# Homework 2

We work with simple (no loops, no parallel edges) undirected graphs. Given a graph $G$ and two vertices $u, v \in V(G)$, the *shortest-path distance* between $u$ and $v$, denoted by $d_G(u,v)$, is the number of edges in a shortest path between $u$ and $v$, or $\infty$ if $u$ and $v$ are in distinct connected components.

Given a graph $G$ and a set $S = \{v_1, v_2, \dots, v_h\} \subseteq V(G)$, the *distance vector* $d_S(v)$ of $v \in V(G)$ is the vector of $(\mathbb{N} \cup \infty)^h$ whose $i$-th entry is $d_G(v, v_i)$. A *resolving set* of $G$ is a subset $S \subseteq V(G)$ such that every vertex $v$ of $G$ has a distinct distance vector $d_S(v)$. Note that distance vectors are *vectors* not multisets, assuming an arbitrary but fixed ordering of $S$. So, for instance, $(1,3,1,4)$ and $(1,4,3,1)$ are *distinct*, since the distance to the second vertex of $S$ is 3 in the former case, and 4 in the latter. The *metric dimension* $\mathrm{md}(G)$ of a graph $G$ is the size of a smallest resolving set of $G$.

We may say that a vertex $v$ of $G$ is *resolved* by $S \subseteq V(G)$ (or that $S$ *resolves* $v$) if for every $w \in V(G) \setminus \{v\}$, $d_S(v) \neq d_S(w)$. A resolving set is then a set resolving every vertex. Observe that $S = \{v_1, v_2, \dots\}$ resolves every vertex in $S$, since the distance vector of no other vertex than $v_i \in S$ has a 0 entry at position $i$.

**- Exercise 1 -** Give a family of connected graphs with bounded treewidth and unbounded metric dimension. Briefly justify.

**- Exercise 2 -** Give a family of graphs with unbounded treewidth and bounded metric dimension. Briefly justify.

**- Exercise 3 -** Characterize the graphs with metric dimension equal to 1.

**- Exercise 4 -** Deduce that *having metric dimension 1* is $\mathrm{MSO}_1$-definable among graphs, and FO-definable among connected graphs.

A pair of *twins* in a graph $G$ is a pair of vertices $u, v \in V(G)$ (adjacent or not) whose neighborhoods outside $\{u, v\}$ are equal.

**- Exercise 5 -** Show that for every resolving set $S \subseteq V(G)$ of a graph $G$, and pair of twins $u, v \in V(G)$, $S \cap \{u, v\} \neq \emptyset$.

**- Exercise 6 -** Let $G$ be the cycle on $2n$ vertices, for some integer $n \geqslant 2$. Characterize the pair of vertices of $G$ forming a resolving set.

**- Exercise 7 -** Use the two previous questions to show that *having metric dimension 2* is not $\mathrm{MSO}_1$-definable.