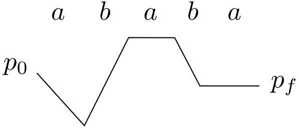
Explorable Automata

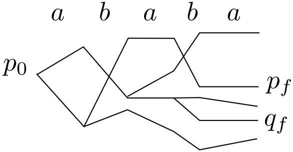
Emile Hazard, Olivier Idir, Denis Kuperberg

P-ACTS seminar, Marne-la-Vallée, May 14th 2025

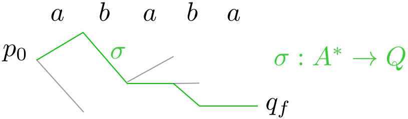
History-Deterministic Automata



Deterministic



Non-deterministic



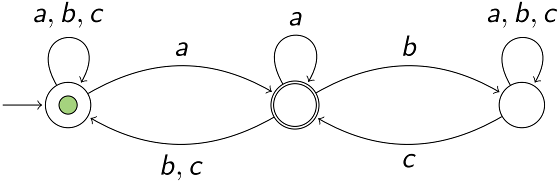
History-Deterministic
(a.k.a. Good-for-Games)

Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:
Adam plays letters:

Eve: resolves non-deterministic choices for transitions



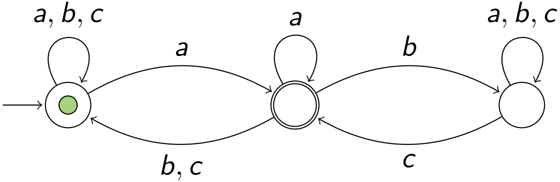
Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters: a

Eve: resolves non-deterministic choices for transitions



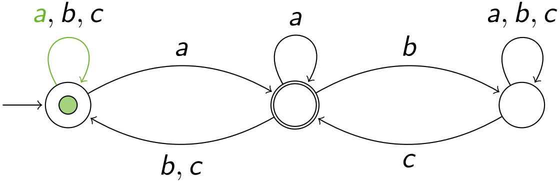
Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters: *a*

Eve: resolves non-deterministic choices for transitions



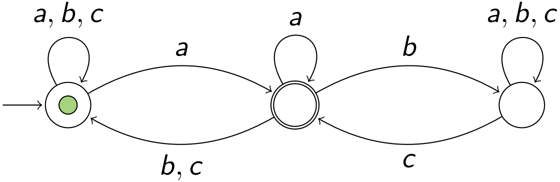
Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters: a a

Eve: resolves non-deterministic choices for transitions



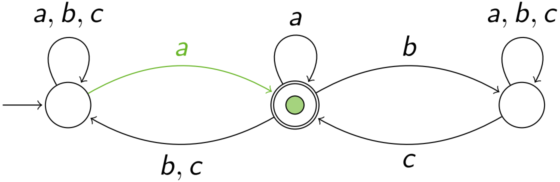
Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters: $a \ \ a$

Eve: resolves non-deterministic choices for transitions



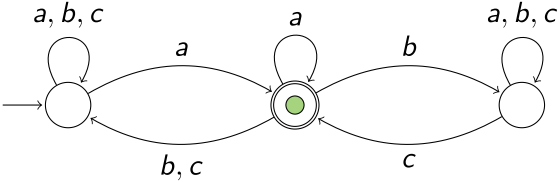
Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters: a a b

Eve: resolves non-deterministic choices for transitions



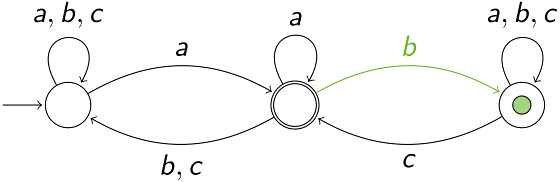
Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters: a a b

Eve: resolves non-deterministic choices for transitions

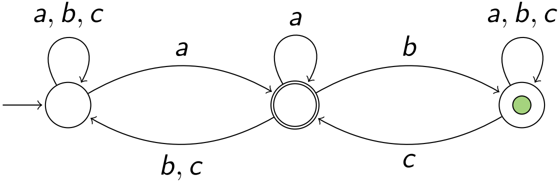


Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:
Adam plays letters: $a \;\; a \;\; b \;\; c$

