

Recognizing Good-for-Games Automata: The G_2 Conjecture

Denis Kuperberg

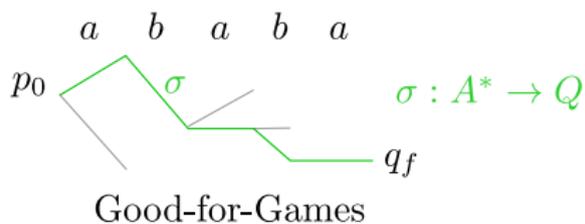
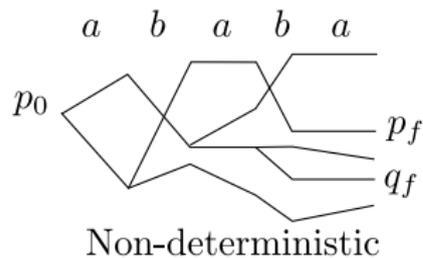
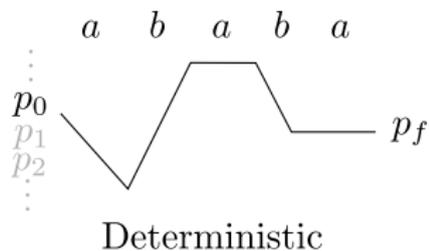
with Marc Bagnol, Udi Boker, Karoliina Lehtinen, Michał Skrzypczak

CNRS, LIP, ENS Lyon

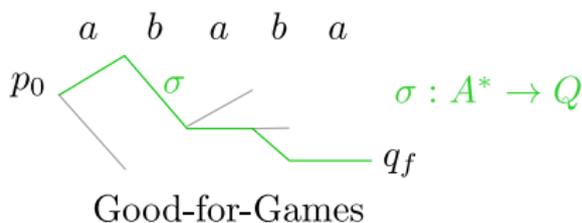
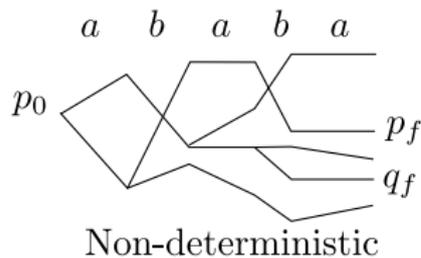
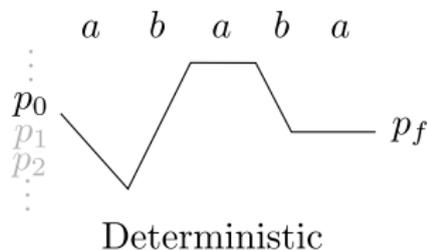
IRIF Seminar

06/11/2020

Good-for-Games Automata



Good-for-Games Automata



Good properties:

- ▶ Compositional soundness [Henziger, Piterman '06] [Colcombet '09]
- ▶ Succinctness [K., Skrzypczak '15]

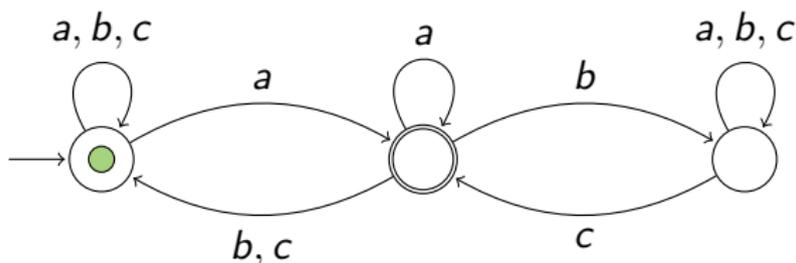
Definition of GFG via a game

A ND automaton on finite or infinite words.

GFG game of \mathcal{A} :

Adam plays letters:

