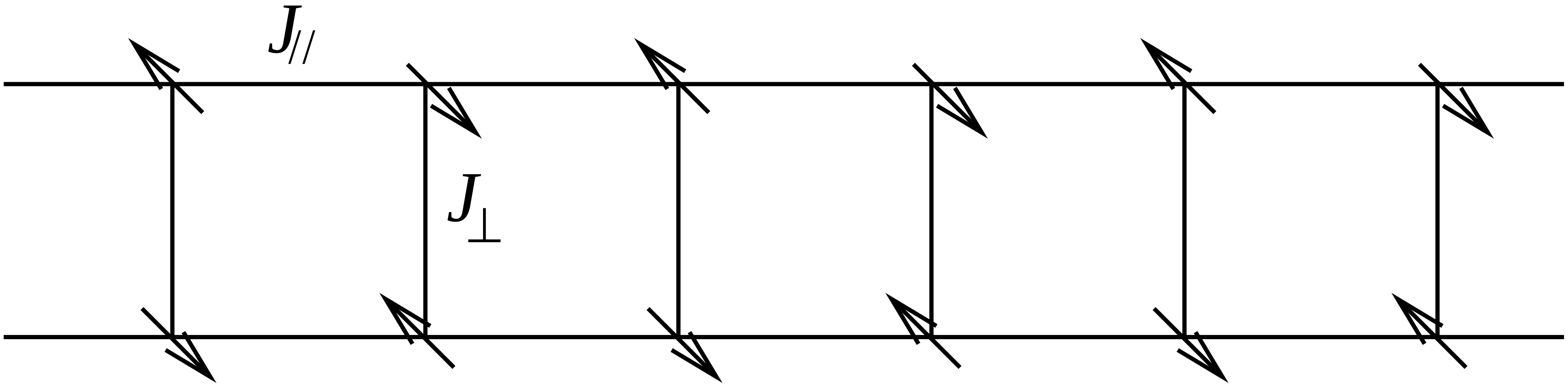
Two leg spin$-\frac{1}{2}$ ladder

$$H = J_{\|} \sum_{\substack{n \\ p=1,2}} (\mathbf{S}_{n,p} \cdot \mathbf{S}_{n+1,p} + J_{\perp} \sum_{n} \mathbf{S}_{n,1} \cdot \mathbf{S}_{n,2}$$

$$(J_{\|} > 0)$$



$\rightarrow$ Presence of a *gap* in spin excitations.

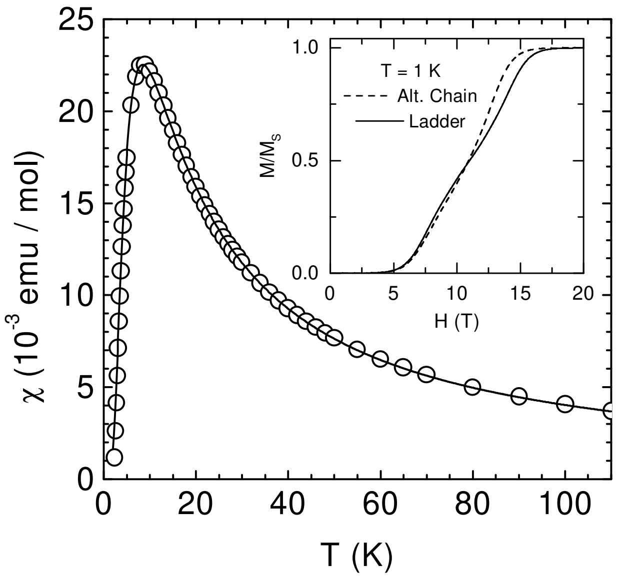
Contrast with the single chain which is *gapless* with a linear "magnon" spectrum $\omega = \frac{\pi}{2} J_{\|} |ka|$.

Relation with the Haldane conjecture (Phys. Rev. Lett. (1983)): *Antiferromagnetic spin chains with integer spins are gapped, but chains with half-odd integer spins are gapless.*

Examples of spin$-\frac{1}{2}$ ladder include:

