Explorable Automata

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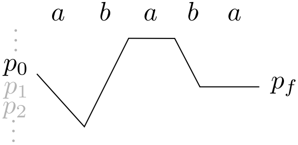
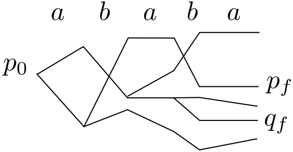
and Marc Bagnol, Udi Boker, Orna Kupferman, Karoliina Lehtinen,
Anirban Majumdar, Michał Skrzypczak, Milla Valnet,. . .

Delta ANR Meeting, Marseille, 30 May 2022

Good-for-Games Automata



Deterministic



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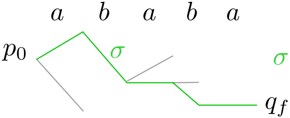
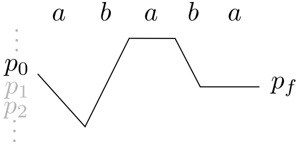


$\sigma : A^* \to Q$

Good-for-Games

Good-for-Games Automata



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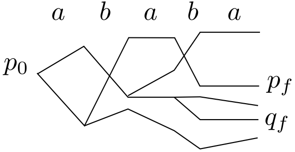

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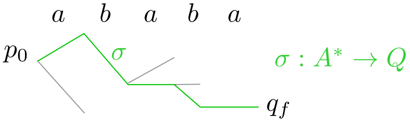


Good-for-Games

Motivations

- Solve Church Synthesis more efficiently
- Intermediate model between Det. and Nondet.

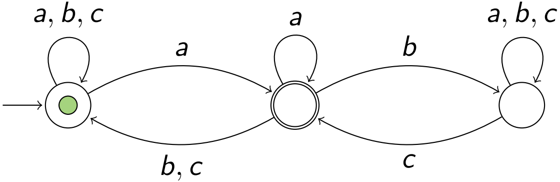
Definition of GFG via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters:

Eve: resolves non-deterministic choices for transitions



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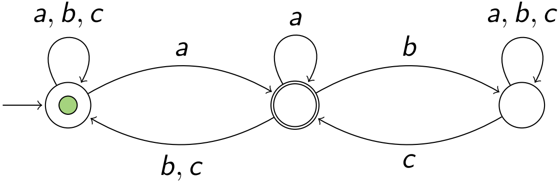

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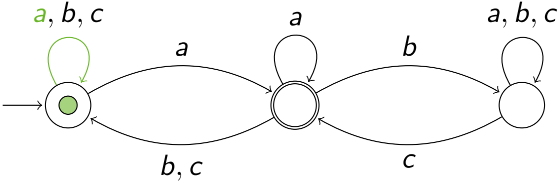

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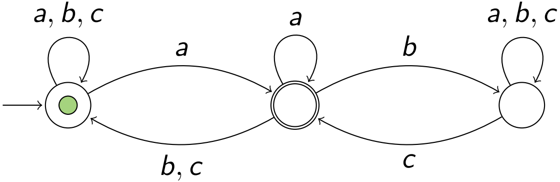

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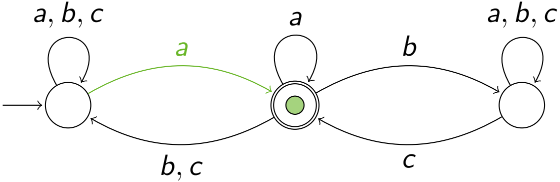

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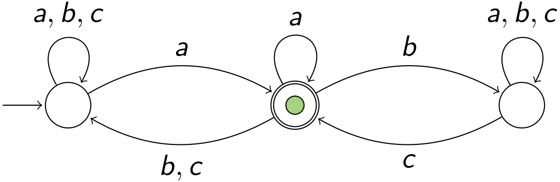