Eve: resolves non-deterministic choices for transitions



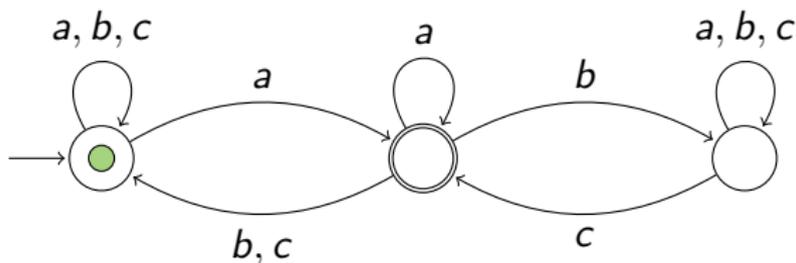
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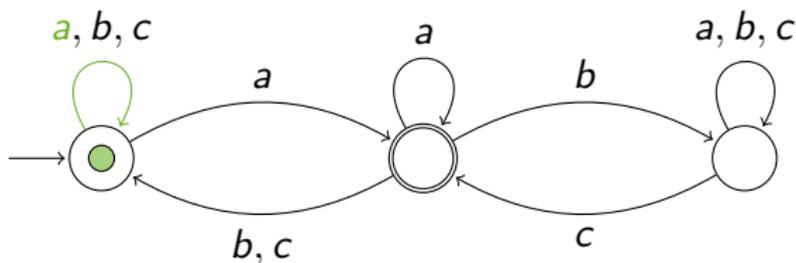
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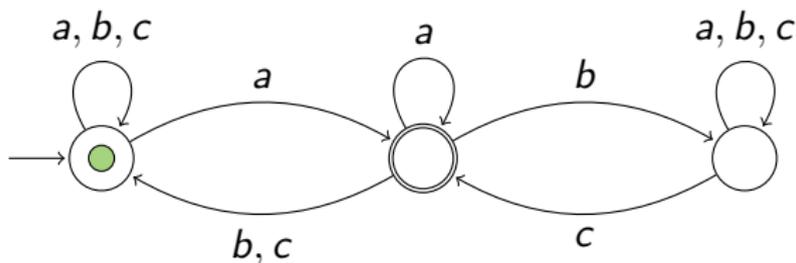
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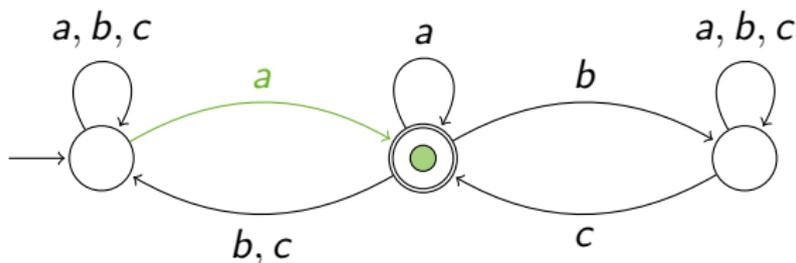
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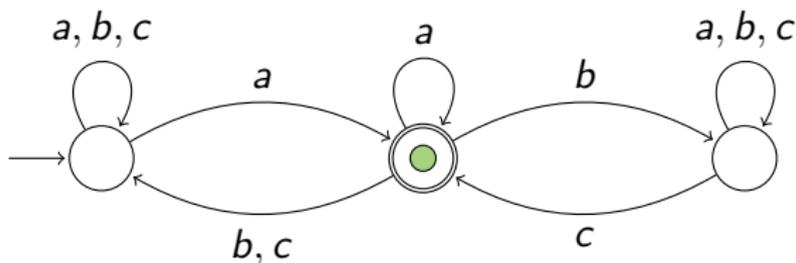
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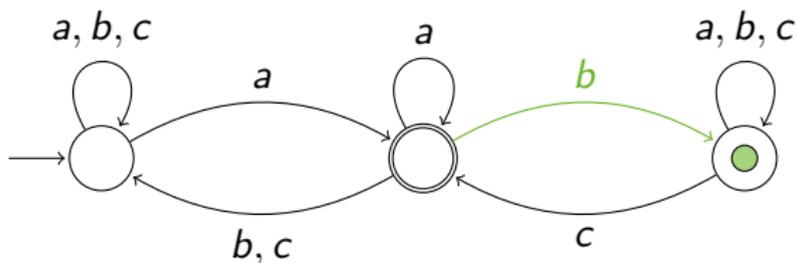
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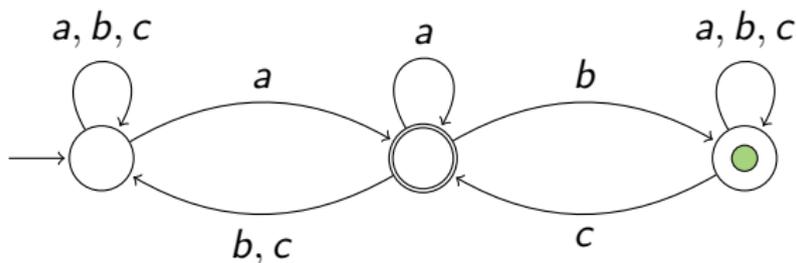
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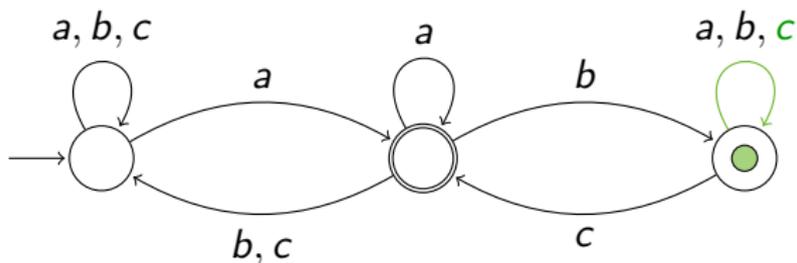
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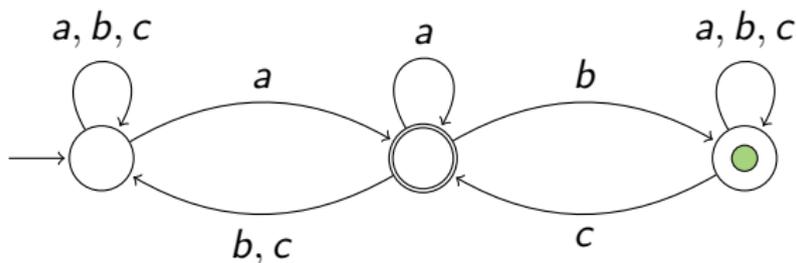
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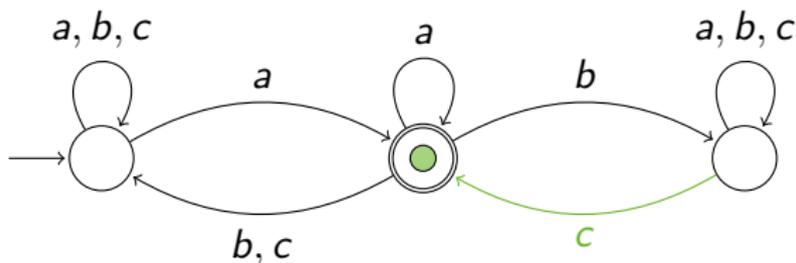
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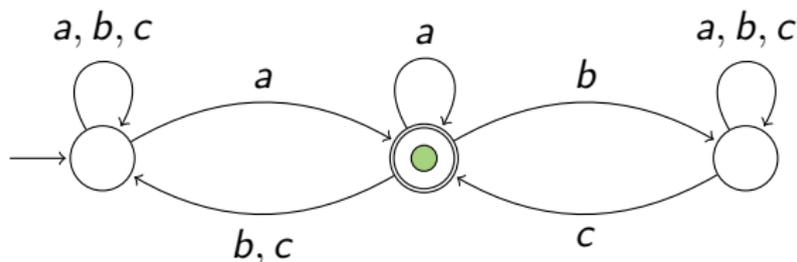
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\Leftrightarrow there is a strategy $\sigma_{\text{GFG}} : A^* \rightarrow Q$ accepting all words of $L(\mathcal{A})$.