Eve: resolves non-deterministic choices for transitions



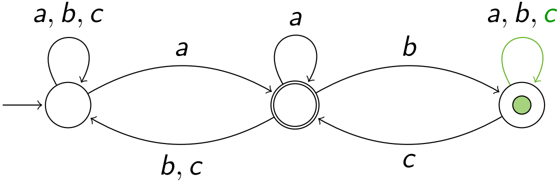
Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters: $a \;\; a \;\; b \;\; c$

Eve: resolves non-deterministic choices for transitions



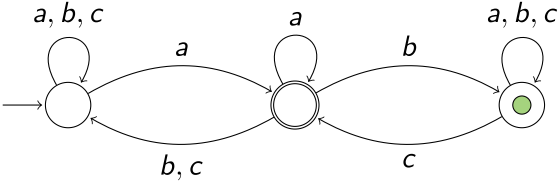
Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters: $a \quad a \quad b \quad c \quad c$

Eve: resolves non-deterministic choices for transitions



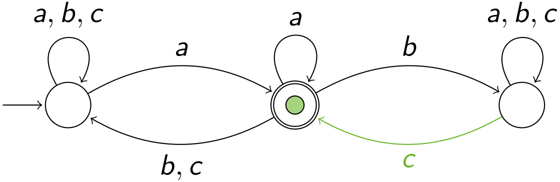
Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters: $a \;\; a \;\; b \;\; c \;\; c$

Eve: resolves non-deterministic choices for transitions



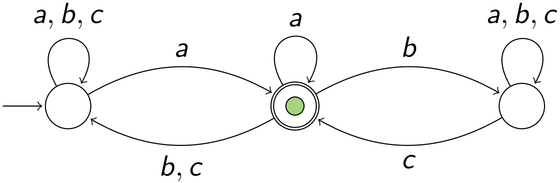
Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:

Adam plays letters: $a \; a \; b \; c \; c \; \ldots \quad = w$

Eve: resolves non-deterministic choices for transitions

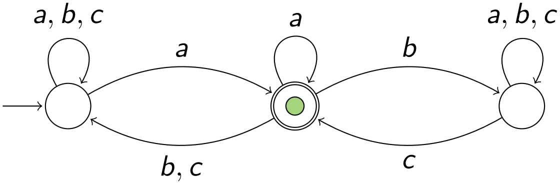


Eve wins if: $w \in L(\mathcal{A}) \Rightarrow$ Run accepting.

Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:
Adam plays letters: $a \quad a \quad b \quad c \quad c \ \dots \quad = \ w$
Eve: resolves non-deterministic choices for transitions



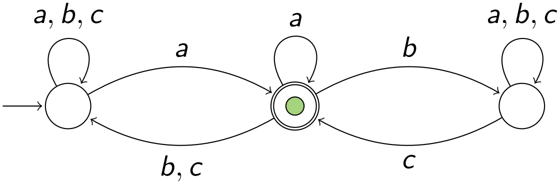
Eve wins if: $w \in L(\mathcal{A}) \Rightarrow$ Run accepting.

$\mathcal{A}$ HD $\Leftrightarrow$ Eve wins the Letter game on $\mathcal{A}$
$\Leftrightarrow$ there is a strategy $\sigma_{\mathrm{HD}} : A^* \rightarrow Q$ accepting all words of $L(\mathcal{A})$.

Definition of HD via a game

$\mathcal{A}$ ND automaton on finite or infinite words.

Letter game of $\mathcal{A}$:
Adam plays letters: $a \;\; a \;\; b \;\; c \;\; c \; \ldots \quad = \; w$
Eve: resolves non-deterministic choices for transitions



Eve wins if: $w \in L(\mathcal{A}) \Rightarrow$ Run accepting.

$\mathcal{A}$ HD $\Leftrightarrow$ Eve wins the Letter game on $\mathcal{A}$
$\Leftrightarrow$ there is a strategy $\sigma_{\mathrm{HD}} : A^* \to Q$ accepting all words of $L(\mathcal{A})$.

Not a parity game! Only ω-regular, hard to solve.

