Bouncing threads for infinitary and circular proofs

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The logic μMLL

Syntax of μMLL

$$\varphi, \psi := a \mid a^\perp \mid \perp \mid \mathbf{1} \mid X \mid \varphi \parr \psi \mid \varphi \otimes \psi \mid \mu X.\varphi \mid \nu X.\varphi$$

No negation in the syntax,
but dualization operation $\varphi \mapsto \varphi^\perp$ with $(\varphi^\perp)^\perp = \varphi$.

- $X^\perp = X$
- $\parr$ disjunction, $\otimes$ conjunction
- μ least fixpoint, ν greatest fixpoint
- $\mu X.X$ false, $\nu X.X$ true

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Motivation: proofs as programs.

Proof system

Rules of μMLL

$$\frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{}{\vdash \mathbf{1}}\ (\mathbf{1})$$

$$\frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes)$$

$$\frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu) \qquad \frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu)$$

$$\frac{}{\vdash F, F^{\perp}}\ (\mathrm{Ax}) \qquad \frac{\vdash \Gamma, F \qquad \vdash F^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{Cut})$$

Infinite proofs

We allow infinite proofs:

$$\dfrac{\vdash \mu X.X}{\vdash \mu X.X}\ (\mu) \qquad\qquad \dfrac{\vdash \nu X.X}{\vdash \nu X.X}\ (\nu)$$

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However, only the right one should be valid.
We add a validity condition.

Threads

Rules with threads

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Infinite threads, validity

Example with $F = \nu X.(X \otimes (\mu Y.X))$.

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Theorem (BDS 2016)

This system is sound, and admits cut-elimination

A proof with cut

Problem: Cuts are not well-managed by the validity condition.

$$\dfrac{\dfrac{\dfrac{}{\vdash \nu X.X, \mu X.X}\,(\mathrm{Ax})}{\vdash \nu X.X, \mu X.X}\,(\mu) \qquad \dfrac{\dfrac{\vdash \nu X.X}{\vdash \nu X.X}\,(\nu)}{\vdash \nu X.X}\,(\nu)}{\vdash \nu X.X}\,(\mathrm{Cut})$$

Cut-elimination

$$\dfrac{\vdash \nu X.X}{\vdash \nu X.X}\,(\nu)$$

A proof with cut

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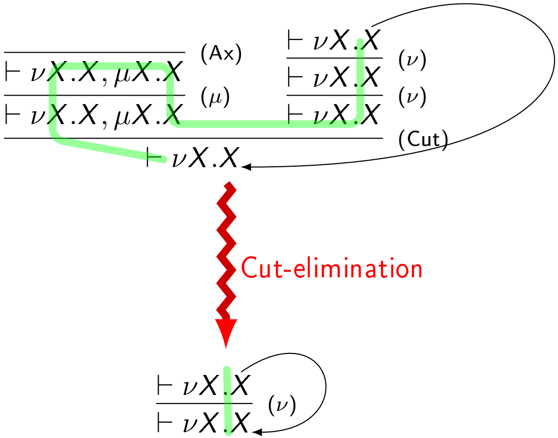
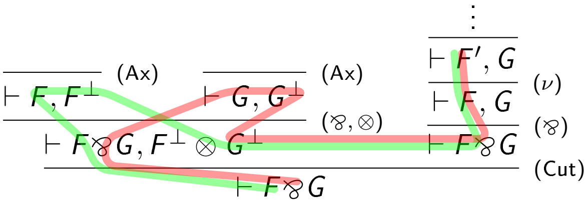
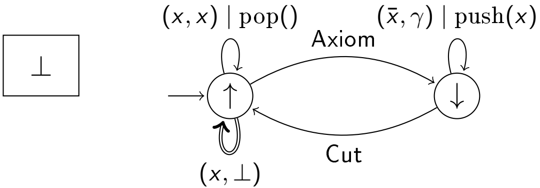


Bouncing threads: matching condition

Not all bouncing threads are coherent with cut-elimination.

