Lecture 10. Parameterized Hardness.

Polynomial Karp reductions: (produce only instance) $\rho : x \in \Sigma^* \to \rho(x) \in \Sigma^*$
from Π_1 to Π_2 ($\Pi_1, \Pi_2 \subseteq \Sigma^*$)

1) ρ polytime computable
2) $x \in \Pi_1 \iff \rho(x) \in \Pi_2$

- Denoted $\Pi_1 \xrightarrow[p]{} \Pi_2$ ($\Pi_1 \leq_p \Pi_2$)

FPT Karp reductions: algorithm $\rho : (x,k) \in \Sigma^* \times \mathbb{N} \to (x',k') \in \Sigma^* \times \mathbb{N}$
from Π_1 to Π_2

1) ρ is computable in $f(k)|x|^c$ (f is computable function, c is a constant)
2) $(x,k) \in \Pi_1 \iff (x',k') \in \Pi_2$
3) $k' \leq g(k)$ for some computable function g.

- Denoted $\Pi_1 \xrightarrow[FPT]{} \Pi_2$

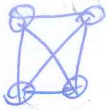
Prop: If $\Pi_1 \xrightarrow[FPT]{} \Pi_2$ and Π_2 is FPT, then Π_1 is FPT.

Pf: $(x,k) \xrightarrow{\text{run } \rho} (x',k') \xrightarrow{\text{run } A_{\Pi_2} \leftarrow h(k)|\cdot|^{c'}}$ answer to Π_1 because of 2)

$f(k)|x|^c$ time for ρ. We also get that $|x'| \leq f(k)|x|^c$

$$f(k)|x|^c + h(k')\left(f(k)|x|^c\right)^{c'} = f(k)|x|^c + \underbrace{h(g(k))f(k)^{c'}}|x|^{cc'} \quad \square$$

Q) k-Clique $\xrightarrow[\text{FPT}]{}$ k-Independent Set ?



$P : (G,k) \longrightarrow (\overline{G},k)$ complement of G $uv \in E(\overline{G}) \Leftrightarrow uv \notin E(G)$

Q) • k-Independent Set $\xrightarrow[\text{FPT}]{}$ k-Vertex Cover ? probably not

k-V.C: $\forall e, \exists u \in S$ such that u is an endpoint of e
$|S| = k$

IS $\rightarrow$ k
S $n-k=k'$
S is a VC of G then $V(G)\setminus S$
$n-k=k' \leq g(k)$

• k-Vertex Cover $\xrightarrow[\text{FPT}]{}$ k-Independent Set ?

$(G,k) \xrightarrow{P}$ solve it YES
NO

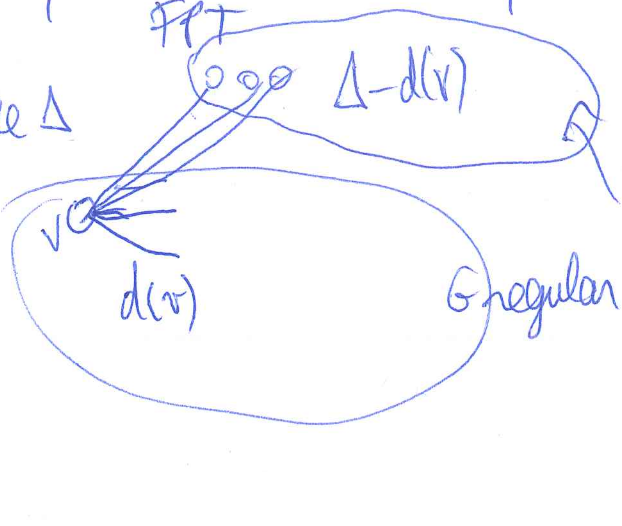
<u>Exercise</u> : $A \xrightarrow[\text{FPT}]{} B \wedge B \xrightarrow[\text{FPT}]{} C \Rightarrow A \xrightarrow[\text{FPT}]{} C$.