An active research area

HD Introduced in [Henzinger, Piterman '06] for synthesis

An active research area

HD Introduced in [Henzinger, Piterman '06] for synthesis
Reintroduced by [Colcombet '09] for cost functions

An active research area

HD Introduced in [Henzinger, Piterman '06] for synthesis
Reintroduced by [Colcombet '09] for cost functions

Exponential succinctness [K.+Skrzypczak '15]

An active research area

HD Introduced in [Henzinger, Piterman '06] for synthesis
Reintroduced by [Colcombet '09] for cost functions

Exponential succinctness [K.+Skrzypczak '15]

Deciding whether $\mathcal{A}$ is HD in PTime?
Co-Büchi [K.+Skrzypczak '15], Büchi [K.+Bagnol '18]

An active research area

HD Introduced in [Henzinger, Piterman '06] for synthesis
Reintroduced by [Colcombet '09] for cost functions

Exponential succinctness [K.+Skrzypczak '15]

Deciding whether $\mathcal{A}$ is HD in PTime?
Co-Büchi [K.+Skrzypczak '15], Büchi [K.+Bagnol '18]
General Parity [Lehtinen+Prakash '25].

Minimizing HD automata

An active research area

HD Introduced in [Henzinger, Piterman '06] for synthesis
Reintroduced by [Colcombet '09] for cost functions

Exponential succinctness [K.+Skrzypczak '15]

Deciding whether $\mathcal{A}$ is HD in PTime?
Co-Büchi [K.+Skrzypczak '15], Büchi [K.+Bagnol '18]

General Parity [Lehtinen+Prakash '25].

Minimizing HD automata
In PTime For transition-based coBüchi [Abu Radi+Kupferman '21], and for Generalized coBüchi [Casares+Idir+K.+Mascle+Prakash '25]

NP-complete for Büchi, state-based.

An active research area

HD Introduced in [Henzinger, Piterman '06] for synthesis
Reintroduced by [Colcombet '09] for cost functions

Exponential succinctness [K.+Skrzypczak '15]

Deciding whether $\mathcal{A}$ is HD in PTime?
Co-Büchi [K.+Skrzypczak '15], Büchi [K.+Bagnol '18]

General Parity [Lehtinen+Prakash '25].

Minimizing HD automata
In PTIME For transition-based coBüchi [Abu Radi+Kupferman '21], and for Generalized coBüchi [Casares+Idir+K.+Mascle+Prakash '25]

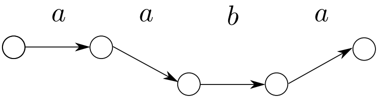
NP-COMPLETE for Büchi, state-based.

Many other questions

- Pushdown, data, stochastic
- Canonical models
- Memory requirements characterizations
- ...

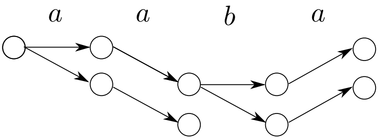
Allowing more runs

Idea: Allow to build several runs, at least one accepting.



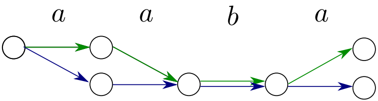
1 state - 1 run

HD



2 states

Width 2
[K. + Majumdar '18 '19]

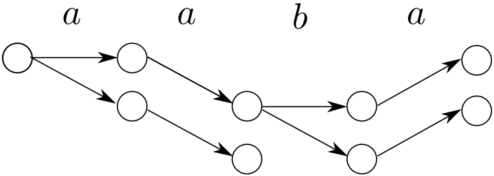


2 runs

2-Explorable
[Hazard+K.+Idir '23 '25]

Width of an automaton

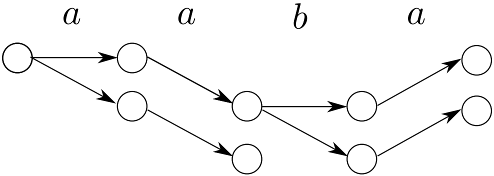
***k*-width game on $\mathcal{A}$:**



Eve wins if $w \in L(\mathcal{A}) \Rightarrow$ her run-DAG contains an accepting run.

Width of an automaton

k-width game on $\mathcal{A}$:

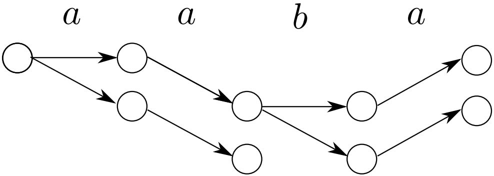


Eve wins if $w \in L(\mathcal{A}) \Rightarrow$ her run-DAG contains an accepting run.

Width of $\mathcal{A}$: Smallest k s.t. Eve wins the k-width game (at most $|Q|$).

