Decidability problems in automata theory

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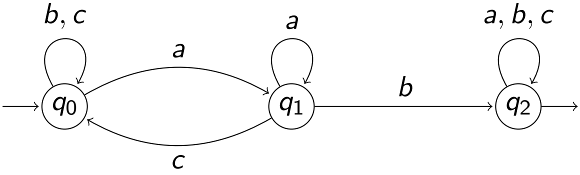
Introduction

- Some natural problems are undecidable.
- For some problems, decidability is open.
- **Finite Automata:** Abstract machine with a lot of decidable properties and equivalent formalisms.
- **Automata theory:**
  Toolbox to decide many problems arising naturally.
  Verification of systems can be done automatically.
  Theoretical and practical advantages.
- **Problem:**
  Decidability is still open for some automata-related problems.

Descriptions of a language



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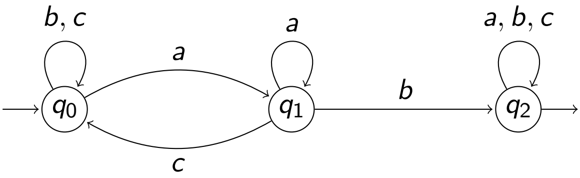

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Other ways than automata to specify $L_{ab} \subseteq \mathbb{A}^*$:

- Regular expression : $\mathbb{A}^* ab \mathbb{A}^*$,
- Logical sentence (MSO) : $\exists x \; \exists y \; a(x) \wedge b(y) \wedge (y = Sx)$.
- Finite monoid : ?

Decision problems

What can we easily decide/compute with automata?

- Emptyness (Hard for Logic !)
- Complementation
- Union/Intersection
- Concatenation
- ...

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What about expressibility in smaller formalisms?
Like FO or star-free expressions?

I.e. given L, is there a FO-formula for L? (resp. star-free expression)

What is a monoid?

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Examples : Groups, $(\mathbb{A}^*, \mathrm{concat})$, $(\{0,1\}, \wedge)$.

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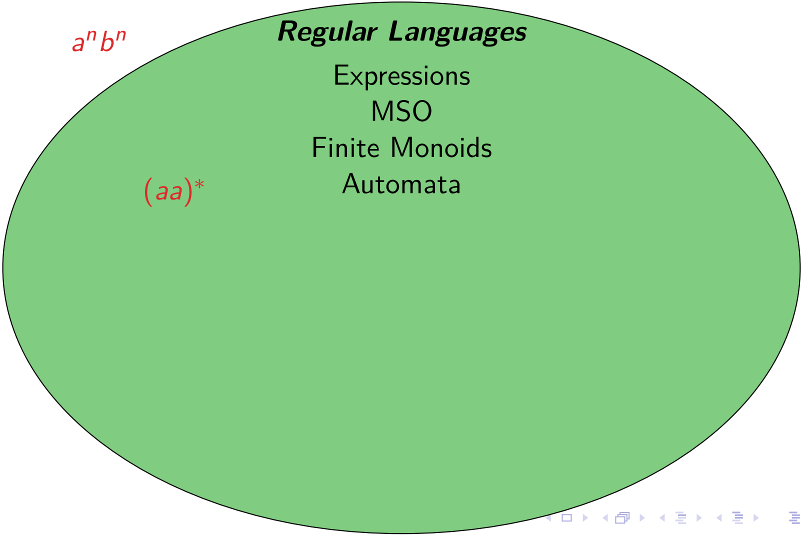
- $\mathbb{A}^* b \mathbb{A}^*$ is recognized by $(\{0,1\}, \wedge)$, with $P = \{0\}$, $h(b) = 0$ and $h(a) = 1$ for $a \neq b$.

Regular Languages

All these formalisms are effectively equivalent.



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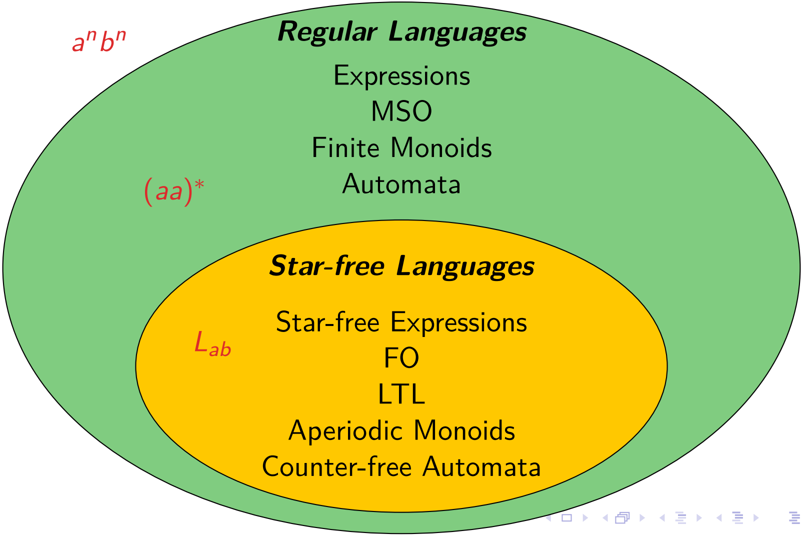

