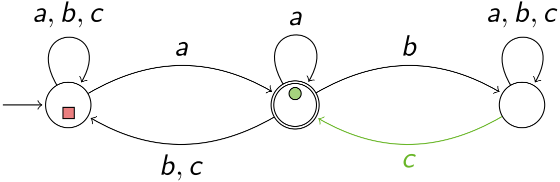

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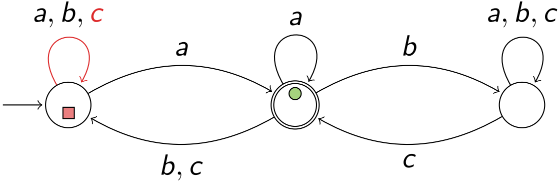

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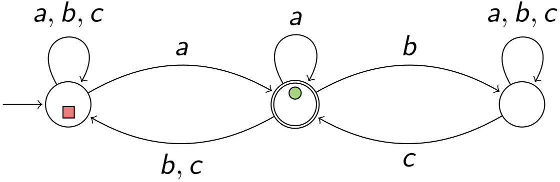

Eve wins if at all times: ■ accepting $\Rightarrow$ ● accepting.

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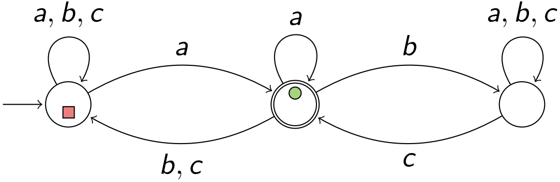

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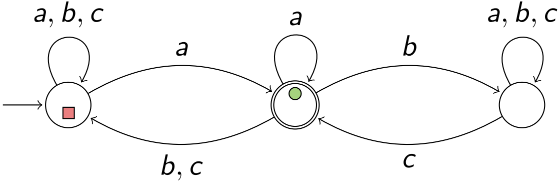

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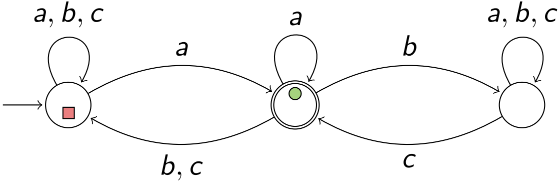

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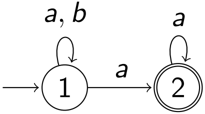

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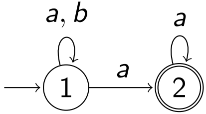

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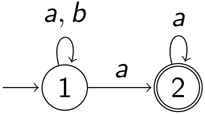

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We need a better abstraction of the Letter game.

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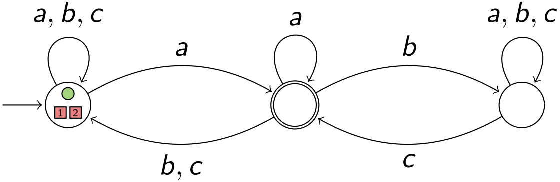

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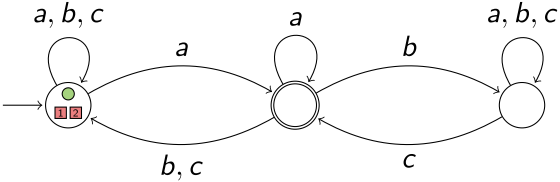

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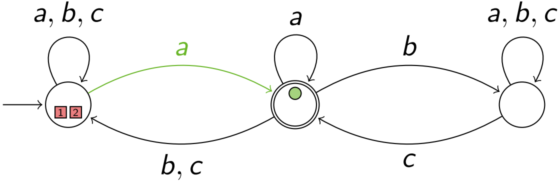

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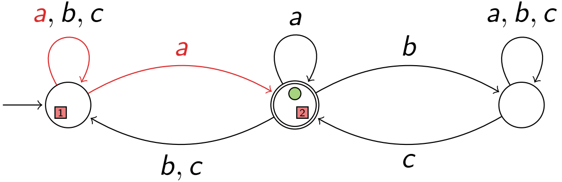