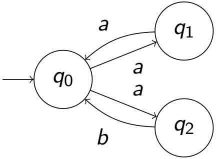
Width of an automaton

***k*-width game on $\mathcal{A}$:**



Eve wins if $w \in L(\mathcal{A}) \Rightarrow$ her run-DAG contains an accepting run.

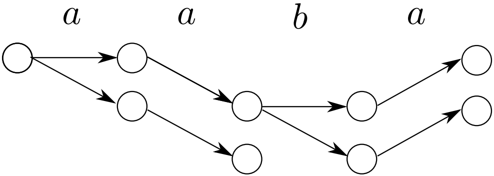
Width of $\mathcal{A}$: Smallest k s.t. Eve wins the k-width game (at most $|Q|$).



A safety NFA of width ?

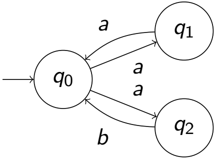
Width of an automaton

***k*-width game on $\mathcal{A}$:**



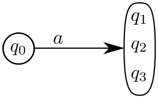
Eve wins if $w \in L(\mathcal{A}) \Rightarrow$ her run-DAG contains an accepting run.

Width of $\mathcal{A}$: Smallest k s.t. Eve wins the k-width game (at most $|Q|$).

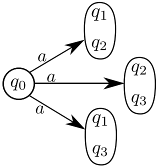


A safety NFA of width 2

k-Determinization

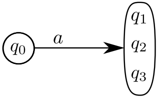


Powerset

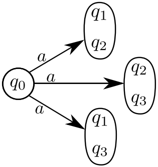


2-Determinization $\mathcal{A}_2$

k-Determinization



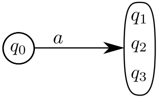
Powerset



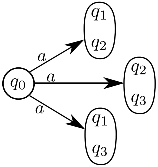
2-Determinization $\mathcal{A}_2$

+ generalization to Breakpoint, Safra: $|\mathcal{A}_k| \approx |\mathcal{A}|^k$

k-Determinization



Powerset



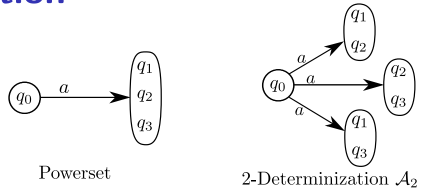
2-Determinization $\mathcal{A}_2$

+ generalization to Breakpoint, Safra: $|\mathcal{A}_k| \approx |\mathcal{A}|^k$

Facts: [K.,Majumdar]

- $\mathrm{width}(\mathcal{A}) \leq k \iff \mathcal{A}_k$ is HD.
- $\mathcal{B} \subseteq \mathcal{A}$ can be tested in $\approx O(n^{\mathrm{width}(\mathcal{A})})$ via a simulation game.

k-Determinization



+ generalization to Breakpoint, Safra: $|\mathcal{A}_k| \approx |\mathcal{A}|^k$

Facts: [K.,Majumdar]

- $\mathrm{width}(\mathcal{A}) \leq k \iff \mathcal{A}_k$ is HD.
- $\mathcal{B} \subseteq \mathcal{A}$ can be tested in $\approx O(n^{\mathrm{width}(\mathcal{A})})$ via a simulation game.

Application: Building a HD aut. from $\mathcal{A}$

Start from $\mathcal{B} = \mathcal{A}$ and $k = 1$;
while $\mathcal{B}$ *is not* HD **do**
    $k := k + 1$;
    $B := \mathcal{A}_k$;
end

Computing the width

Can we compute $k = \text{width}(\mathcal{A})$ to build $\mathcal{A}_k$ directly ?

Theorem [K, Majumdar]

Computing the width of an NFA is $\textsc{ExpTime}$-complete.
Even deciding whether it is $\leq |Q|/2$.

Computing the width

Can we compute $k = \text{width}(\mathcal{A})$ to build $\mathcal{A}_k$ directly ?

Theorem [K, Majumdar]

Computing the width of an NFA is EXPTIME-complete.
Even deciding whether it is $\leq |Q|/2$.

