Is your automaton good for playing games ?

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CNRS, LIP, ENS Lyon

University of Warwick
30/10/2018

Games

Two Players: Eve Adam

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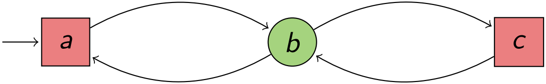
Arena: finite graph $G = (V, E)$, with $V = V_{\bigcirc} \uplus V_{\square}$.



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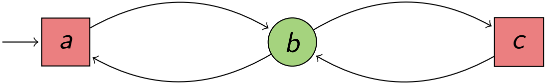

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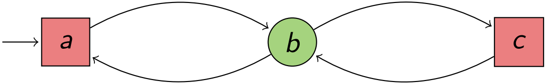

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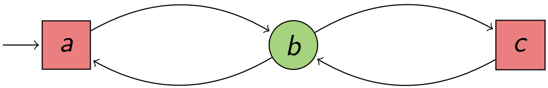

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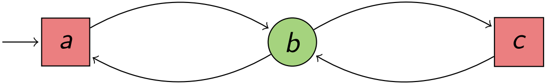

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ω-regular games

Winning condition W: an ω-regular language.

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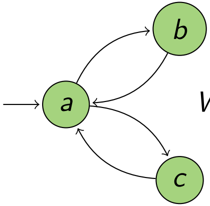

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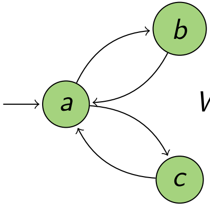

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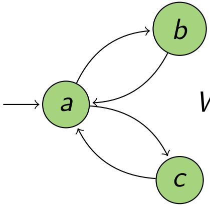

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Applications in Logic, Verification, Proof theory, Complexity, ...

Church Synthesis Problem.

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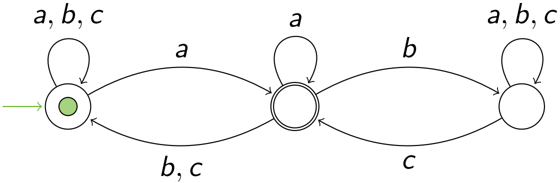
Also works for composition with infinite trees.

Definition of GFG via a game

$\mathcal{A}$ Parity automaton, we associate to it a **GFG game**:

Adam plays letters:

Eve: resolves non-deterministic choices for transitions



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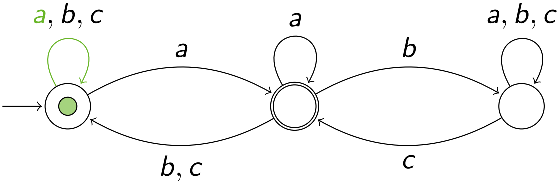

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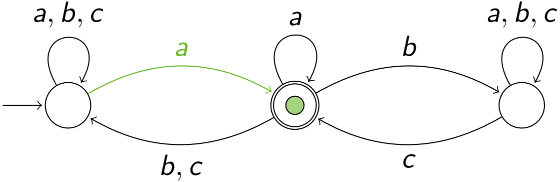

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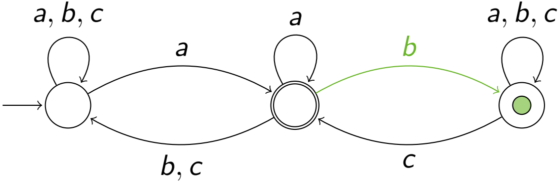

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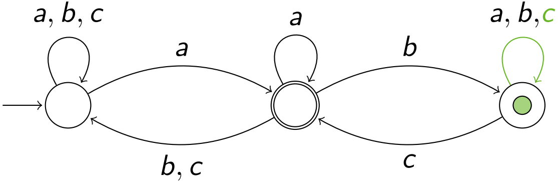

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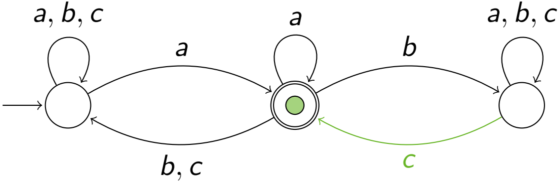

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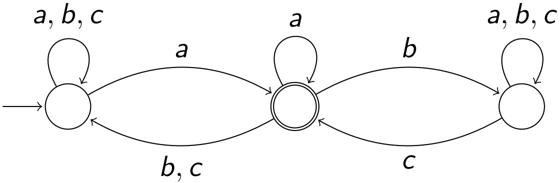