Abstracting the Letter game: infinite words

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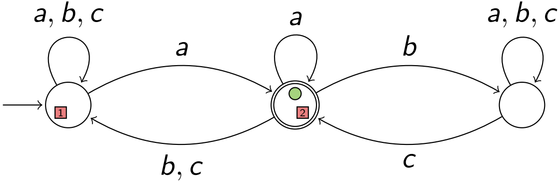

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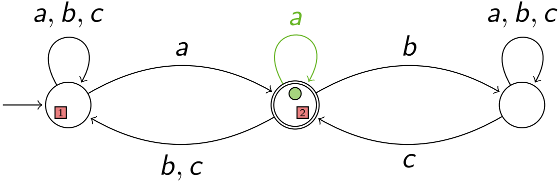

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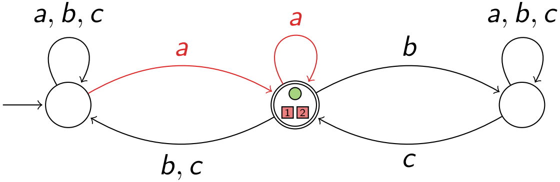

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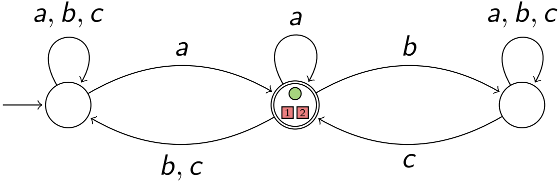

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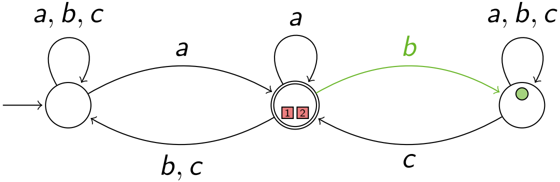

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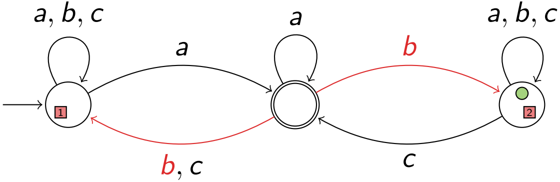

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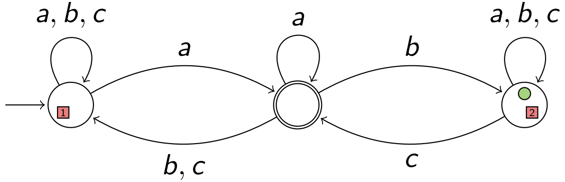

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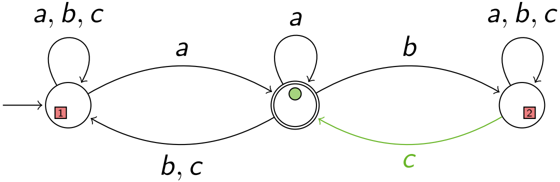

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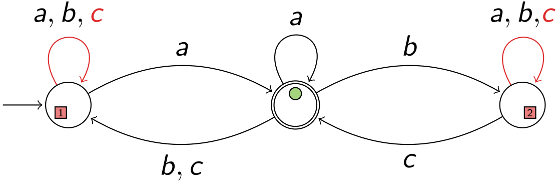

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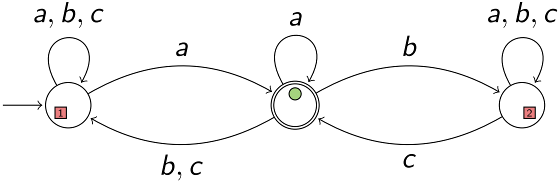

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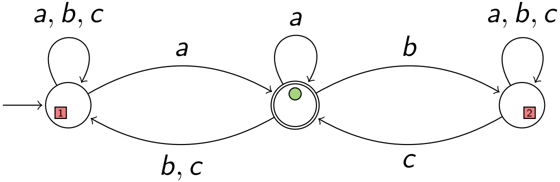

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For all automata $\mathcal{A}$: Eve wins $G_2 \Leftrightarrow \mathcal{A}$ is GFG.

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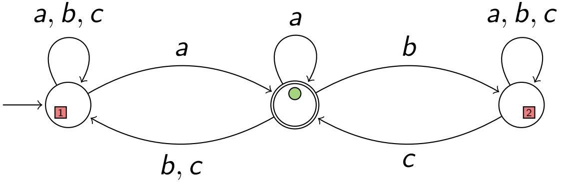

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Solving G_2 is polynomial $\Rightarrow$ Efficient algorithm for GFGness.