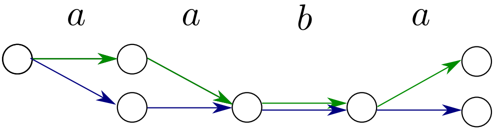
Reduction via a SAT game, introduced by [Robson]
to show EXPTIME-completeness of popular games:

(Japanese rules)

Explorable Automata

***k*-explorability game**:

Adam plays letters, Eve moves k tokens

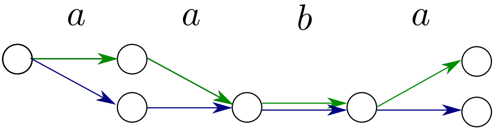


Eve wins if $w \in L(\mathcal{A}) \Rightarrow$ at least one token follows an accepting run.

Explorable Automata

***k*-explorability game**:

Adam plays letters, Eve moves k tokens



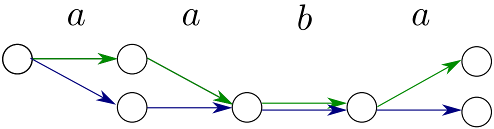
Eve wins if $w \in L(\mathcal{A}) \Rightarrow$ at least one token follows an accepting run.

$\mathcal{A}$ is k-explorable if Eve wins the k-explorability game.

Explorable Automata

***k*-explorability game**:

Adam plays letters, Eve moves k tokens



Eve wins if $w \in L(\mathcal{A}) \Rightarrow$ at least one token follows an accepting run.

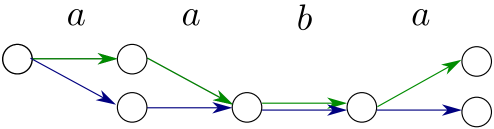
$\mathcal{A}$ is k-explorable if Eve wins the k-explorability game.

$\mathcal{A}$ is explorable if it is k-explorable for some k.

Explorable Automata

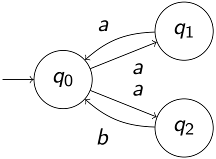
***k*-explorability game**:

Adam plays letters, Eve moves k tokens



Eve wins if $w \in L(\mathcal{A}) \Rightarrow$ at least one token follows an accepting run.

$\mathcal{A}$ is k-explorable if Eve wins the k-explorability game.
$\mathcal{A}$ is explorable if it is k-explorable for some k.



A non-explorable safety NFA

First results

The width result can be lifted:

Theorem [K., Majumdar '18]:

Deciding $|Q|/2$-explorability is $\textsc{ExpTime}$-complete.

First results

The width result can be lifted:

Theorem [K., Majumdar '18]:
Deciding $|Q|/2$-explorability is $\textsc{ExpTime}$-complete.

Can we decide explorability ? If yes, how efficiently ?

First results

The width result can be lifted:

Theorem [K., Majumdar '18]:
Deciding $|Q|/2$-explorability is $\textsc{ExpTime}$-complete.

Can we decide explorability ? If yes, how efficiently ?

Original motivation: progress on the HDness problem

First results

The width result can be lifted:
Theorem [K., Majumdar '18]:
Deciding $|Q|/2$-explorability is $\textsc{ExpTime}$-complete.

Can we decide explorability ? If yes, how efficiently ?

Original motivation: progress on the HDness problem

How many tokens might be needed in explorable automata ?

A related paper

Similar questions in [Betrand et al 2019: Controlling a population]

k-**population game**: Arena like *k*-explorability game on NFA,
Goal of Adam: bring all tokens to a sink state.

A related paper

Similar questions in [Betrand et al 2019: Controlling a population]

k-**population game**: Arena like k-explorability game on NFA,
Goal of Adam: bring all tokens to a sink state.

Population Control Problem (PCP): $\exists k$ s.t. Eve wins ?

A related paper

Similar questions in [Betrand et al 2019: Controlling a population]

k-**population game**: Arena like *k*-explorability game on NFA,
Goal of Adam: bring all tokens to a sink state.

Population Control Problem (PCP): $\exists k$ s.t. Eve wins ?

Results in [Bertrand, Dewaskar, Genest, Gimbert, Godbole]:

- The PCP is ExpTime-complete
- Doubly exponentially many tokens might be needed.

A related paper

Similar questions in [Betrand et al 2019: Controlling a population]

k-**population game**: Arena like *k*-explorability game on NFA,
Goal of Adam: bring all tokens to a sink state.

Population Control Problem (PCP): $\exists k$ s.t. Eve wins ?

Results in [Bertrand, Dewaskar, Genest, Gimbert, Godbole]:

- The PCP is ExpTime-complete
- Doubly exponentially many tokens might be needed.

Our goal: Generalize to Explorability, but

- Game harder to solve: the input word has to be in $L(\mathcal{A})$
- Must deal with acceptance conditions on infinite words.