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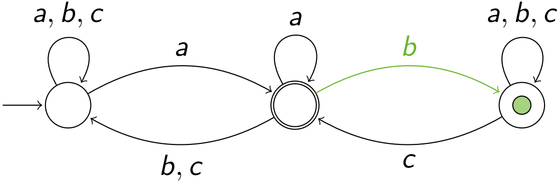

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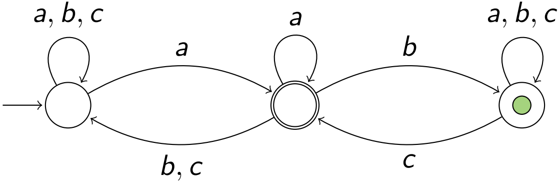

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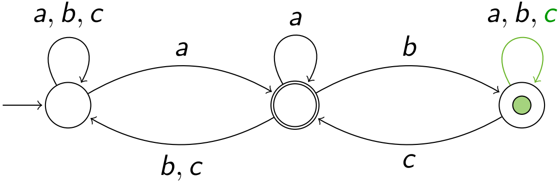

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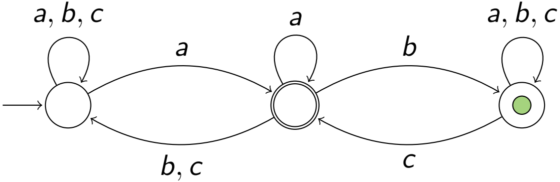

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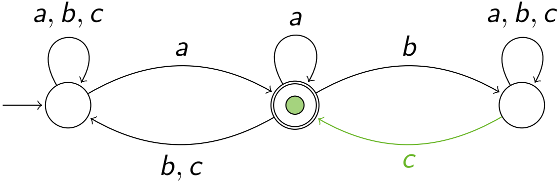

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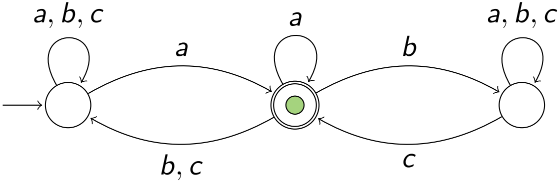

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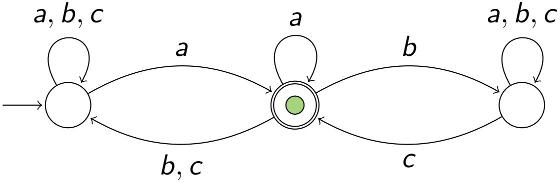

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$\mathcal{A}$ GFG $\Leftrightarrow$ Eve wins the Letter game on $\mathcal{A}$
$\Leftrightarrow$ there is a strategy $\sigma_{\mathrm{GFG}} : A^* \to Q$ accepting all words of $L(\mathcal{A})$.

First facts

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Every deterministic automaton is GFG.

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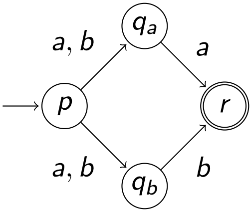
Some non-GFG automaton:

$$L = (a + b)(a + b)$$



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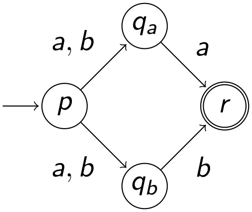

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Determinizable by removing some transitions.

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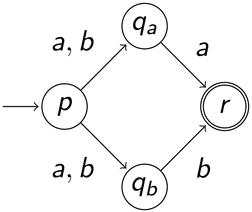

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Fact

DBP $=$ "GFG *with a positional strategy*".
$\rightarrow$ *Every* DBP *automaton is* GFG.

Some GFG automata

Theorem

On finite words, $\mathrm{DBP} = \mathrm{GFG}$.