Example with F, G arbitrary:





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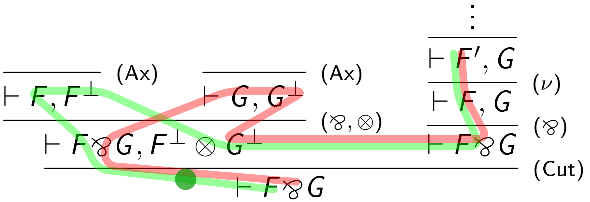

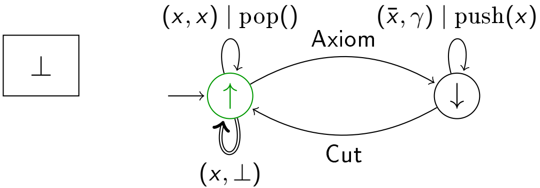

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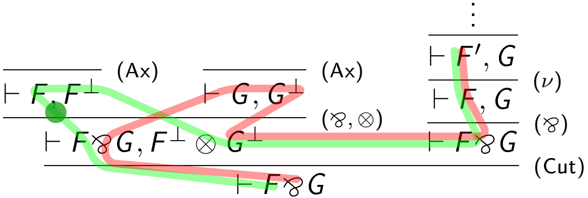

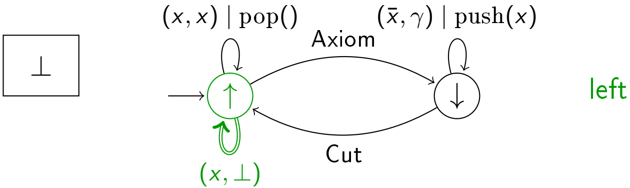

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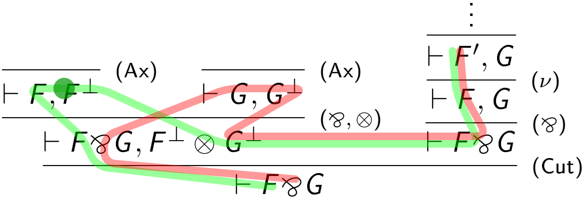

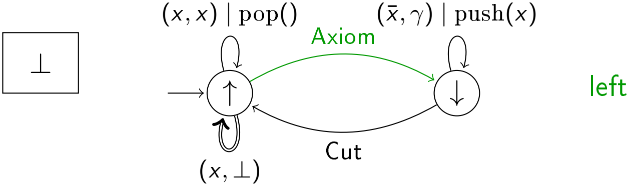

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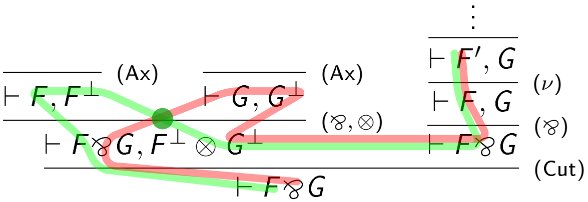

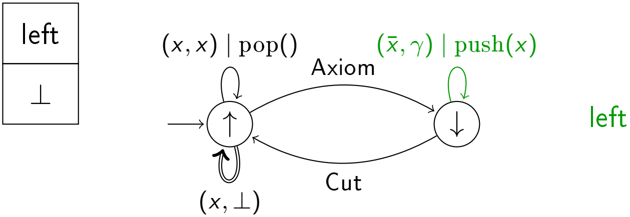

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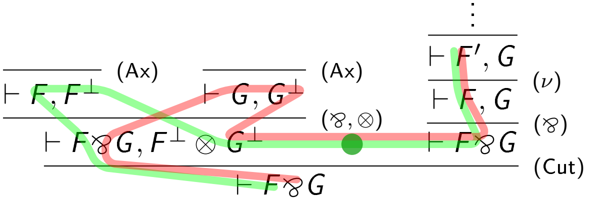

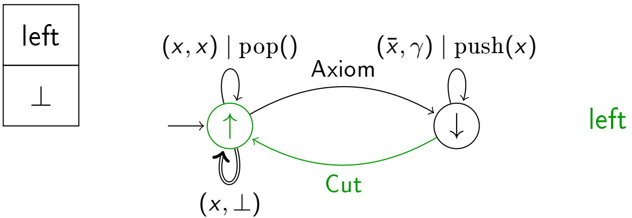

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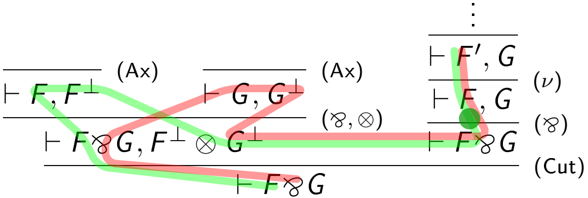
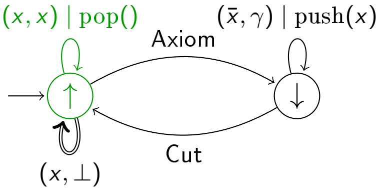