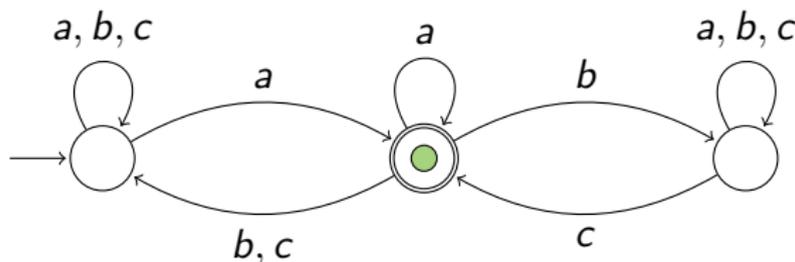
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Not a parity game! Only ω -regular, hard to solve.

A few facts on GFG automata

Fact

Every deterministic automaton is GFG.

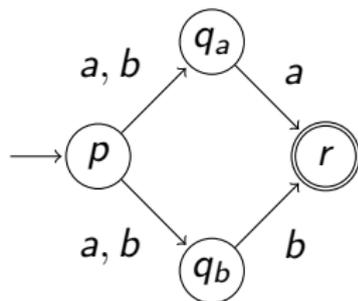
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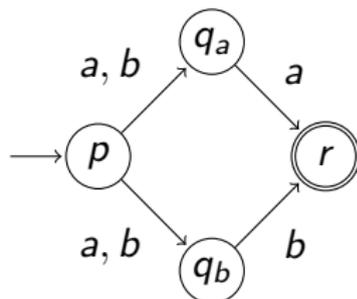
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Remark:

GFG automata can be used in algorithms without knowing σ_{GFG} .
The strategy σ_{GFG} “hides” the determinism.

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So GFGness is decidable, but can we do better than EXPTIME ?

On finite words

Theorem (Löding)

The GFGness problem is in P for finite words automata.