Classical world SAT
Param world k-Clique / k-Independent Set.

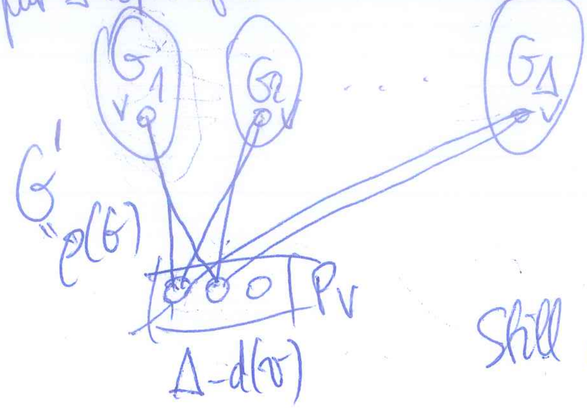
Q) k-Clique $\xrightarrow[\text{FPT}]{}$ k-Clique restricted to regular graphs

$\exists \Delta, \forall v \in V(G), d(v) = \Delta$

maximum degree Δ



put Δ copies of G



all copies of G

$\forall v \in V(G)$ introduce P_v of size $\Delta - d(v)$
link every copy of v to every vertex of P_v

Still asking for a clique of size k.

$\omega(G)$: max size of a clique in G

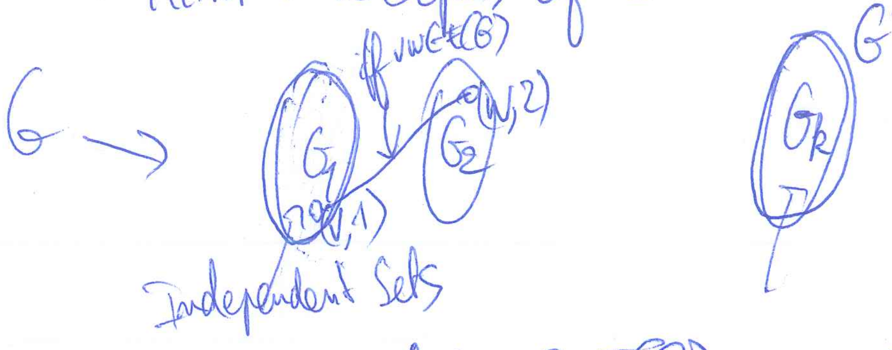
Correctness: $\omega(G) \geq k \iff \omega(G') \geq k$ $\quad (\omega(G) = \omega(G'))$

- $\omega(G') \geq \omega(G)$ because G is an induced subgraph of G'.
- assume $k \geq 3$: $\omega(G) \geq \omega(G')$ - no vertex of P_v in a triangle
  - no edge between the copies
  - $\hookrightarrow$ cliques of size ≥ 3 are within one copy.

<u>MultiColored</u> k-Clique: $G = (V_1 \uplus V_2 \uplus \dots \uplus V_k, E)$
Is there a k-Clique intersecting every V_i exactly once.

~~Multic~~ k-Clique $\xrightarrow[\text{FPT}]{\text{Regular}}$ MultiColored k-Clique? Regular

Hint: k "copies" of G.



