Positive first-order logic on words

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Highlights 2020

Upward-closed languages

Alphabet: $A = 2^{\Sigma}$
Letter from A = Set of atoms from Σ.

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Definition (Upward-closed languages)

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$$uav \in L \text{ and } a \subseteq b \Longrightarrow ubv \in L.$$

Semantic notion.

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Example

On $\Sigma = \{1, 2\}$:

- $L_0 = A^*\{1,2\}A^*$ is upward-closed.
- $L_1 = A^*\{1\}A^*$ is not upward-closed.

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- no negation: all predicates appear positively.
- atomic predicate $a^\uparrow(x)$ with $a \in \Sigma$: label of x contains a.

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Does this characterize upward-closed FO-definable languages ?
(Syntax vs Semantics)

Our result

Theorem (Syntax $\subsetneq$ Semantics)

There exists an upward-closed FO language on $\Sigma = \{1, 2, 3\}$ that is not FO$^+$-definable.

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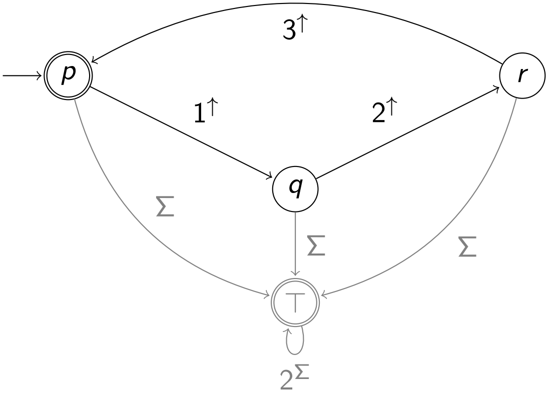

Background: Lyndon's theorem

Zoom out: FO with arbitrary relational signature, on all structures.

Theorem (Lyndon 1959)

FO-definable and upward-closed $\Leftrightarrow$ *FO^+-definable.*

φ *preserved by surjective morphisms* $\Leftrightarrow$ *equivalent to a positive formula.*

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Lyndon's theorem fails on finite structures:

- *on signature* $(4, 3, 3, 3, 3, 2, 1, 1)$ *[Ajtai Gurevich 1987]*
  (lattices, probabilities, number theory, topology)
- *on signature* $(2, 2)$ *[Stolboushkin 1995]*
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- *on signature* $(2, 1, 1, 1)$ *[This work]*
  (E-F games on words, easier)

Ongoing work

Open problem

Can we decide whether a regular language is FO^+-definable ?

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Thanks for your attention !