On finite words

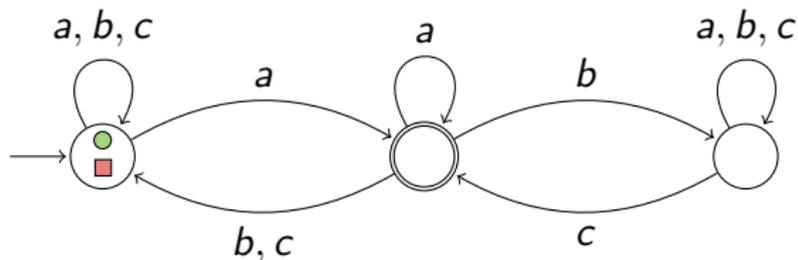
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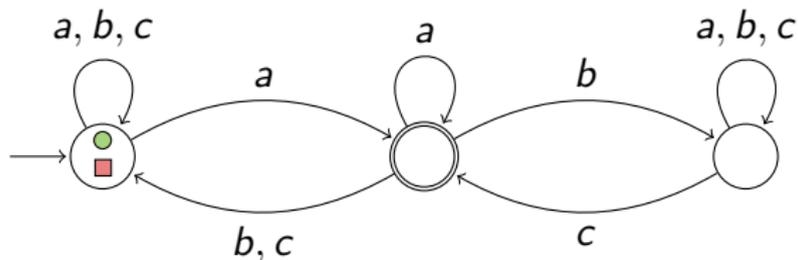
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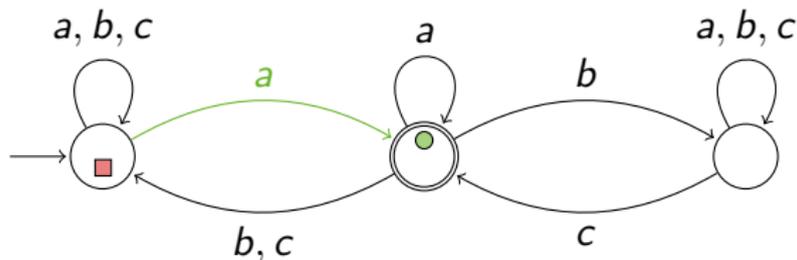
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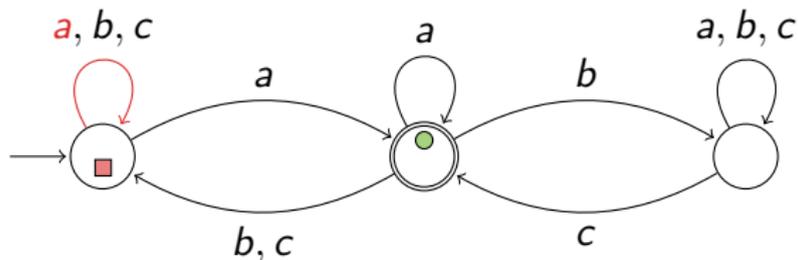
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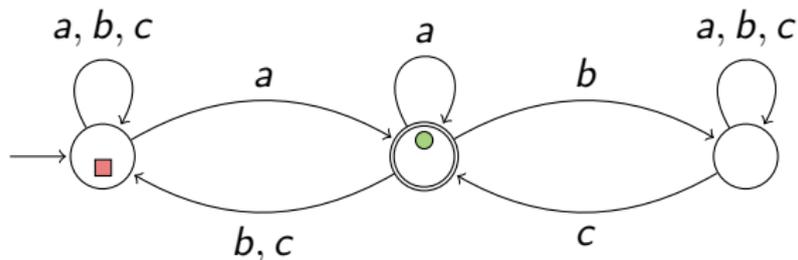
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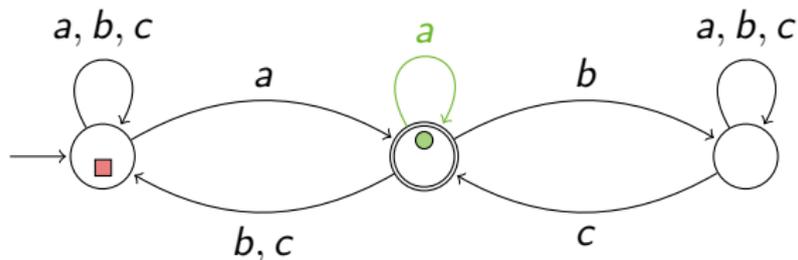
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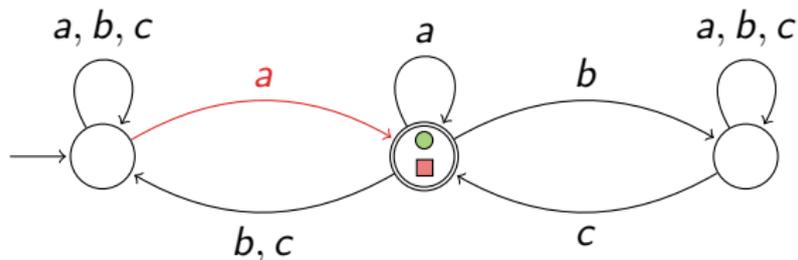
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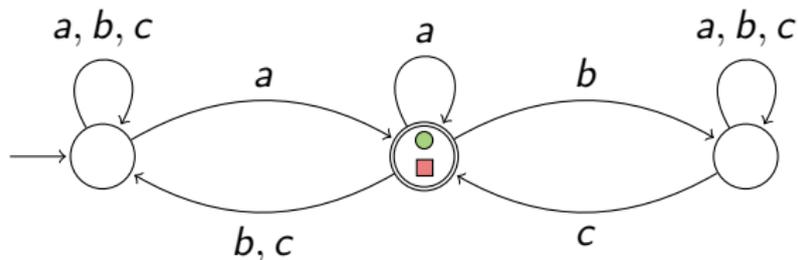
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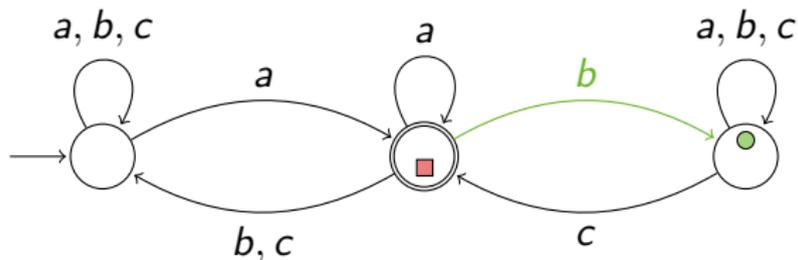
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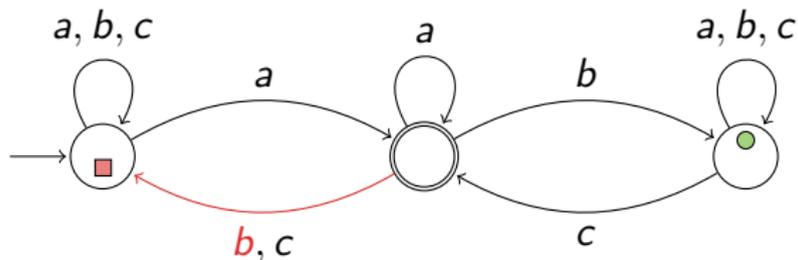
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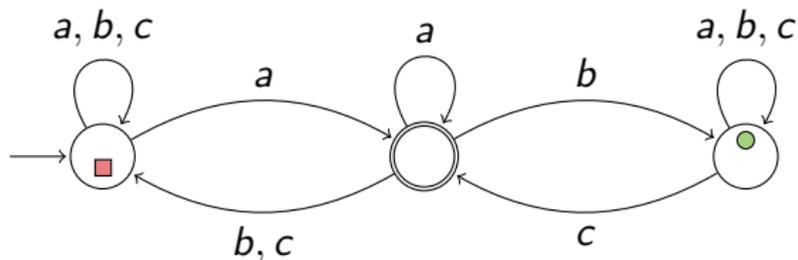
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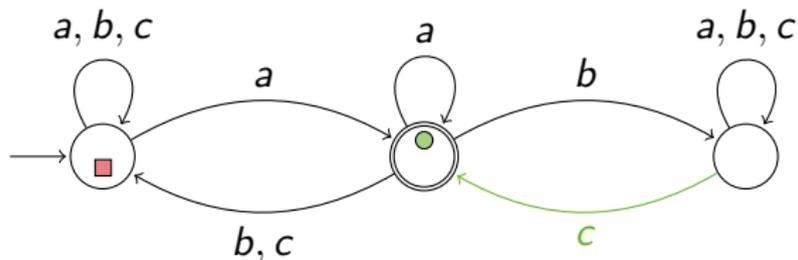
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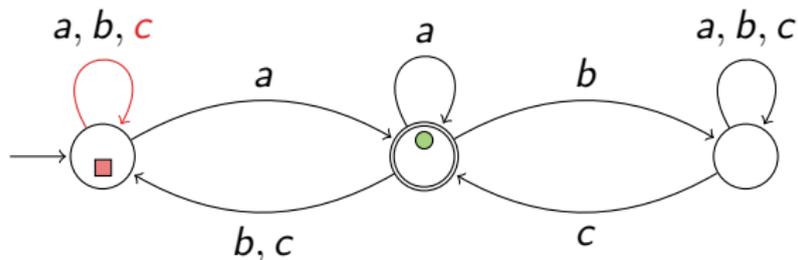
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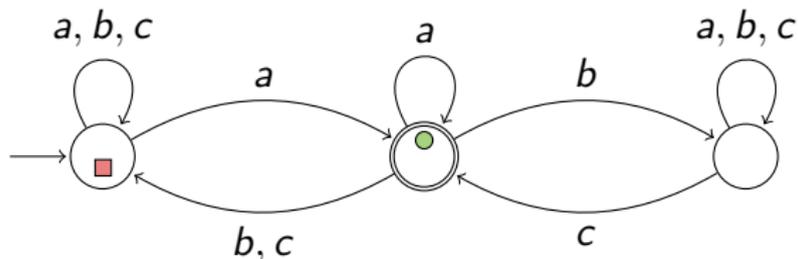
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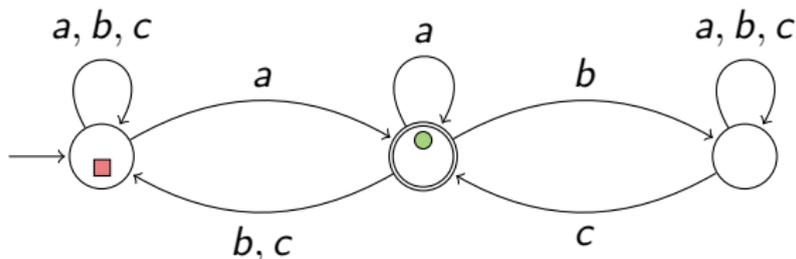
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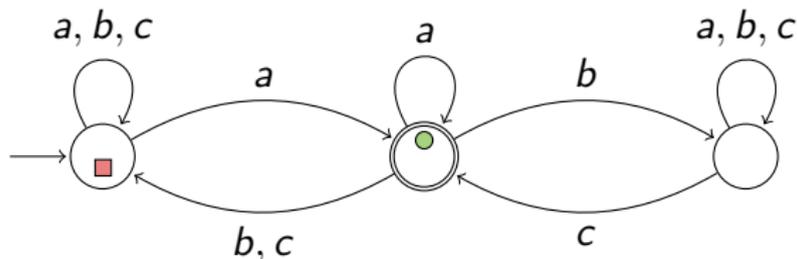
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G_1 is a safety game, solvable in polynomial time.