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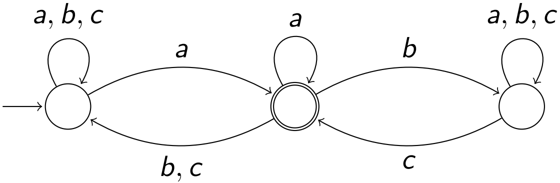

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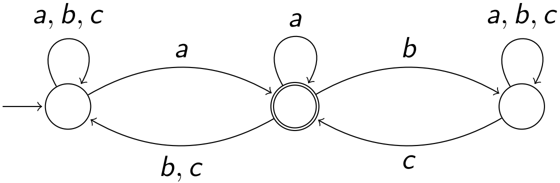

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$\mathcal{A}$ **GFG** means that there is a strategy $\sigma_{\mathrm{GFG}} : A^* \to Q$, for accepting all words of $L(\mathcal{A})$.

A few facts on GFG automata

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Every deterministic automaton is **GFG**.

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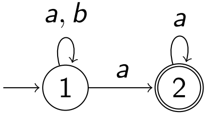

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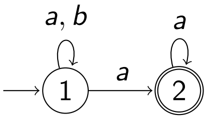

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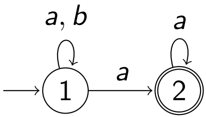

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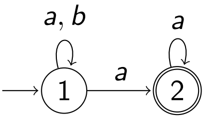
*There are some non-***GFG** *automata:* 

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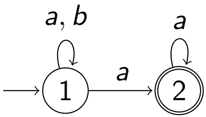

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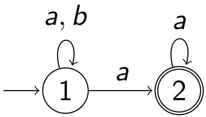

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$\rightarrow$ we can ignore the strategy and still use $\mathcal{A}$ as GFG in algorithms.

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Lemma: For Reachability/Safety conditions, **GFG** $\approx$ Det.

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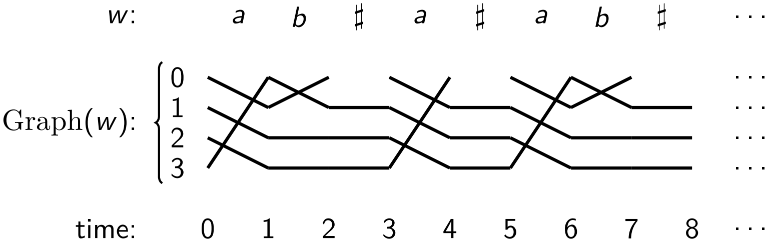

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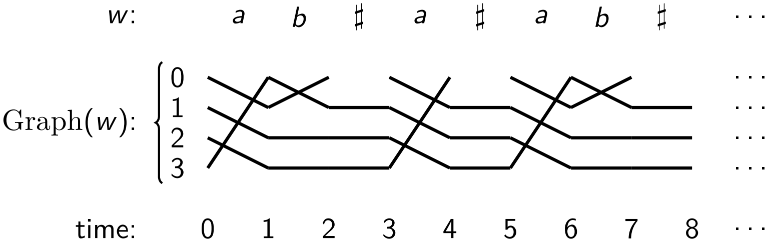

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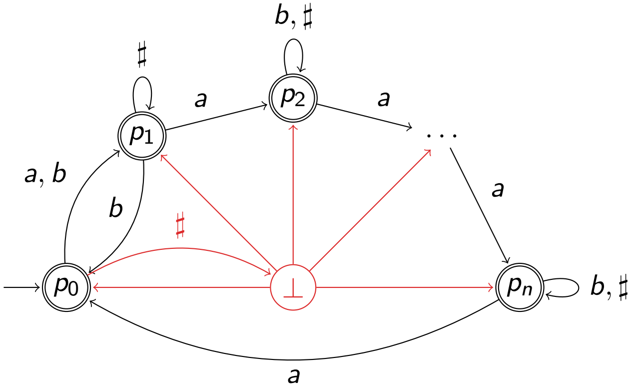
Any Det Parity automaton for L_n needs $\dfrac{2^{\frac{n}{2}}}{n+1}$ states !

A small GFG automaton for L_n

GFG coBüchi automaton:



Idea: Try paths one after the other.

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GFGness problem: input $\mathcal{A}_{\mathrm{ND}}$, is it **GFG**?

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Parity Games $\equiv$ GFGness for universal Parity automata.