1. Automata theory
2. **Hard decision problems**
3. Regular Cost Functions
4. Formalisms on finite words

Algebraic approach to membership problems

Given a class of languages $\mathcal{C}$, is there an algorithm which given an automaton for L, decides whether $L \in \mathcal{C}$?

Theorem (Schützenberger 1965)

It is decidable whether a regular language is star-free, thanks to the equivalence with aperiodic monoids (i.e. without groups).

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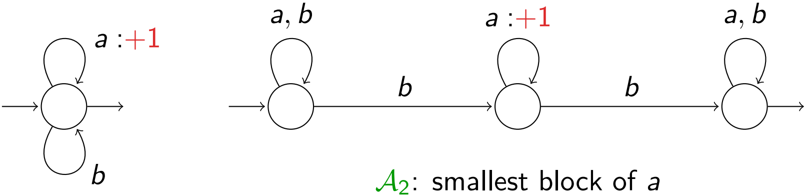
Finite Power Problem: Given L, is there n such that

$$(L+\varepsilon)^n = L^* \ ?$$

There is no known algebraic characterization,
other technics are needed to show decidability.

Distance Automata



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$\mathcal{A}_1$: number of a

Unbounded: There are words with arbitrarily large value (on any run).

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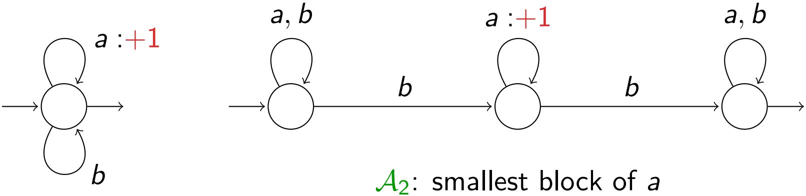

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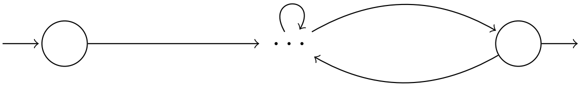
Theorem (Hashiguchi 82, Kirsten 05)

Boundedness is decidable for distance automata.

Deciding **Boundedness** for distance automata $\Rightarrow$ solving finite power problem.

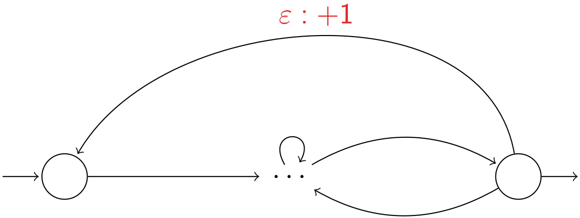
Reductions from Finite Power to Boundedness

1. Start with an automaton for L.



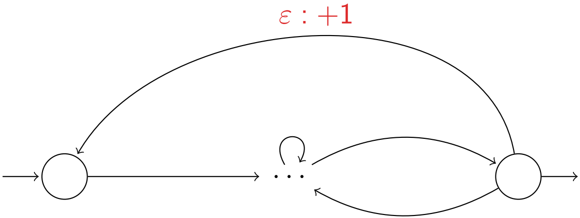
Reductions from Finite Power to Boundedness

1. Start with an automaton for L.
2. Add increment ε-transitions from final states to initial.



Reductions from Finite Power to Boundedness

1. Start with an automaton for L.
2. Add increment ε-transitions from final states to initial.
3. Decide boundedness



Problems solved using counters

- ***Finite Power*** (finite words) [Simon '78, Hashiguchi '79]
  Is there n such that $(L + \varepsilon)^n = L^*$?
- ***Fixed Point Iteration*** (finite words)
  [Blumensath+Otto+Weyer '09]
  Can we bound the number of fixpoint iterations in a MSO formula ?
- ***Star-Height*** (finite words/trees)
  [Hashiguchi '88, Kirsten '05, Colcombet+Löding '08]
  Given n, is there an expression for L, with at most n nesting of Kleene stars?
- ***Parity Rank*** (infinite trees)
  [reduction in Colcombet+Löding '08, decidability open, deterministic input Niwinski+Walukiewicz '05]
  Given $i < j$, is there a parity automaton for L using ranks $\{i, i+1, \ldots, j\}$?