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Büchi: see infinitely many states (or transitions) from F .

coBüchi: eventually see only states (or transitions) from F .

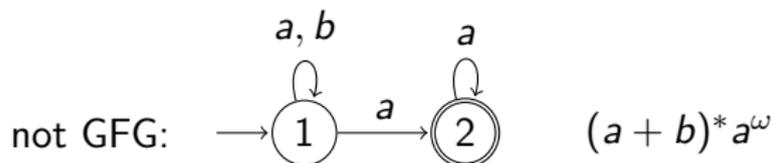
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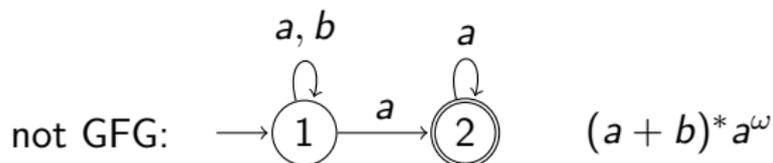
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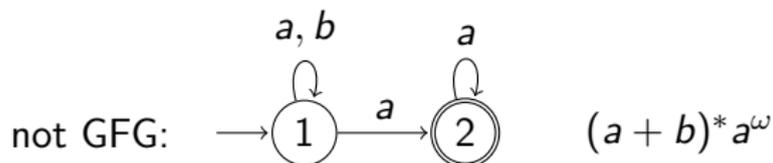
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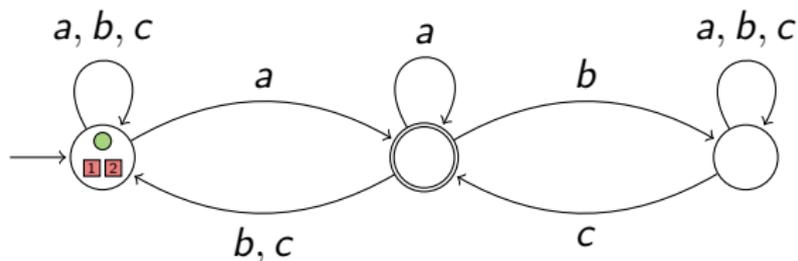
We need a better abstraction of the GFG game.