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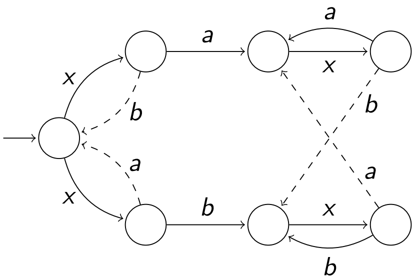
Theorem ([Boker, K., Kupferman, Skrzypczak '13])
On infinite words, DBP $\subsetneq$ GFG.



A GFG coBüchi automaton for $(xa + xb)^*[(xa)^\omega + (xb)^\omega]$.

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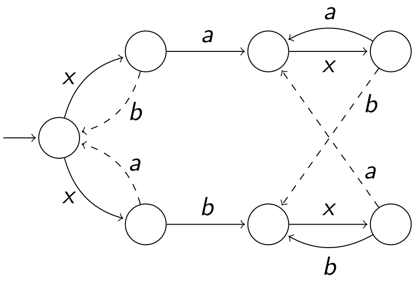

A GFG coBüchi automaton for $(xa + xb)^*[(xa)^\omega + (xb)^\omega]$.

State-blowup to determinize can be **Exponential** [K., Skrzypczak '15].

Application: Inclusion testing

Simulation game $\mathcal{B} \leq_s \mathcal{A}$: Each round:

- Adam chooses $q \xrightarrow{a} q'$ in $\mathcal{B}$
- Eve has to replicate $p \xrightarrow{a} p'$ in $\mathcal{A}$

Eve wins if $\mathcal{B}$ accepts $\Longrightarrow$ $\mathcal{A}$ accepts

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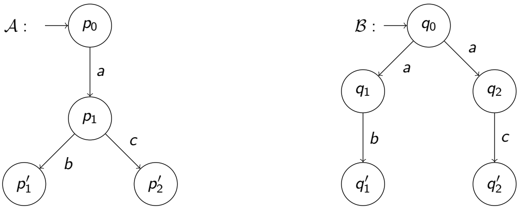

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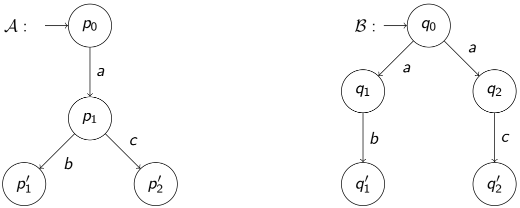

$$\mathcal{B} \leq_s \mathcal{A} \quad \text{but} \quad \mathcal{A} \not\leq_s \mathcal{B}$$

Lemma: If $\mathcal{A}$ GFG, then $\mathcal{B} \leq_s \mathcal{A}$ is equivalent to $L(\mathcal{B}) \subseteq L(\mathcal{A})$

Recognizing GFG automata

Complexity of the GFGness problem:
Input: A nondeterministic automaton $\mathcal{A}$
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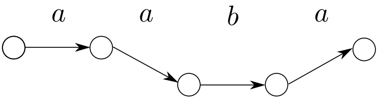
To tackle these questions, we generalize the notion of GFG...

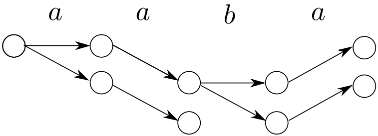
Allowing more runs

Idea: Allow to build several runs, at least one accepting.



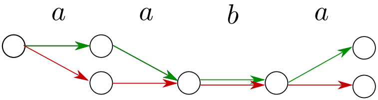
1 state - 1 run

GFG



2 states

Width 2

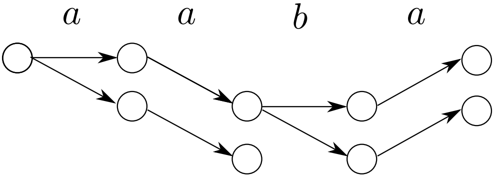


2 runs

2-Explorable

Width of an automaton

k-width game on $\mathcal{A}$:



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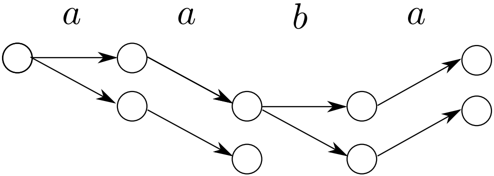

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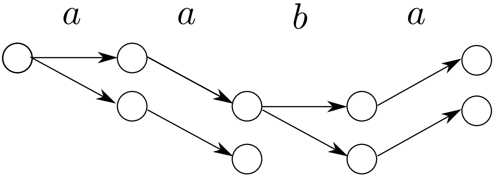

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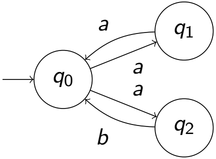

A safety NFA of width ?

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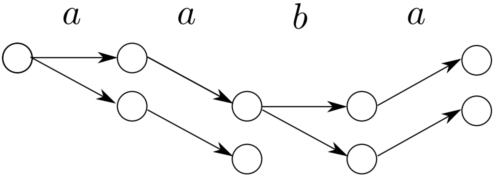

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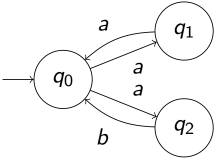
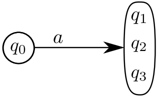
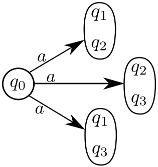


A safety NFA of width 2

k-Determinization



Powerset



2-Determinization $\mathcal{A}_2$

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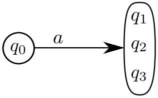

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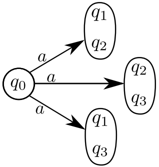

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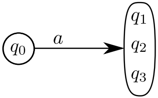
+ generalization to Breakpoint, Safra: $|\mathcal{A}_k| \approx |\mathcal{A}|^k$

k-Determinization



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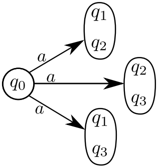

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Facts: [K.,Majumdar]

- $\mathrm{width}(\mathcal{A}) \leq k \iff \mathcal{A}_k$ is GFG.
- $\mathcal{B} \subseteq \mathcal{A}$ can be tested in $\approx O(n^{\mathsf{width}(\mathcal{A})})$ via a simulation game.

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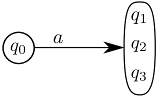

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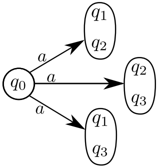

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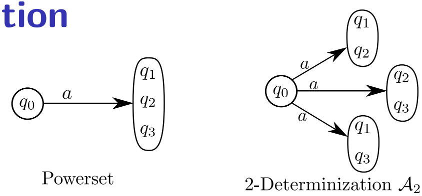
Application: Building a GFG aut. from $\mathcal{A}$

Start from $\mathcal{B} = \mathcal{A}$ and $k = 1$;
while $\mathcal{B}$ *is not GFG* **do**
    $k := k + 1$;
    $B := \mathcal{A}_k$;
end

k-Determinization



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Application: Building a GFG aut. from $\mathcal{A}$

Start from $\mathcal{B} = \mathcal{A}$ and $k = 1$;
while $\mathcal{B}$ *is not GFG* **do**
  $k := k + 1$;
  $B := \mathcal{A}_k$;
end

⚠ We need to test GFGness !

Computing the width

Can we compute $k = \mathrm{width}(\mathcal{A})$ to build $\mathcal{A}_k$ directly ?

Theorem [K, Majumdar]

Computing the width of an NFA is $\textsc{ExpTime}$-complete.
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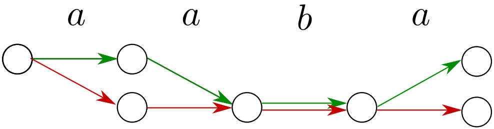
Reduction via a SAT game, introduced by [Robson]
to show $\textsc{ExpTime}$-completeness of popular games:

(Japanese rules)

Explorable Automata

We now bound the number of runs.