Results

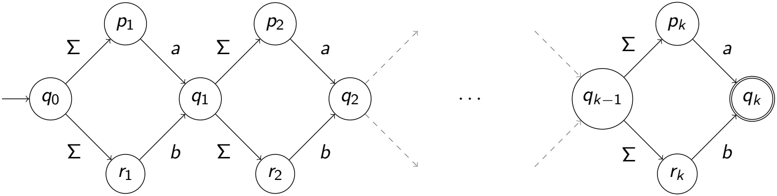
Theorems [Hazard, K. 2023]

Explorability is $\textsc{ExpTime}$-complete for NFA, Büchi.
Doubly exponentially many tokens might be needed.

Results

Theorems [Hazard, K. 2023]

Explorability is EXPTIME-complete for NFA, Büchi.
Doubly exponentially many tokens might be needed.

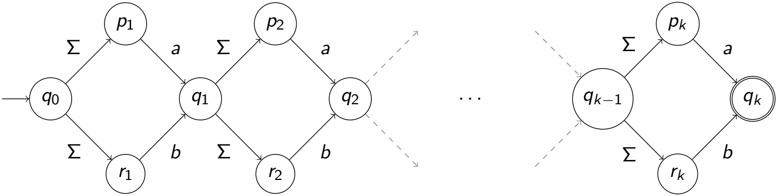


NFA needing exponentially many tokens.

Results

Theorems [Hazard, K. 2023]

Explorability is EXPTIME-complete for NFA, Büchi.
Doubly exponentially many tokens might be needed.



NFA needing exponentially many tokens.

Theorems [Idir, K.]

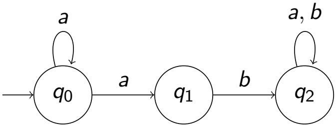
Explorability is EXPTIME for coBüchi, $[0,2]$-Parity.

ω-explorability

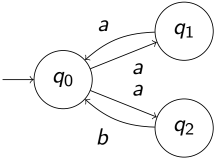
What happens if we allow a countable infinity of tokens ?

ω-explorability

What happens if we allow a countable infinity of tokens ?



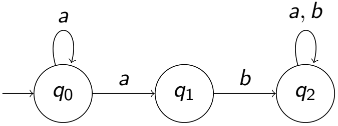
not explorable but
ω-explorable



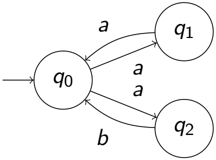
not ω-explorable

ω-explorability

What happens if we allow a countable infinity of tokens ?



not explorable but
ω-explorable



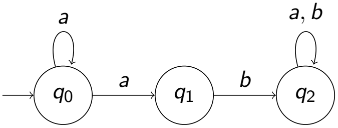
not ω-explorable

Intuition:

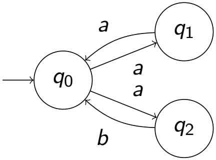
Non-ω-explorable: Adam can always kill any run

ω-explorability

What happens if we allow a countable infinity of tokens ?



not explorable but
ω-explorable



not ω-explorable

Intuition:

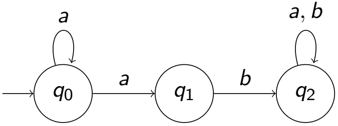
Non-ω-explorable: Adam can always kill any run

Theorem [Hazard, K. 2023]

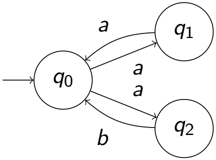
ω-explorability is EXPTIME-complete for safety, coBüchi.

ω-explorability

What happens if we allow a countable infinity of tokens ?



not explorable but
ω-explorable



not ω-explorable

Intuition:

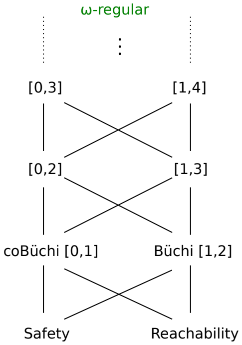
Non-ω-explorable: Adam can always kill any run

Theorem [Hazard, K. 2023]

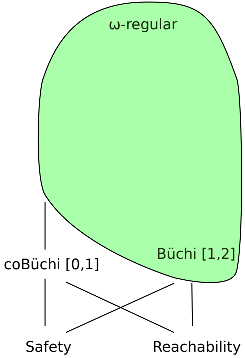
ω-explorability is EXPTIME-complete for safety, coBüchi.

