Homework 1

We consider *cops and robber* games and their relation to graph decompositions. This offers a new interpretation of width parameters, which can be useful to understand these invariants. In particular, they provide a dynamical viewpoint. Notably, decompositions correspond to the strategy of the cops, while the dual parameters (such as brambles) correspond to the strategy of the robber.

In what follows, we assume that there is a team of k cops $\mathcal{C} = \{c_1, \dots, c_k\}$ trying to catch a robber r in a graph G. The robber occupies a vertex of G. She is able to move *at any time* from her current vertex to another vertex following (at infinite speed) a path in the induced subgraph $G - \mathcal{C}$, i.e., she cannot go through a vertex occupied by a cop. Each cop moves by helicopter, and can either be in the sky (outside of the graph, and ready to be redeployed) or landed at a vertex of G.

A move for the cops team consists of a chosen (possibly empty) subset of those on the ground taking off, and a chosen (possibly empty) subset of those in the sky landing at chosen vertices. Since the robber can freely move, she is able to move just before the cops land, hence a cop cannot capture her by simply landing on her vertex, unless all neighbors of r are occupied. A move for the robber is to relocate at a vertex accessible via a path of unoccupied vertices. A winning strategy for the cops is one that forces that a cop lands at the vertex occupied by the robber while she cannot escape. A winning strategy for the robber is one that allows her to avoid being caught indefinitely (and exists if and only if there is no winning strategy for the cops). The *cop number* of G is the minimum number of cops for which the cops team has a winning strategy. We consider two other variants of the cop number:

- Lazy cop number: Once a cop lands on a vertex of G, he no longer moves.
- Invisible cop number: The robber is invisible. The cops cannot use the position of the robber to plan their strategy. Equivalently, the robber has access to the strategy of the cops to decide hers.

Example: on a 4-vertex path, the lazy cop number is 3, while the cop number and invisible cop number are equal to 2. On a tree with a single vertex of degree 3 made by attaching at the middle of a long path another long path, the cop number is 2, and the invisible cop number is 3.

- Exercise 1 - Compute the cop number, the lazy cop number, and the invisible cop number, on these graphs:

- the cliques and complete bipartite graphs;
- the path of length k;
- the complete binary tree with depth k (on $2^k - 1$ vertices);
- the 3×3 grid.

- Exercise 2 - Show that the lazy cop number, invisible cop number, and cop number of a graph G can be upperbounded by a function of the treedepth, the pathwidth, and the treewidth of G, respectively.

- Exercise 3 - Show how a bramble gives a strategy for the robber in the case of the (normal) cop number. Deduce that your upper bound of the cop number in the previous question is tight.

The last question is not directly related (do not use cops and robber game). However, in light of the previous question, it shows that a winning strategy for the robber in the normal game has a short (yet non constructive) description.

- Exercise 4 - Show that if the bramble number is k, there is one bramble of order k and size $f(k)$. Hint: use the Graph Minor theorem.