$\boxed{\perp}$



left

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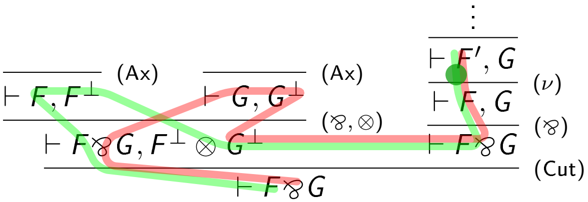
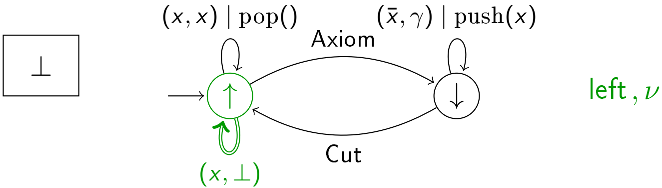




Bouncing threads: visible part

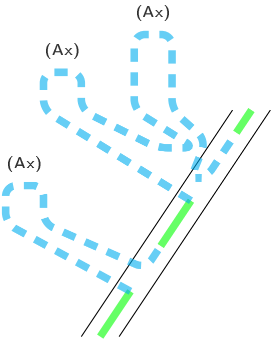
Visible part: survives the cut-elimination.
Hidden part: Must satisfy matching constaints.

$$\dfrac{\dfrac{\dfrac{}{\vdash F, F^{\perp}}\,(\mathrm{Ax}) \qquad \dfrac{}{\vdash G, G^{\perp}}\,(\mathrm{Ax})}{\vdash F \parr G, F^{\perp} \otimes G^{\perp}}\,(\parr, \otimes) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\vdash F', G}}{\vdash F, G}\,(\nu)}{\vdash F \parr G}\,(\parr)}{\vdash F \parr G}\,(\mathrm{Cut})$$

Bouncing thread valid: ∞ ν-unfoldings in visible part.

Valid branches

Branch B valid: $\exists$ valid thread with visible part included in B.



Proof valid: all infinites branches are valid.
Reason: guarantee productivity of cut-elimination algorithm.

Theorem

Soundness and cut-elimination still hold.

Decidability of the validity condition ?

Given a circular proof, decide validity ?

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Answer: NO.

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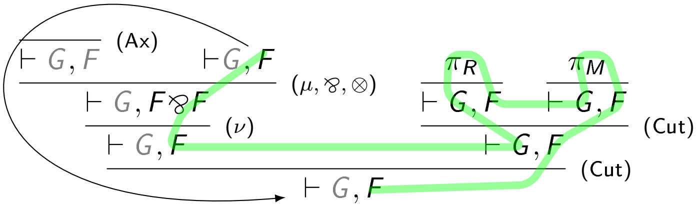
Answer: NO.

Ingredients:

- Minsky Machines: 2-counter finite-state deterministic machine with Increment, Decrement, Zero test
- Simulate the run of a machine with a bouncing thread on $F = \nu X.(X \parr X)$
- If terminates, rewind computation to erase extra constraints



Zooming on the encoding

Encoding details:

- $F = \nu X.(X \parr X)$, $\quad G = F^{\perp} = \mu X.(X \otimes X)$.
- Current state p encoded in node of the graph.
- Counter values n, m encoded in a constraint stack $\mathbf{l}^n\mathbf{r}\mathbf{l}^m\mathbf{r}$.

<u>Example:</u> Encoding $p \xrightarrow{Inc_1} q$

$$
\dfrac{(q) \vdash G, \mathbf{F} \qquad \dfrac{\dfrac{\dfrac{}{\vdash G_{\mathbf{l}}, F}\,(\mathrm{Ax}) \qquad \dfrac{}{\vdash G_{\mathbf{r}}}\,(\infty)}{\vdash G_{\mathbf{l}} \otimes G_{\mathbf{r}}, F}\,(\otimes)}{\vdash \mathbf{G}, F}\,(\mu)}{(p) \vdash G, F}\,(\mathrm{Cut})
$$

$$\mathbf{l}^n\mathbf{r}\mathbf{l}^m\mathbf{r} \mapsto \mathbf{l}^{n+1}\mathbf{r}\mathbf{l}^m\mathbf{r}$$

A hierarchy of decidable conditions

Height of a bouncing thread: parameter $\approx$ "memory".

Height = maximal height of the stack of the pusdhown automaton.

k-proof: valid proof using only threads of height $\leq k$.

Theorem

Every valid circular proof is a k-proof for some $k \in \mathbb{N}$

Theorem

For all $k \in \mathbb{N}$, it is decidable whether a circular proof is a k-proof.