***k*-explorability game**:



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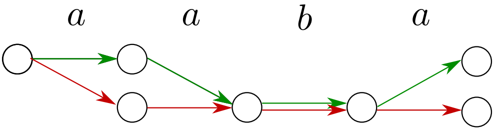

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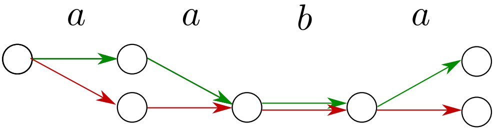

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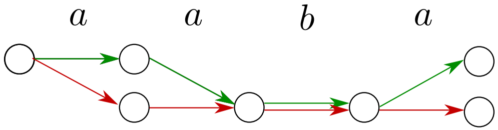

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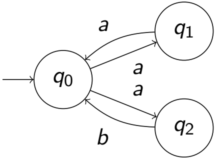

A ?-explorable safety NFA

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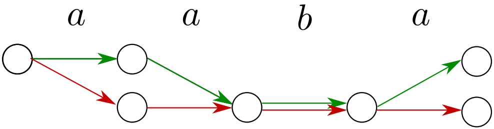

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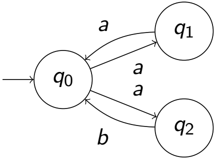

A non-explorable safety NFA

First results

Theorem: Deciding $|Q|/2$-explorability is still $\textsc{ExpTime}$-complete.

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How many tokens might be needed in explorable automata ?

A related paper

Similar questions in [Betrand et al 2019: Controlling a population]

k-**population game**: Arena like *k*-explorability game on NFA,
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Our goal: Generalize to Explorability, but

- Game harder to solve: the input word has to be in $L(\mathcal{A})$
- Must deal with acceptance conditions on infinite words.

Results

Theorems [Hazard, K.]

Explorability Problem is ExpTime-complete for NFA, Büchi.
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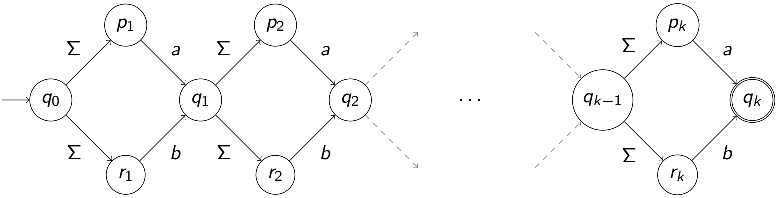

NFA needing exponentially many tokens.

ω-explorability

What happens if we allow a countable infinity of tokens ?

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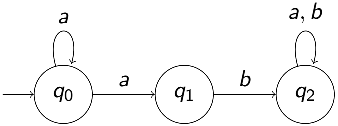

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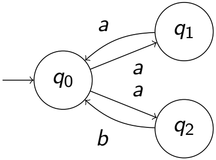

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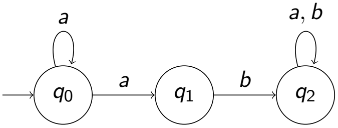

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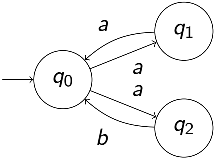

not ω-explorable

Intuition:
Non-explorable: Environment can kill a run **chosen by System**
Non-ω-explorable: Environment can kill a run **of its choice**

Results on ω-explorability

Facts:

- any NFA is ω-explorable,
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Only up to coBüchi for now... Duality with explorability problem.

Current and future work

- Complexity of (ω)-explorability for parity conditions ?
- Complexity of k-explorability with k in binary?
- Studying GFG and explorable models in other frameworks.
- Practical applications, experimental evaluations.
- $\textsc{PTime}$ GFGness for parity automata.
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Thanks for your attention!