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This talk:

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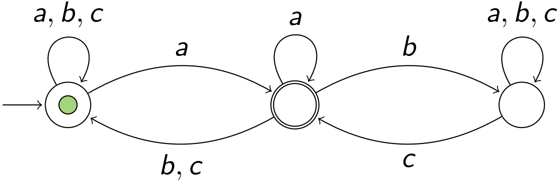
Recall: $\mathcal{A}$ is GFG $\Leftrightarrow$ Eve wins the GFG game on $\mathcal{A}$.



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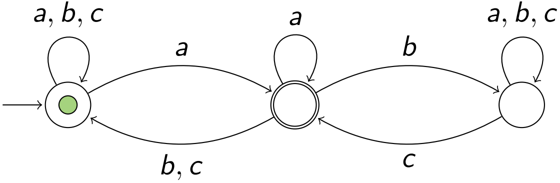

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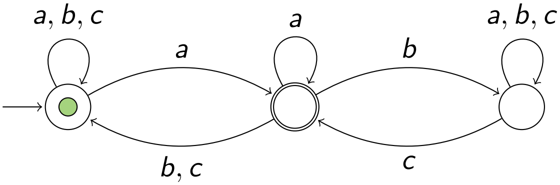

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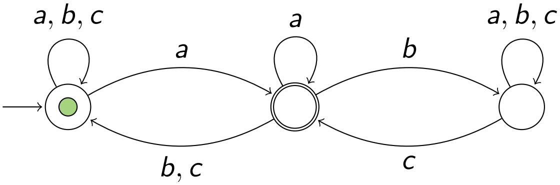

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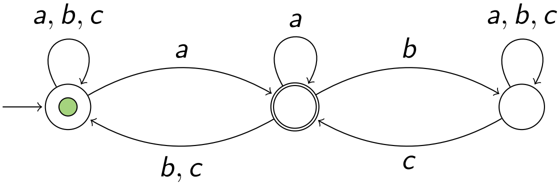

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Upper bound: EXPTIME (by computing $\mathcal{A}_{\mathrm{Det}}$)

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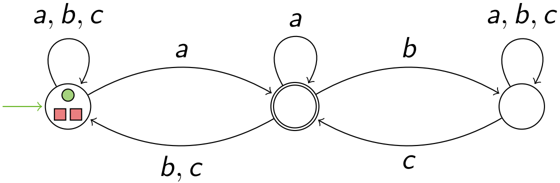
The game G_2:
Adam plays letters:

Eve: moves one token Adam: moves two tokens



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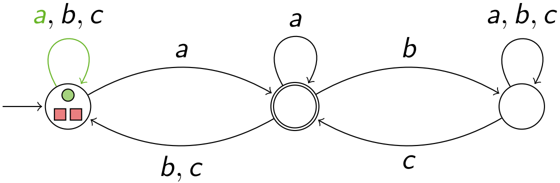

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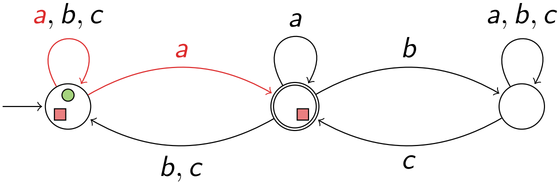

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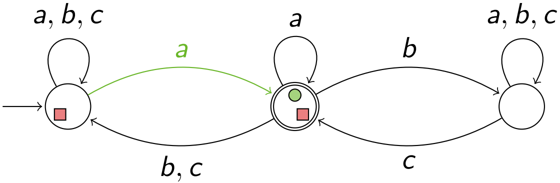

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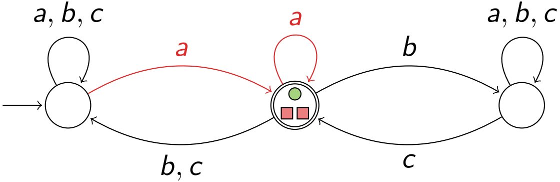

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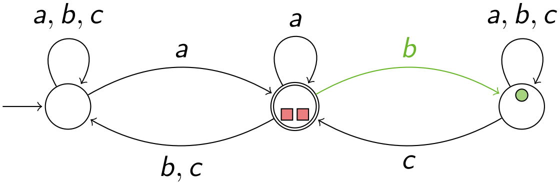

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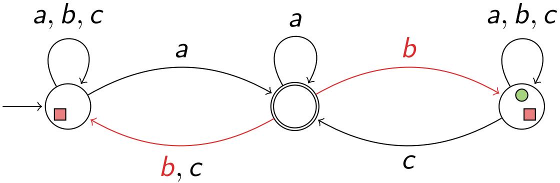