Decidability open for Büchi.

Expressivity of explorable automata

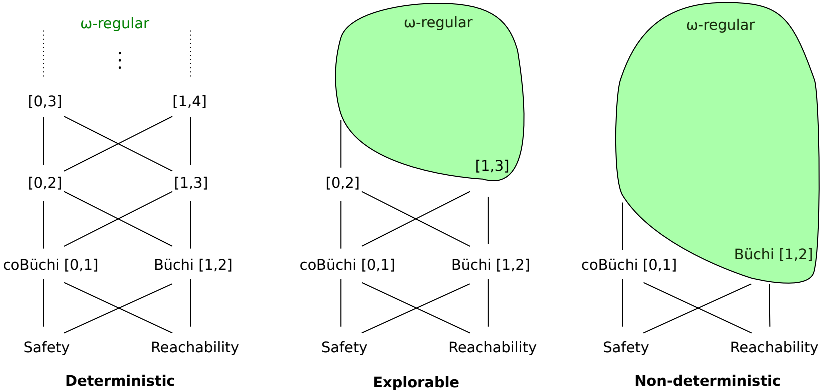


Deterministic

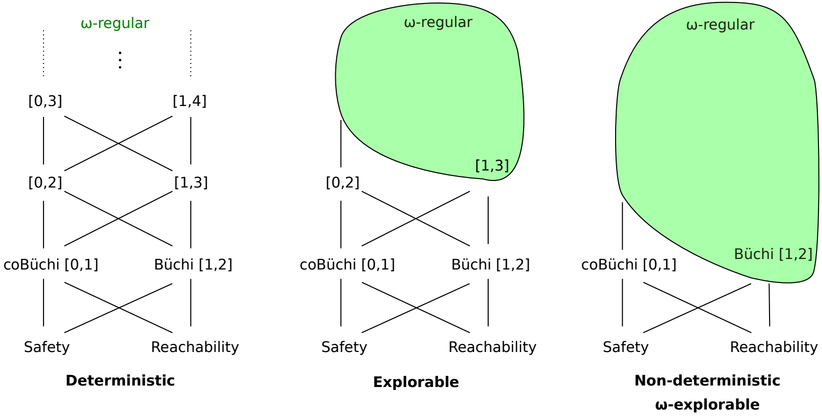


Non-deterministic

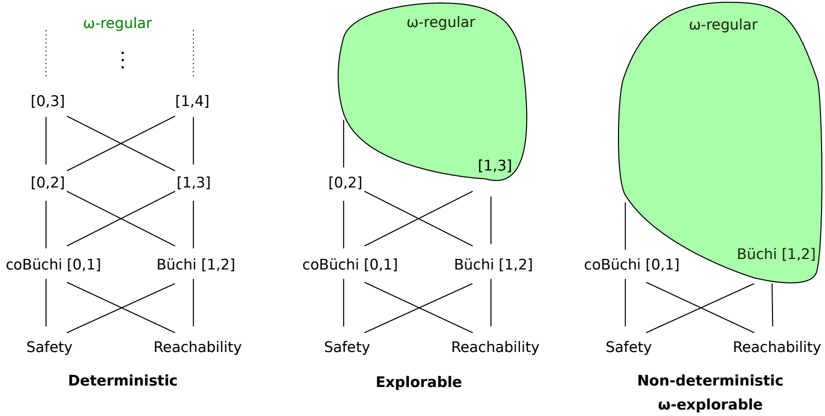
Expressivity of explorable automata



Expressivity of explorable automata



Expressivity of explorable automata



Theorem (Idir, K.)

$[1,3]$-*explorability decidable* $\Leftrightarrow$ *Parity explorability decidable*
Büchi ω-*explorability decidable* $\Leftrightarrow$ *Parity* ω-*explorability decidable*

Future work

- Open decidability: $[1, 3]$-expl., Büchi ω-expl.
- Complexity of k-expl. with k in binary?
- Studying HD and expl. models in other frameworks.
- Practical applications, experimental evaluations.
- ...

Future work

- Open decidability: $[1, 3]$-expl., Büchi ω-expl.
- Complexity of k-expl. with k in binary?
- Studying HD and expl. models in other frameworks.
- Practical applications, experimental evaluations.
- ...

Thanks for your attention!