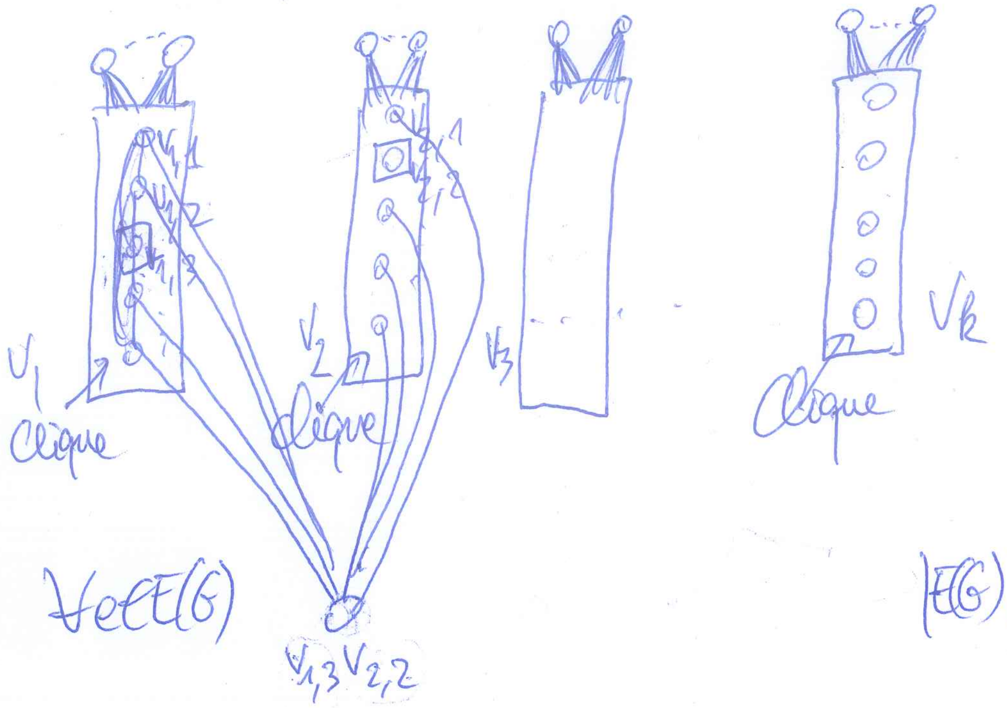
$\forall uv \in E(G)$, we add all edges between (u,i) and (v,j) with $i \neq j$. $i,j \in [k]$

- If S is a clique of size k in G
  $\{v_1, \dots v_k\} \longrightarrow$ take $(v_1,1), (v_2,2), \dots (v_k,k)$ clique in G'.
- If S is a multicolored k-clique in G' $(v_1,1), (v_2,2) \dots (v_k,k) \longrightarrow v_1, \dots v_k$ in G

Multicolored k-Ind Set $\xrightarrow{FPT}$ k-Dominating Set
~~Clique~~

- k vertices S such that $N[S] = V(G)$

$(G = V_1 \uplus \ldots \uplus V_k, E)$



$|E(G)|$ such vertices.

$v_{i,a}v_{j,b}$ is linked to all vertices of $V_i \cup V_j$
but $v_{i,a}$ and $v_{i,b}$

Correctness.

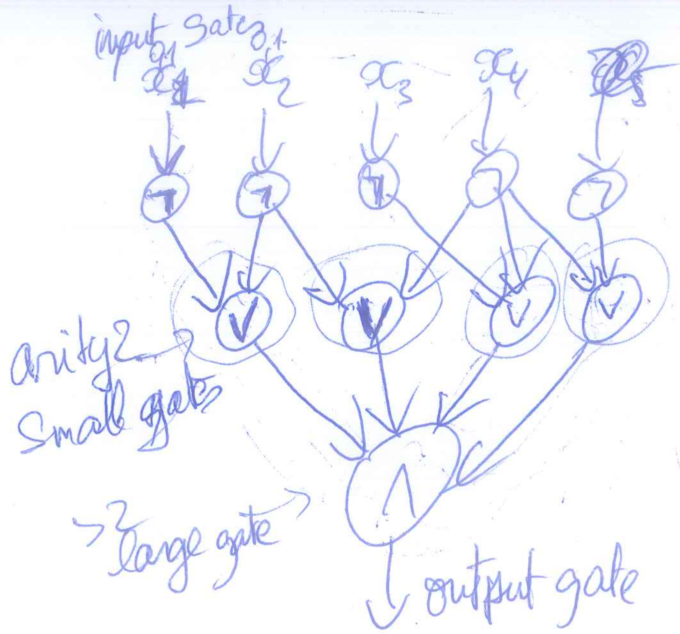
- If S is a multi. ind set in G
  $\shortparallel$
  $v_{1,a_1}, v_{2,a_2} \ldots v_{k,a_k}$
  also DS in G'.