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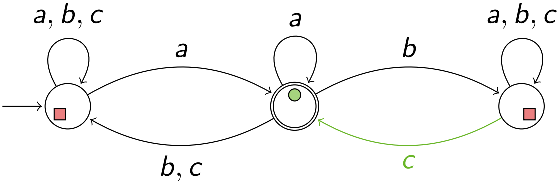

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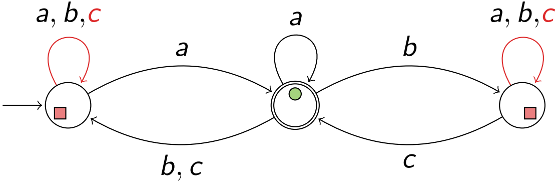

Abstracting the GFG game

The game G_2:

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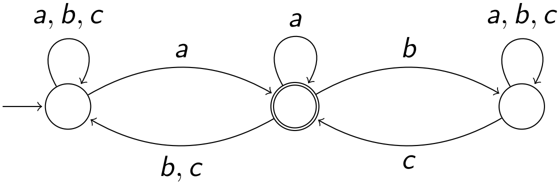

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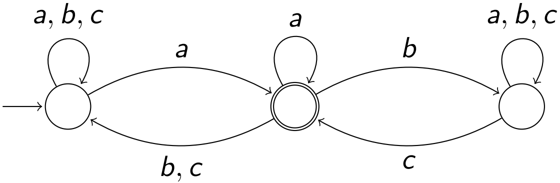


Eve wins if: ■$_1$ or ■$_2$ accepts $\Rightarrow$ ● accepts.

Goal: Show that Eve wins $G_2 \Leftrightarrow$ Eve wins the GFG game.

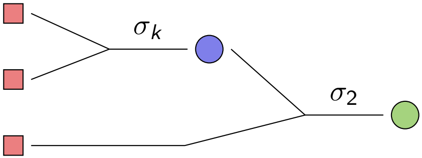
Proof sketch

Lemma

Eve wins $G_2 \Leftrightarrow$ Eve wins G_k for all k.

Proof sketch: $G_2 \Rightarrow G_3$

- play a *virtual token* ● against the first 2 ■
- play G_2 strategy against ● and last ■



Main proof sketch for $G_2 \Leftrightarrow \mathrm{GFG}$

Assume:

- Adam wins the GFG game with finite-memory strategy τ_{GFG}.
- Eve wins $G_2 \Rightarrow$ wins G_k with strategy σ_k, for a big k.

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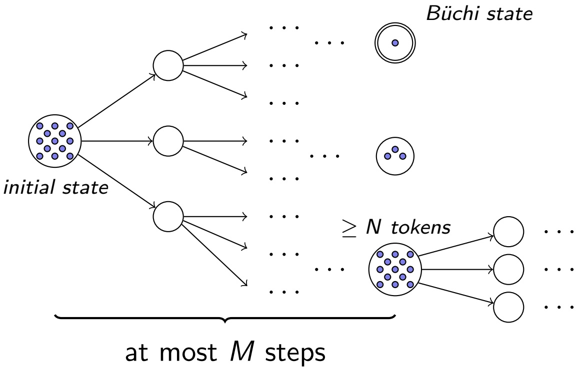
Build strategy for Eve against τ_{GFG}:

- move k virtual tokens ● against τ_{GFG}
- play σ_k against these k tokens



Building GFG automata

From a language L, how to obtain $\mathcal{A}_{\mathrm{GFG}}$ for L ?

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Incremental construction [K., Majumdar 2018]:
Partial k-Determinization of $\mathcal{A}_{\mathrm{ND}}$
Increase k until $\mathcal{A}$ is **GFG**.
Worst case: $k = |\mathcal{A}|$, reach the full determinization construction.

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Directly from LTL formula φ [Iosti, K. (unpublished)]
Adapted for Chruch Synthesis
For now: Fragment of coBüchi languages

Worst-case blowup result: GFG is doubly exponential in $|\varphi|$.

Heuristic and modular, can be combined with other approaches.

Conclusion

Results

- **GFG** automata are intermediate between **Det** and **ND**.
- They are useful to solve complex games.
- They can bring an exponential advantage compared to determinism.
- For Büchi and coBüchi conditions, they are recognized efficiently.

Perspectives

- $G_2 \Leftrightarrow$ GFG for Parity automata ?
- Complexity of the Parity GFGness Problem ?
- New ways to build **GFG** automata ?
- Generalization to other models (alternating, pushdown, ...) ?