Abstracting the GFG game

The game G_2 :

Adam plays letters:

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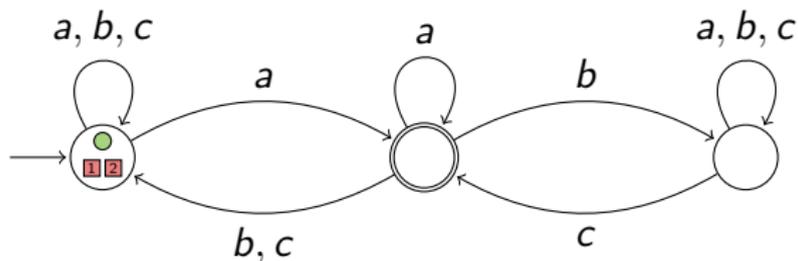


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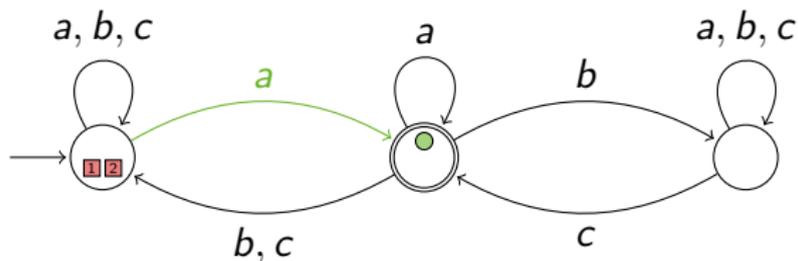


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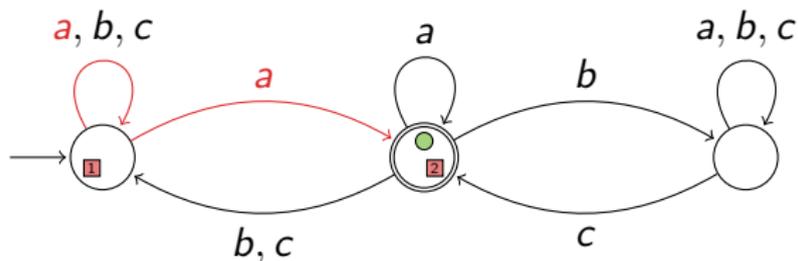


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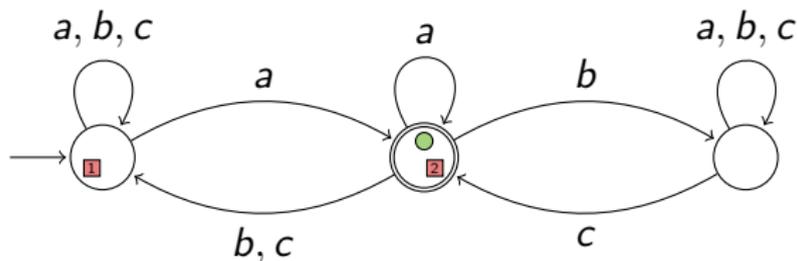


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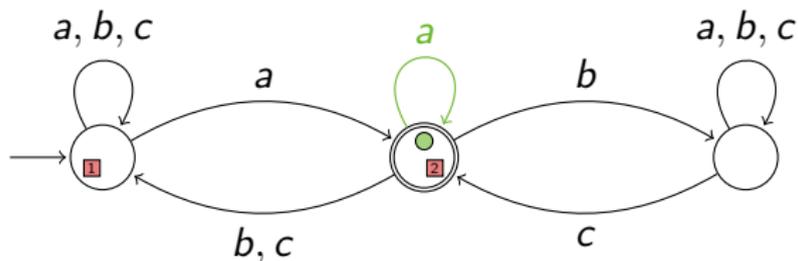


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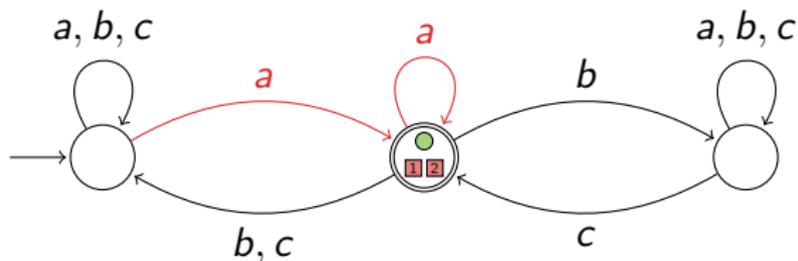


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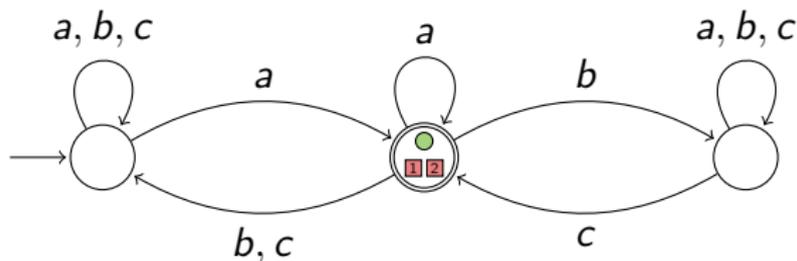