- Sr_2CuO_3 ($J_{||}$ =1795K, $J_{\perp}$ =809K) [Azuma et al. Phys. Rev. Lett. **73** 3463 (1994)]
- $Sr_{14}Cu_{24}O_{41}$ ($J_{||} \simeq$1800K,$J_{\perp} \simeq$900K) [Eccleston et al. Phys. Rev. Lett. **81** 1702 (1998)]
- CaV_2O_5 ($J_{||}$ =122K, $J_{\perp}$ =608K)[Isobe et al. J. Phys. Soc. Jpn. **67** 755 (1998)]
- $[(DT-TTF)_2Au(mnt)_2]$ ($J_{||}$ =83K, $J_{\perp}$ =142K)[Rovira et al. Angew. Chem. Int. Engl. Ed. **36** 2323 (1997)]
- $(5IAP)_2CuBr_4$ ($J_{||}$ =3.8K, $J_{\perp}$ =12.9K)[C. P. Landee et al. Phys. Rev. B **63**, 100402(R) (2001)]
- $(C_5H_{12}N)_2CuBr_4$ ($J_{||}$ =0.75K, $J_{\perp}$ =13.3K)[Watson et al. Phys. Rev. Lett. **86** 5168 (2001)]

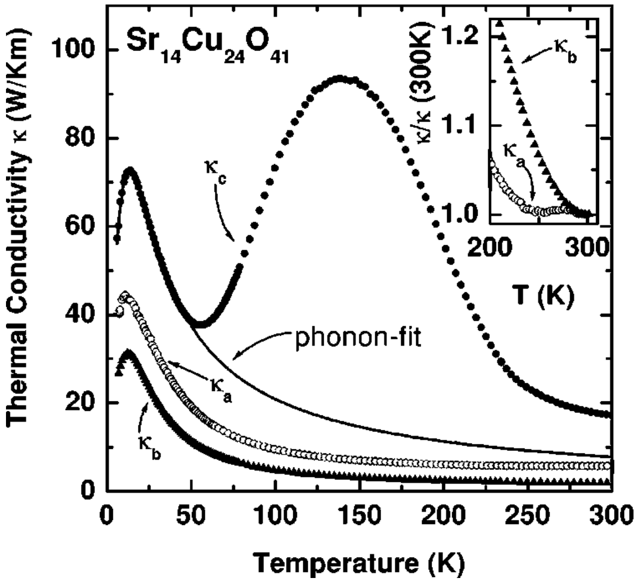
Signature of the spin gap in magnetic susceptibility



From Watson et al. Phys. Rev. Lett. **86** 5168 (2001)

$(C_5H_{12}N)_2CuBr_4$

Thermal transport measurements



C. Hess et al. Phys. Rev. B **64** 184305 (2001)

Very high contribution to thermal conductivity from spin excitations.

Effective field theory of the two-leg ladder

Valid in the limit $J_\perp, T \ll J_\|$.

For $J_\perp = 0$: two decoupled antiferromagnetic spin 1/2 chains

Low energy field theory given by bosonization: (see e. g. H. J. Schulz Phys. Rev. B **34** 6372 (1986))

$$H = H_1 + H_2$$

$$H_p = \int \frac{dx}{2\pi} \left[u(\pi \Pi_p)^2 + u(\partial_x \phi_p) \right]$$

$$[\phi_p(x), \Pi_{p'}(x')] = i\delta(x - x')\delta_{pp'}$$

$$u = \frac{\pi}{2} J_\| a$$

$$a = \text{Lattice spacing.}$$

$$S_p^+(x = na) = (S_{n,p}^x + iS_{n,p}^y)/a = \frac{e^{i\sqrt{2}\theta_a(x)}}{\pi a} \left[(-)^{x/a} + \cos\sqrt{2}\phi_p(x) \right]$$

$$S_p^z(x = na) = S_{n,p}^z/a = -\frac{1}{\pi\sqrt{2}} \partial_x \phi_p + (-)^{x/a} \frac{\cos\sqrt{2}\phi_p}{\pi a}$$

$$\theta_p(x) = \pi \int^x \Pi_p(x') dx'$$

interladder interaction

$H_{inter.}^{q=\pi} = \int dx \frac{J_\perp}{2\pi^2 a} \left[2\cos\sqrt{2}(\theta_1 - \theta_2) + \cos\sqrt{2}\phi_1 \cos\sqrt{2}\phi_2\right]$
scaling dimension =1 $\Rightarrow$ relevant perturbation $\Rightarrow$ spin gap. [Strong, Millis Phys. Rev. Lett. **69** 2419 (1992)/ H. J. Schulz (1986)]

there is also a marginal term:

$H_{inter.}^{marginal} = J_\perp \int dx \cos\sqrt{2}(\theta_1 - \theta_2)\cos\sqrt{2}\phi_1 \cos\sqrt{2}\phi_2 + \partial_x\phi_1\partial_x\phi_2$
scaling dimension = 2 $\Rightarrow$ less relevant to the formation of the spin gap.

