CR 13: Graph Decompositions
Homework 2

Date: 28/10/2021, to hand in by 15/11/2021
Total Marks: 20

- You can handwrite or type your composition.
- You can use (without a proof) all theorems stated in class.
- For each question, you may of course consider the statements of the previous questions as true even if you could not prove them.
- We encourage you to draw figures whenever you feel that they will be useful for your reader.

1 Twin-width of some interval graphs (14 marks)

For every positive integer n, the graph $U_{n,n}$ has n^2 vertices, vertex set, say, $\{v_{i,j} \mid 1 \leqslant i, j \leqslant n\}$, and edge set $\{v_{i,j}v_{i,j'} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \neq j' \leqslant n\} \cup \{v_{i,j}v_{i+1,j'} \mid 1 \leqslant i \leqslant n-1,\ 1 \leqslant j' \leqslant j \leqslant n\}$.

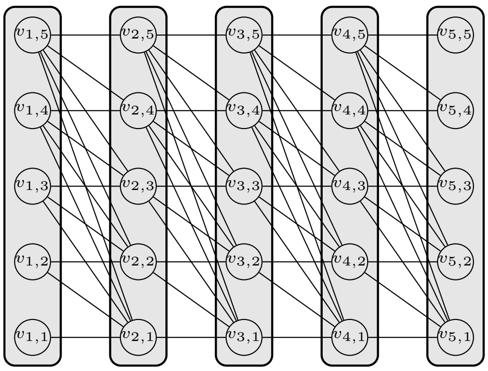

Figure 1: The graph $U_{5,5}$. Within each shaded area, all the vertices are pairwise adjacent. These cliques are not represented with actual edges, for the sake of legibility.

Q.1) Show that for every positive integer n, the graph $U_{n,n}$ is a unit interval graph. 1 mark

Q.2) Show that the clique, independent set, and path on n vertices are each an induced subgraph of $U_{n,n}$. 2 marks

Q.3) Prove that every unit interval graph on n vertices is an induced subgraph of $U_{n,n}$.

Hint: observe that the three graphs of the previous question even appear when selecting exactly one vertex per "row" of Figure 1. Try and generalize this fact to any unit interval graph H; you can start by partitioning $V(H)$ into cliques. 2 marks

Q.4) Use the previous question to show that unit interval graphs have twin-width at most 2. 3 marks

Q.5) Show that every unit interval graph on at most 5 vertices has twin-width at most 1.

Hint: recall that cographs (which are exactly the graphs avoiding a four-vertex path as an induced subgraph) have twin-width 0, and that every interval graph is chordal. 2 marks

Q.6) Show that interval graphs representable with intervals of two distinct lengths have bounded twin-width.

Hint: think of the characterization of bounded twin-width via mixed minors. You may use with or without a proof the fact that if three intervals of length x with increasing left endpoints I_1, I_2, I_3 and three intervals of length y with increasing left endpoints J_1, J_2, J_3 are such that $I_1 \cap J_3 \neq \emptyset$ and $I_3 \cap J_1 \neq \emptyset$, then it holds that $I_2 \cap J_2 \neq \emptyset$. 4 marks

2 Twin-width of map graphs (6 marks)

A *map graph* is the intersection graph of simply connected closed regions (that is, homeomorphic to closed disks) of the Euclidean plane with pairwise disjoint interiors. The geometric realization of the vertices of a map graph are traditionally called countries, and by assumption here, have no holes. As countries have disjoint interiors, two countries intersect if and only if their boundaries do. Unlike planar graphs, map graphs may have arbitrarily large cliques.

For every positive integer n, let J_n be the graph on $2n + 3$ vertices obtained from a clique on $2n$ vertices partitioned into two vertex sets A, B both of size n, by adding three pairwise adjacent vertices a, b, c such that a is adjacent to every vertex of A but to no vertex of B, b is adjacent to every vertex of B but to no vertex of A, and c is adjacent to exactly one vertex in A and exactly one vertex in B.

Q.7) Show that for every positive integer n, J_n is a map graph. 1.5 marks

The next question is *not* connected to the first one.

Q.8) Show that map graphs have bounded twin-width.

Hint: use without a proof the fact that planar graphs have bounded twin-width. 4.5 marks