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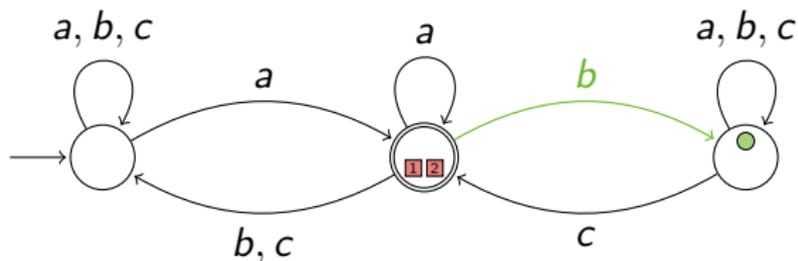


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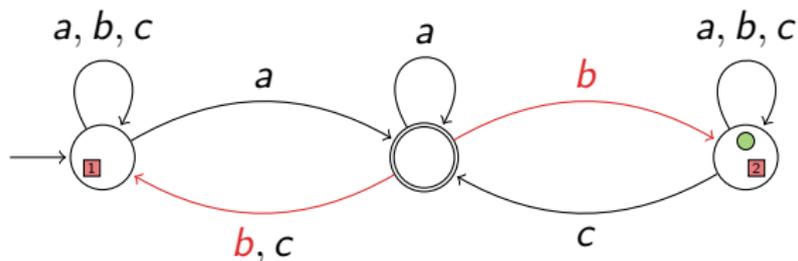


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The game G_2 :

Adam plays letters: a a b

Eve: moves one token \bullet , Adam: moves two tokens $\boxed{1}$, $\boxed{2}$

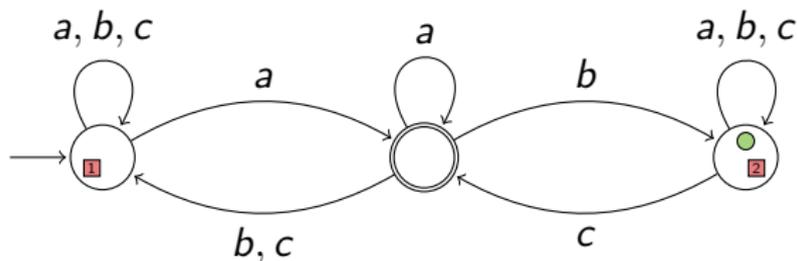


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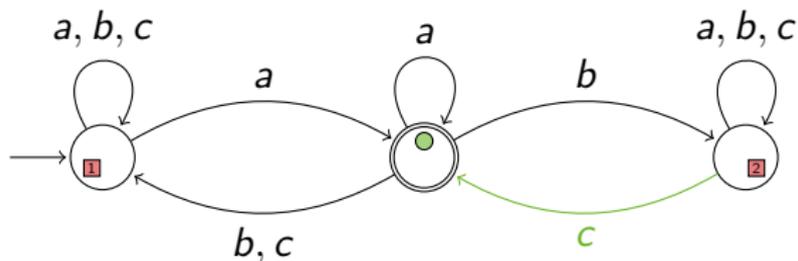


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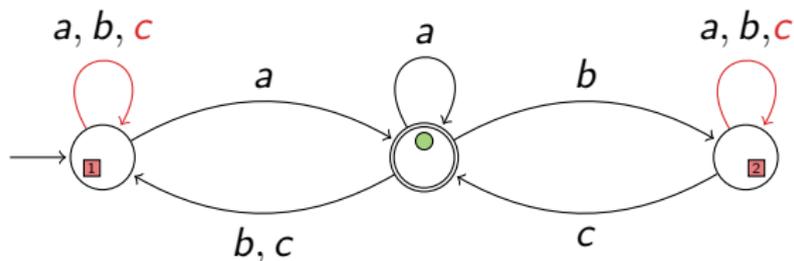


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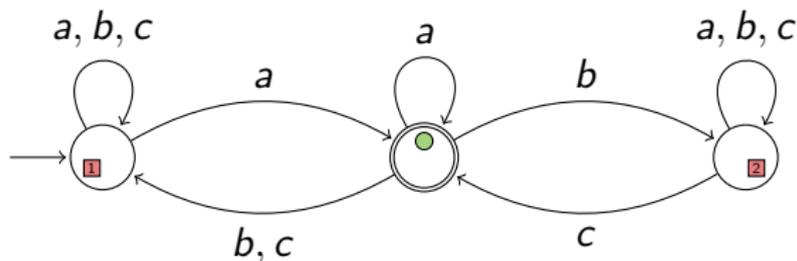


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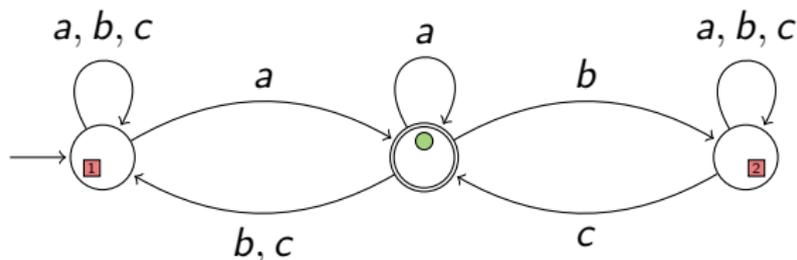
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Theorem (Bagnol, K. '18)

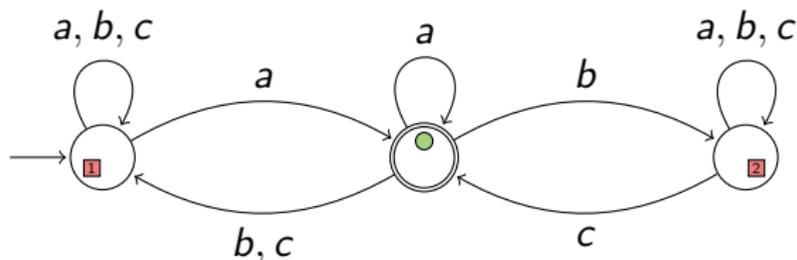
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Solving G_2 is polynomial \Rightarrow Efficient algorithm for GFGness.