Theory of Regular Cost Functions

Aim: General framework for previous constructions.

- Generalize from languages $L : \mathbb{A}^* \to \{0, 1\}$
  to functions $f : \mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$
- Accordingly generalize automata, logics, semigroups, in order to obtain a theory of regular cost functions, which behaves as well as possible.
- Obtain decidability results thanks to this new theory.

Cost automata over words

Nondeterministic finite-state automaton $\mathcal{A}$

+ finite set of counters
  (initialized to 0, values range over $\mathbb{N}$)

+ counter operations on transitions
  (increment `I`, reset `R`, check `C`, no change ε)

Semantics: $[\![\mathcal{A}]\!] : \mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$

Boundedness relation

"$[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$": undecidable [Krob '94]

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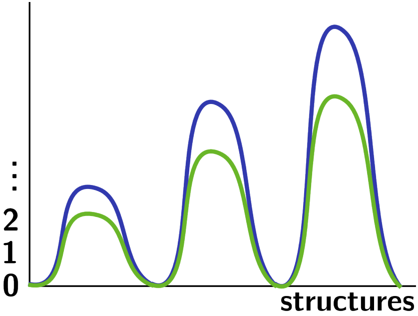
for all subsets U, $[\![\mathcal{A}]\!](U)$ bounded iff $[\![\mathcal{B}]\!](U)$ bounded



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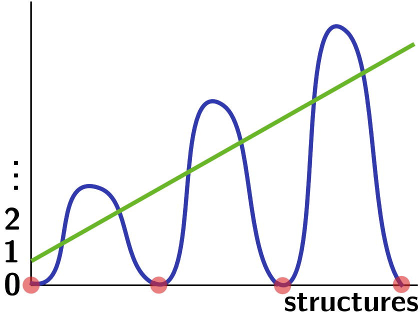

$[\![\mathcal{A}]\!] \not\approx [\![\mathcal{B}]\!]$

Therefore we always identify two functions if they are bounded on the same sets.

Example

For any function f, we have $f \approx 2f \approx \exp(f)$.
But $(u \mapsto |u|_a) \not\approx (u \mapsto |u|_b)$, as witnessed by the set a^*.

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Theorem (Colcombet '09, following Hashiguchi, Leung, Simon, Kirsten, Bojańczyk+Colcombet)

Cost automata ⇔ Cost logics ⇔ Stabilisation monoids.
For some suitable models of Cost Logics and Stabilisation Monoids, extending the classical ones.
Boundedness decidable.

All these equivalences are only valid up to $\approx$.
It provides a toolbox to decide boundedness problems.

Languages as cost functions

A language L is represented by its characteristic function

$$\chi_L(u) = \begin{cases} 0 & \text{if } u \in L \\ \infty & \text{if } u \notin L \end{cases}$$

Cost function theory strictly extends language theory.

All theorems on cost functions are in particular true for languages.

Research program: Study cost function theory, and generalise known theorems from languages to cost functions.

1. Automata theory
2. Hard decision problems
3. Regular Cost Functions
4. **Formalisms on finite words**

Classical Logics on Finite Words

- Linear Temporal Logic (LTL) over $\mathbb{A}^*$:

$$\varphi := a \mid \Omega \mid \neg\varphi \mid \varphi \vee \psi \mid \mathbf{X}\varphi \mid \varphi\mathbf{U}\psi$$

$$\varphi\mathbf{U}\psi: \quad \begin{matrix} \varphi & \varphi & \varphi & \varphi & \varphi & \varphi & \varphi & \varphi & \psi & & \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{matrix}$$

Future operators $\mathbf{G}$ (Always) and $\mathbf{F}$ (Eventually).
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- MSO: FO with quantification on sets, noted X, Y.

Generalisation: cost LTL

- CLTL over $\mathbb{A}^*$:

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Negations pushed to the leaves, to guarantee monotonicity.

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- "Error variable" N is unique, shared by all occurences of $\mathbf{U}^{\leq N}$.
- $\mathbf{G}^{\leq N}\varphi$: φ is false at most N times in the future ($\varphi \mathbf{U}^{\leq N} \Omega$).

Generalisation : Cost FO and Cost MSO

- CFO over $\mathbb{A}^*$:

  $$\varphi := a(x) \mid x = y \mid x < y \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x \varphi \mid \forall x \varphi \mid \forall^{\leq N} x \varphi$$

  Negations pushed to the leaves, to guarantee monotonicity.
- As before, N unique free variable.
- $\forall^{\leq N} x \varphi(x)$ means φ is false on at most N positions.
- CMSO extends CFO by allowing quantification over sets.