- If S is a dominating set in G'
  $\hookrightarrow$ again S is a multi independent set
  $S \ddot{\iota} z e \, k$.

- k-Dominating Set $\xrightarrow{FPT}$ k-IS ?

- W_hierarchy

Booleán Circuits



Q) What is this circuit doing?

large gate: gate with >2 inputs

$\bigwedge \neg x_i \vee \neg x_j$

depth: max length of a directed path from an input to the output gate.

weft: max # of large gates on a directed path from an input gate to the output gate.

weaving

Weighted Boolean Circuit Satisfiability. (WCS)

Circuit C with n input gates.
Is there a weight-k input so that 1 is output by C?
i.e, with at most k inputs 1

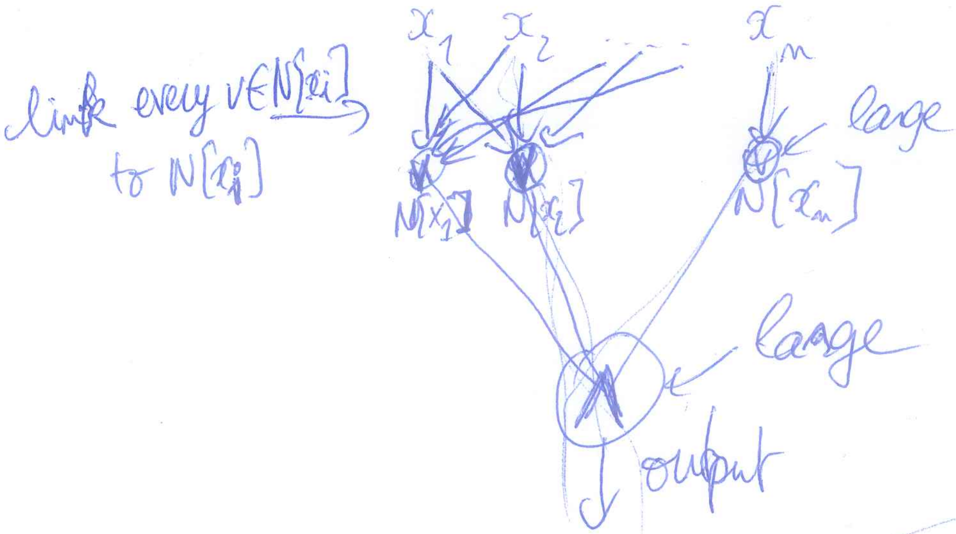
$WCS_{t,d}$ = restricted to circuits of weft t and depth d.

$W[t]$ = all the problems FPT reducible to $WCS_{t,d}$ for some constant d.

$\Pi \xrightarrow[FPT]{} WCS_{t,d}$

k-Ind. Set is in W[1]. It's actually W[1]-complete.