Proof sketch

Game G_k : k tokens \blacksquare . Some \blacksquare_i accepts \Rightarrow \bullet must accept.

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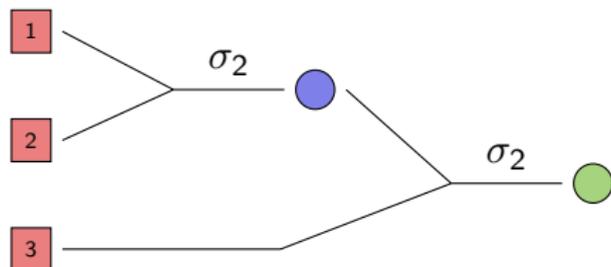
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Eve wins $G_2 \Leftrightarrow$ Eve wins G_k for all $k \geq 2$.

Proof sketch: $G_2 \Rightarrow G_3$

- ▶ play a *virtual token* \circ against \square_1 and \square_2 .
- ▶ play G_2 strategy against \circ and \square_3 .



Main proof sketch for $G_2 \Rightarrow \text{GFG on Büchi}$

Assume for contradiction:

- ▶ Eve wins G_2 , so Eve wins G_k with strategy σ_k , for a big k .
- ▶ Adam wins the GFG game with finite-memory strategy τ_{GFG} .

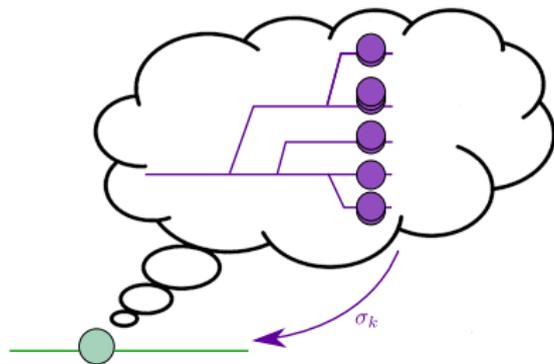
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Idea for a strategy against τ_{GFG} in the GFG game:

- ▶ move k virtual tokens uniformly
- ▶ play σ_k against these k tokens



Trick: Word from $\tau_{\text{GFG}} \Rightarrow$ one Büchi for some \bullet every M steps.
 \rightsquigarrow \bullet wins against τ_{GFG} , contradiction.

G_2 for coBüchi

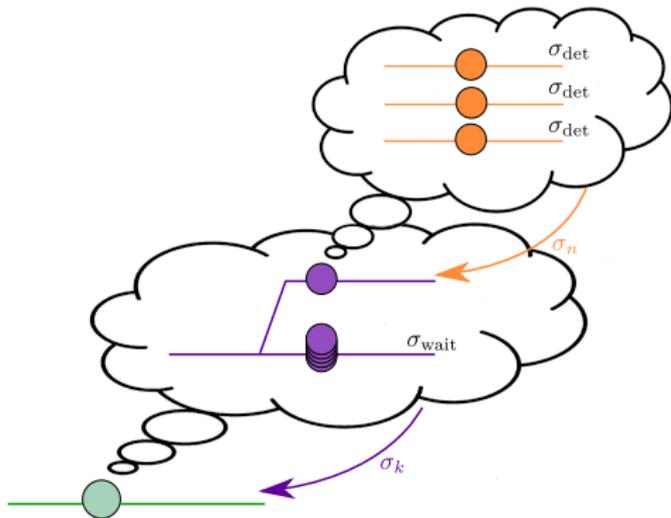
Theorem (unpublished)

G_2 characterizes *GFGness* for coBüchi automata.

G_2 for coBüchi

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The main conjecture

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Alternating **GFG** automata are also interesting.

[Colcombet '13] [Boker, Lehtinen '19] [Boker, K., Lehtinen, Skrzypczak '20]

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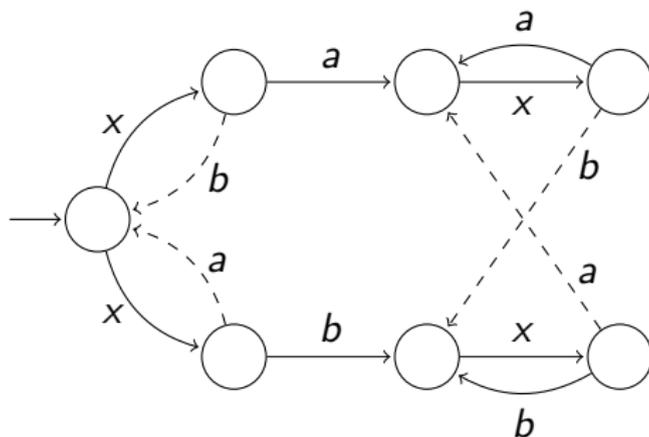
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Thanks for your attention!

Some GFG automata

Theorem ([Boker, K., Kupferman, Skrzypczak '13])

Some GFG automata cannot be determinized by removing transitions.



A GFG coBüchi automaton for $(xa + xb)^*[(xa)^\omega + (xb)^\omega]$.

Theorem (K., Skrzypczak '15)

There are some GFG automata such that determinization costs an exponential blow-up in number of states.