Fermionization

[Shelton, Nersesyan,Tsvelik Phys. Rev. B **53** 8521 (1996)]

Introduce $\phi_\pm = \frac{1}{\sqrt{2}}(\phi_1 \pm \phi_2)$ and $\Pi_\pm = \frac{1}{\sqrt{2}}(\phi_1 \pm \phi_2)$

$\Rightarrow$ Rewrite Hamiltonian:

$$H = H_+ + H_-$$

$$H_+ = u \int \frac{dx}{2\pi} \left[(\pi \Pi_+)^2 + (\partial_x \phi_+)^2 \right] + J_\perp \int \cos 2\phi_+$$

$$H_- = u \int \frac{dx}{2\pi} \left[(\pi \Pi_-)^2 + (\partial_x \phi_-)^2 \right] + J_\perp \int \cos 2\phi_- + 2J_\perp \cos 2\theta_-$$

Introduce Majorana fermions

$$e^{i(\theta_+ - r\phi_+)} = \xi_r^1 + i\xi_r^2$$

$$e^{i(\theta_- - r\phi_-)} = \xi_r^3 + i\xi_r^0$$

fermionized Hamiltonian

$$H = H_t + H_s$$

$$H_t = -i\frac{u}{2}\int dx \sum_r \vec{\xi}_r \cdot \partial_x \vec{\xi}_r + m_t \int dx \vec{\xi}_+ \cdot \vec{\xi}_-$$

$$H_t = -i\frac{u}{2}\int dx \sum_r \xi_r^0 \cdot \partial_x \xi_r^0 + m_t \int dx \xi_+^0 \cdot \xi_-^0$$

$$m_t = \frac{J_\perp}{2\pi}, m_s = -3m_t$$

$$\vec{xi} = (\xi^2, \xi^2, \xi^3)$$

Massive Majorana fermions $E_t(k) = \pm\sqrt{(uk)^2 + m_t^2} \Rightarrow$ Spin gap.

similar description (without H_s) applicable to spin-1 chains.

Linear response definition of thermal current

$$H_t = \int dx \mathcal{H}(x)$$

Energy conservation:

$$\partial_t \mathcal{H} + \partial_x j_E(x) = 0$$

Total current:

$$J_E = \int dx j_E(x).$$

Thermal conductivity:

$$\kappa(T,\omega) = \frac{1}{2LT^2} \int \langle \{J_E(t), J_E(0)\}_+ \rangle e^{i\omega t} dt$$

Within bosonization

$$J_E = u^2 \int dx (\Pi_+ \partial_x \phi_+ + \Pi_- \partial_x \phi_-)(x)$$

J_E is proportional to the total momentum operator P

$P =$ infinitesimal generator of the translations

Translational invariance $\Rightarrow i\frac{dJ_E}{dt} = [J_E, H] = 0$

$$\Rightarrow \kappa(T,\omega) = \tilde{\kappa}(T)\delta(\omega)$$

where $\tilde{\kappa}(T) = \frac{\pi}{LT^2}\langle J_E^2 \rangle$

General argument applying also to all gapped systems described by the massive sine Gordon model such as:

- XXZ spin chain with $J_z > J$
- $J_1 - J_2$ frustrated spin chain
- explicitly dimerized spin chain

With Majorana fermions

$$J_E = -i\frac{u^2}{4}\sum_{a=0}^{4}\sum_{r=\pm}\left[\xi_r^a \partial_x \xi_r^a - (\partial_x \xi_r^a)\xi_r^a\right]$$

J_E identifies again with the total momentum operator

Free fermions description allows an exact calculation.

$$\tilde{\kappa}(T) = \frac{3}{8T^2}\int dk \frac{u^4 k^2}{\cosh^2 \frac{\sqrt{(uk)^2+m_t^2}}{2T}} + \frac{1}{8T^2}\int dk \frac{u^4 k^2}{\cosh^2 \frac{\sqrt{(uk)^2+(3m_t)^2}}{2T}}$$

At low temperature $T \ll J_\perp$:

$$\tilde{\kappa}(T) \sim 3u\sqrt{\frac{\pi m_t^2}{2T}} e^{-m_t/T}$$

At “high” temperature $J_\perp \ll T \ll J_\parallel$:

$$\tilde{\kappa}(T) = 2\frac{\pi^2 uT}{3}$$

Classical interpretation

Massive relativistic quasiparticles with mass m,

$$\epsilon(k) = \sqrt{(uk)^2 + m^2}.$$

Energy current: $\epsilon(k)\frac{\partial\epsilon}{\partial k} = u^2 k$

At low temperature, low density, statistics does not matter $\Rightarrow$ Maxwell Boltzmann.

Particles are collisionless so J_E is time independent.

$$\tilde{\kappa}(T) \sim \frac{1}{T^2}\int dk k^2 e^{-\epsilon(k)/T} \sim \frac{1}{\sqrt{T}} e^{-m/T}$$

This result should be valid not only for the ladder or the spin-1 chain, but for all systems described by the massive sine-Gordon model.