Why G_2 is powerful: The game G_k

Game G_k: k tokens $\square$. Some $\boxed{i}$ accepts $\Rightarrow$ $\bigcirc$ must accept.

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Lemma

Eve wins $G_2 \Leftrightarrow$ *Eve wins* G_k *for all* $k \geq 2$.

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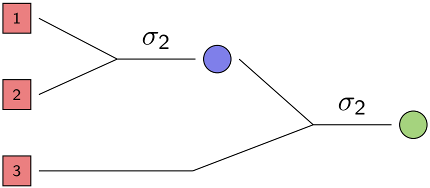
Proof sketch: $G_2 \Rightarrow G_3$

- play a *virtual token* $\bigcirc$ against $\boxed{1}$ and $\boxed{2}$.
- play G_2 strategy against $\bigcirc$ and $\boxed{3}$.



Explorable automata

Definition (k-Letter game)

k-letter game: Letter game where Eve moves k tokens instead of one. She wins if
$w \in L \Rightarrow$ at least one token follows an accepting run.

Definition

$\mathcal{A}$ is k-GFG if Eve wins the k-Letter game.
$\mathcal{A}$ is **Explorable** if it is k-GFG for some k.

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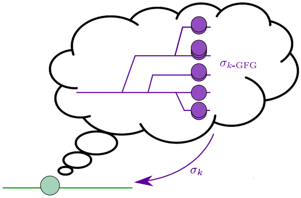
Theorem (Unpublished)

If $\mathcal{A}$ is explorable: Eve wins $G_2 \Leftrightarrow \mathcal{A}$ is GFG.



What is known about the G_2 conjecture

G_2 characterizes GFGness on

- Explorable parity automata [Unpublished]
- LimSup, LimInf automata [Boker, Lehtinen, on Arxiv]
- Büchi automata [Bagnol, K. '18]
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Theorem ([Boker, K., Lehtinen, Skrzypcak, on Arxiv])

If the G_2 conjecture is true on non-deterministic automata, it is true on alternating automata.

Main proof sketch for $G_2 \Rightarrow \mathrm{GFG}$ on Büchi

Assume for contradiction:

- Eve wins G_2, so Eve wins G_k with strategy σ_k, for a big k.
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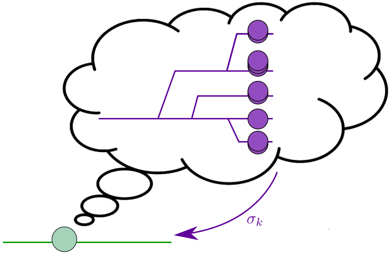
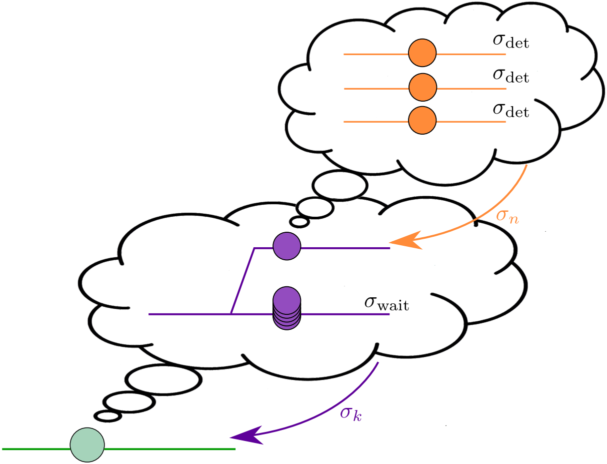
Idea for a strategy against τ_{GFG} in the Letter game:

- move k virtual tokens uniformly
- play σ_k against these k tokens



Trick: Word from τ_{GFG} $\Rightarrow$ one Büchi for some ● every M steps.
$\rightsquigarrow$ ● wins agains τ_{GFG}, contradiction.

G_2 for CoBüchi



Current and future work

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  - Succinctness
  - Complexity
- Practical applications, experimental evaluations.
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- Prove or disprove the G_2 conjecture for parity automata.
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