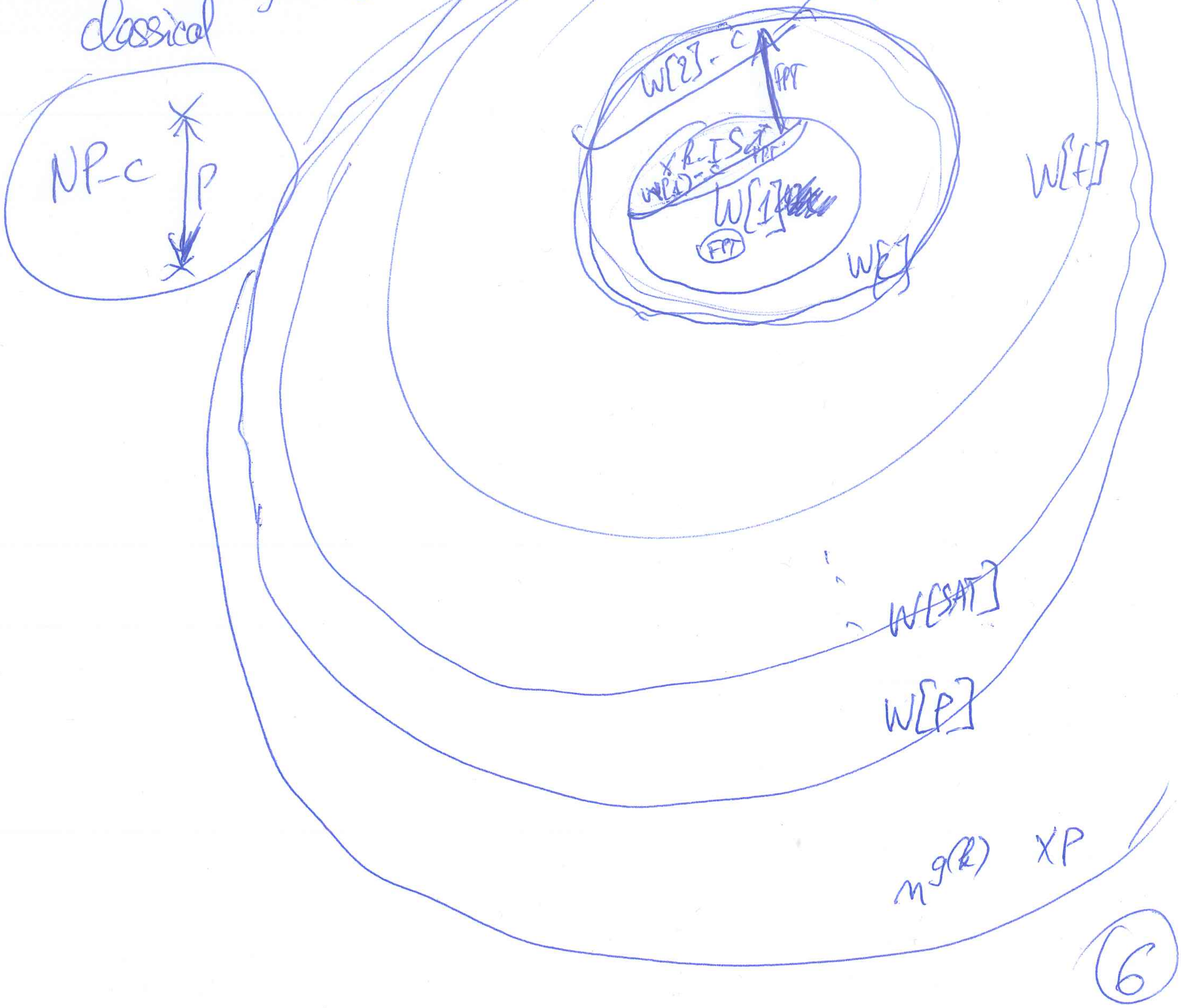
Q) What about k-Dominating set? Show that it's in W[2].



imp

$\forall v \in V(G)$ $N[v] \cap S \neq \emptyset$ (S = DS)

k_DS is in W[2].
actually W[2]-c.



k-Set Cover: $S_1, \dots, S_m$ over universe U finite, size n, find k sets $S_{a_1}, \dots, S_{a_k}$ among $S_1, \dots, S_m$ such that $\bigcup_{i=1}^{k} S_{a_i} = U$

Q) k-Set Cover in W-hierarchy?

k-IS is W[1]-c
k-DS is W[2]-c

$$k\text{-DS} \overset{FPT}{\Longleftrightarrow} k\text{-Set Cover}$$

$V(G)$
$v_1, \dots, v_n \longrightarrow N[v_1], \dots, N[v_n], \quad U = V(G)$

$$\bigcup_{i=k}^{k} N[v_{a_i}] = V(G) \Longleftrightarrow N[\{v_{a_1}, \dots, v_{a_k}\}] = V(G)$$