Anomalous transport by *solitons* as in some classical models [Livi et al. Phys. Rep. (2002)].

Comparison with exact results

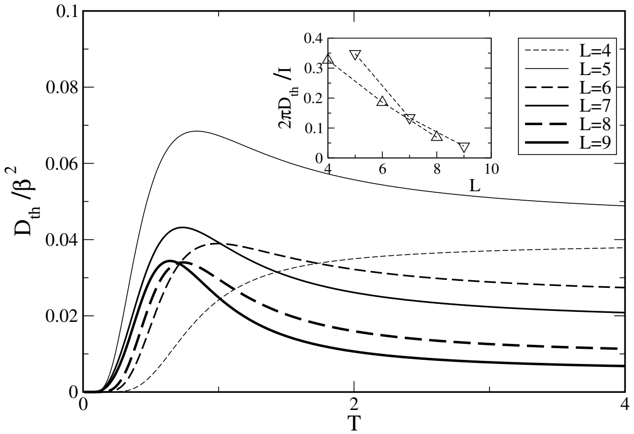
K. Sakai and A. Klümper have studied the XXZ chain in the regime:

•$J = J_z$ gapless and found $\tilde{\kappa}(T) = \frac{\pi^2 u T}{3}$ (J. Phys. A **35** 2173 (2002))

•$J_z > J$ $\tilde{\kappa}(T) \sim \frac{1}{\sqrt{T}} e^{-m/T}$ with m the spinon mass (cond-mat/0307227)

In agreement with the classical argument.

Comparison with numerical simulations



From X. Zotos `cond-mat/0304629`

see also F. Heidrich-Meisner et al. `cond-mat/0304595`, Phys. Rev. B **66**, 140406(R) (2002)

$D_{th} = \tilde{\kappa}, L =$ length of ladders

Drude weight goes to zero in the thermodynamic limit.

Explanation for discrepancy:

Marginal interactions ?

$\rightarrow$ Do not modify the thermal current

$\rightarrow$ Commute with the momentum operator

$\Rightarrow$ only renormalize $\tilde{\kappa}(T)$ $\Rightarrow$ NO! Irrelevant band curvature terms ? $\propto (\partial_x \phi)^3 + \ldots$

$rightarrow$ Modify thermal current J_E= Momentum operator $+ \ldots$

$\rightarrow$ Still commute with the momentum operator

$\Rightarrow$ preserve the part of the thermal current proportional to momentum operator $\Rightarrow$ NO ! [Can be shown from the Mazur inequality; Zotos, Naef and Prelovsek Phys. Rev. B **55**, 11029 (1997); Kawakami and Fujimoto Phys. Rev. Lett. 90, 197202 (2003)]

On the lattice

Translations do not form a continuous group $\Rightarrow$ Nonexistence of a conserved momentum operator

Integrable models such as the XXZ model have an infinity of conserved currents $\Rightarrow$ Mazur's inequality implies a nonzero Drude weight $\Rightarrow$ results for these models are not generic.

Field theories lead also to non-generic predictions as they artificially add a conserved momentum operator.

Umklapp processes ?

Rosch and Andrei Phys. Rev. Lett. (2000)

In the context of the 1D extended Hubbard model

To escape the conservation of momentum current, introduce an incommensurate Umklapp process to destroy translational invariance: $g_3 \cos(\sqrt{8}\phi_\rho + \delta x)$

Actually insufficient since $\Pi_\rho \partial_x \phi_\rho \to \Pi_\rho(\partial_x \phi_\rho + \delta/\sqrt{8})$ still gives a conserved current.

$\to$ Fabricate a second Umklapp incommensurate with the first one coming from $\cos(\sqrt{2}\phi_\rho + \delta x/2)\partial_x \phi_\rho$.

$\to$ However, this would also violate momentum conservation in the case of the *integrable* Hubbard model with only on-site interaction.

$\Rightarrow$ Does not solve the discrepancy !

Conclusions

→ Transition between diffusion and ballistic transport at a temperature T^* ($T^* = \infty$ in the integrable models) ? (Proposed by Zotos but unlikely) → Presence of anomalous transport such as superdiffusion $\langle x^2 \rangle \sim t^\gamma$ with $1 < \gamma < 2$?
Exists in classical models whose continuous counterpart has solitons, such as Fermi-Pasta-Ulam chains which map on KdV equations in the continuum but not in sine-Gordon chains
→ Divergent diffusion coefficient for $T \to 0$? Quasi-ballistic transport, experimental diffusion coefficient determined by defects, interladder interactions, interactions with phonons.