Semantics of Cost Logics

From formula to cost function:

Formula $\varphi \longrightarrow$ cost function $[\![\varphi]\!] : \mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$, defined by

$$[\![\varphi]\!](u) = \inf\{n \in \mathbb{N} : \varphi \text{ is true over } u \text{ with } n \text{ as error value}\}$$

Example on alphabet $\{\epsilon, Request, Grant\}$:

$\mathbf{G}(Request \implies \bot\mathbf{U}^{\leq N} Grant)$.

If φ is a classical formula for L, then $[\![\varphi]\!] = \chi_L$.

Aperiodic Monoids

Aperiodic Monoids

Theorem (McNaughton-Papert, Schützenberger, Kamp)

Aperiodic Monoids $\Leftrightarrow$ *FO* $\Leftrightarrow$ *LTL* $\Leftrightarrow$ *Star-free Expressions.*

We want to generalise this theorem to cost functions.
The problems are:

- No complementation $\Rightarrow$ No Star-free expressions.
- Deterministic automata are strictly weaker.
- Heavy formalisms (semantics of stabilisation monoids).
- New quantitative behaviours.
- Original proofs already hard.

Aperiodic cost functions

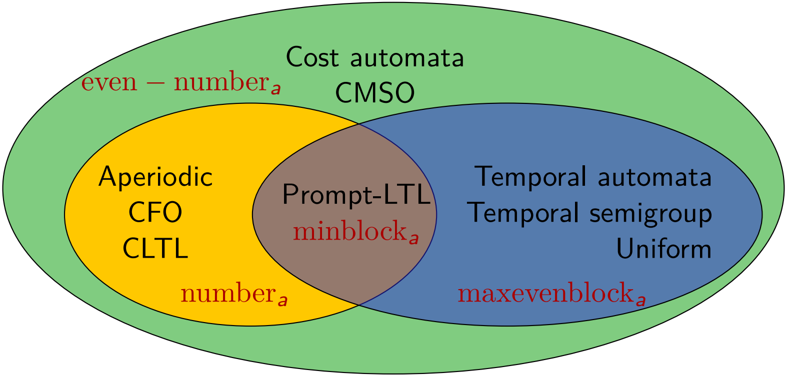
Theorem (K. STACS 2011)

Aperiodic stabilisation monoid $\Leftrightarrow$ *CLTL* $\Leftrightarrow$ *CFO.*

Proof Ideas:

- Generalisation of Myhill-Nerode $\Rightarrow$ Syntactic object.
- Induction on $(|M|, |\mathbb{A}|)$.
- Extend functions to sequences of words.
- Use bounded approximations.
- Extend CLTL with Past operators, show Separability.

Cost Functions on finite words



Decidability of membership and effectiveness of translations [K+Colcombet+Lombardy ICALP '10, K. STACS '11].
Generalization of Myhill-Nerode Equivalence [K. STACS '11].
Boundedness of CLTL is PSPACE-complete [K. LMCS].

As for regular languages, theory of regular cost functions can be extended to

- Infinite words,
- Finite trees,
- Infinite trees.

Led to rich developments, results for both functions and languages, but also complications and open problems...
Main open problem : Decidability of boundedness on infinite trees?

Conclusion

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- Robust quantitative extension of regular language theory.
- Embeds proof using different kind of automata with counters.
- Rich quantitative behaviours occur.
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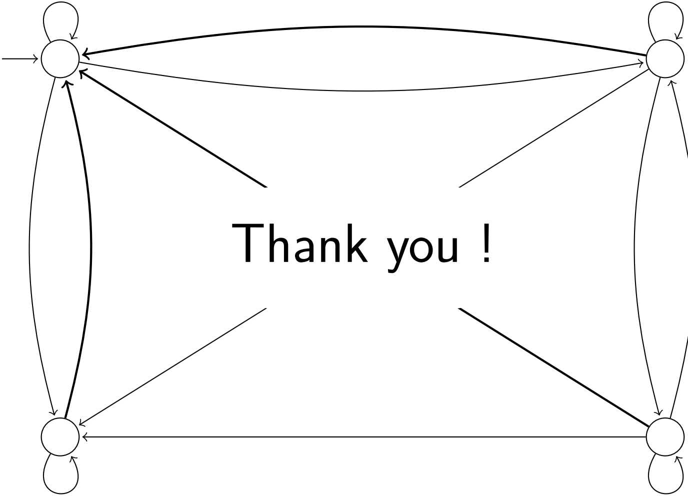
Current challenges and related works:

- Main open problem: decide boundedness on infinite trees. Application to language theory.
- Link with other formalisms, as MSO+U of Bojańczyk.
- Decide properties of cost automata, like optimal number of counters.
- Fine study of approximations (Daviaud)
- Alternative formalisms: IST, profinite words


Thank you !