More quantitative approach

Under FPT $\neq$ W[1], k-Clique is not FPT

cannot rule out $f(k)\, n^{\log\log\log k}$

Exponential Time Hypothesis, Strong Exponential-Time Hypothesis

ETH, SETH.

best algorithm for k-SAT $\rightarrow \delta_k = \inf\{\delta \mid k\text{-SAT is solvable in } O(2^{\delta n})\}$

ETH: $\delta_3 > 0$. widely believed

SETH: $\lim_{k \to +\infty} \delta_k = 1$ much more debatable $2^{n/2}$ with quantum computers.

3-SAT: $1.3\ldots^n$ exists.

k-SAT: $2^{(1-\frac{c}{k})n}$

SETH $\Rightarrow$ ETH $\Rightarrow$ FPT $\neq$ W[1] use ETH to derive something for Independent Set. (MIS)

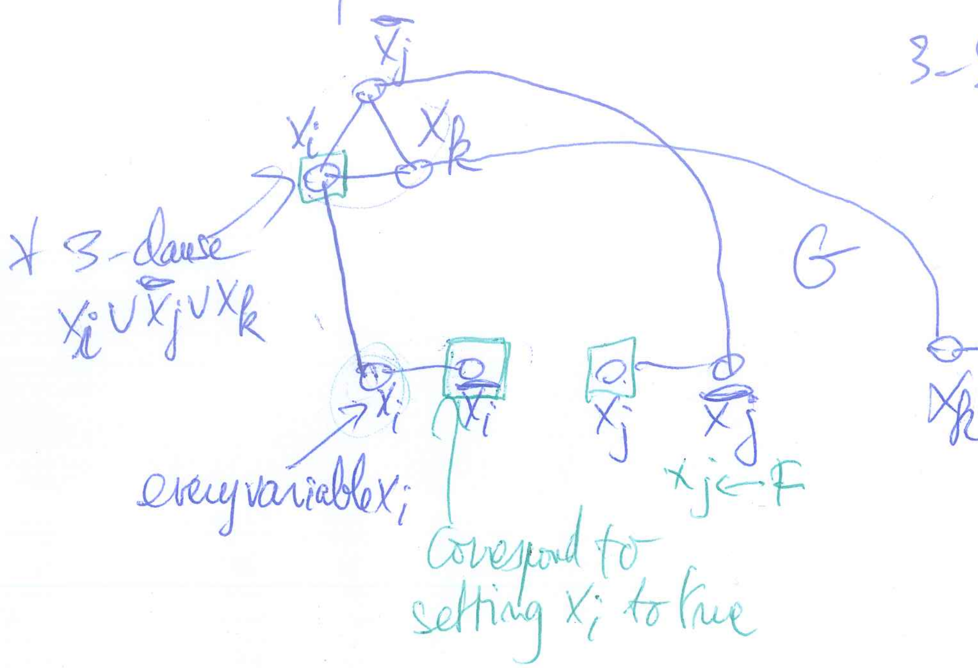
3_SAT cannot be solved better than $2^{\delta_3 n}$ $\delta_3 > 0$

↪ no $2^{o(n)}$

n: # variables m: # clauses

N, M

φ 3_SAT $\xrightarrow{P}$ MIS



3_SAT is satisfiable $\iff \exists$ ind. set of size N+M in G

φ has n variables, m clauses $O(n^3)$

$$2n + 3m = O(n^3) = N$$

var.

3_SAT has no $2^{o(n)}$_time algorithm

↪ MIS has no $2^{o(N^{1/3})}$_time alg.

Maybe there is no $\boxed{2^{o(N)}}$?

$f(k)\, n^{O(1)}$