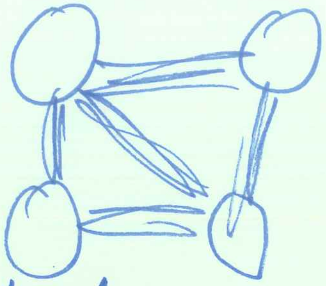
CROSS-COMPOSITIONS

$(x_1, x_2, \dots, x_t) \longmapsto (y, k)$

(some NP-h problem) equivalent wrt some polynomial equ. rela.

* runs P-time : $|y| = \text{poly}\left(\sum^{t} |x_i|\right)$
* $k \leq \text{poly}\left(\max_i |x_i| + \log t\right)$
* (y, k) positive $\iff$ x_i positive for at least one $i \in [t]$



natural ideas :

1) disjoint union ?
   complete join ?

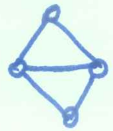
2) NP-h reductions? → adapt

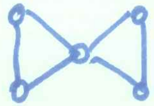
3) cross-compose from the same problem you want kernel lower bound for

Non-trivial examples

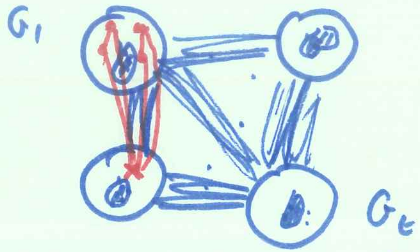
Maximum Independent Set in H-free graphs

 → $O(k^2)$ vertices

 → $O(k^3)$ vertices

 butterfly → ?

NO PK for MIS in butterfly-free graphs (unless ...)



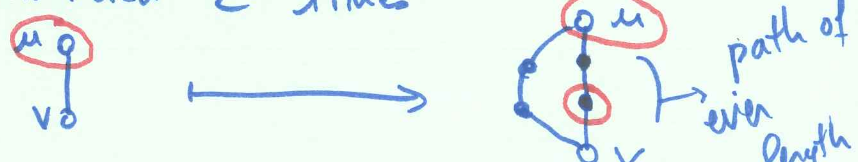
~~disjoint un..~~

~~complete join~~ has butterflies

NP-hardness for MIS in H-free graphs

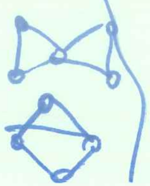
G (any graph) $\longmapsto \ddot{G}$: every edge is subdivided 2 times

size of a MIS ↑ $\alpha(G)$



$\alpha(\ddot{G}) = \alpha(G) + |E(G)|$

$\ddot{G}$ is △-free



=> MIS remains NP-h in ~~any~~ H-free graphs ∀ H different from:

→ path

→ subdivision of star

MIS has no PK in ⋈-free graphs

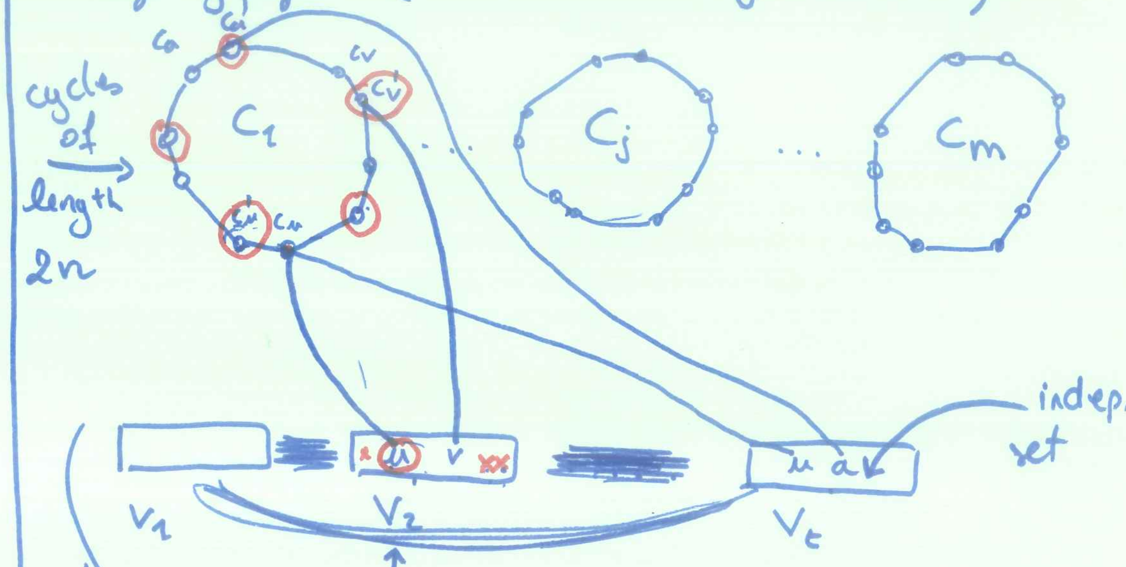
$G_1, \dots, G_t$; $k_1, \dots, k_t$ } → arbitrary graphs.

Polynomial equivalence relation:

* $|V(G_i)| = |V(G_j)| =: n$
* $|E(G_i)| = |E(G_j)| =: m$ $\quad \forall\, i,j \in [t]$.
* $k_i = k_j =: k$

Cross-composition from MIS in general graphs to MIS(k) in ⋈-free graphs

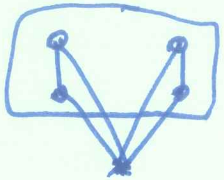
edge gadgets (same for every instance)



set of vertices of G_2

instance selector + vertex selector

$2K_2$



G^* : resulting graph

k^* : $k + m \cdot n = \mathrm{poly}(|G_i| + k_i)$

* G^* is butterfly-free
* G^* has an indep. set of size $\geq k^*$

iff $\exists\, i \in [t]$: G_i has an indep. set $\geq k$

Second example

Disjoint Factors

input: string s over an alphabet $A=\{1,\dots,k\}$ ~~$|A|\geq 2$~~

question: does s admit ~~an~~ a ~~k-subset~~ k-disjoint factor decomposition

→ $s_1,\dots,s_k$: s_i substring of s

s_i doesn't overlap with s_j $\forall i\neq j$

$|s_i|\geq 2$

s_i starts and ends with "i"

$s = 2\,\boxed{3\,2\,3}\,2\,1\,\boxed{1\,2\,1}\,3\,\boxed{2\,1\,2}$ $\quad A=\{1,2,3\}$

$\boxed{11}$ $\quad\boxed{2112}\quad$ $11\,\boxed{2112}$

- Disjoint Factors is FPT (k)
  - → enumerate order of factors $O^*(k!)$
  - → dynamic prog. $= O^*(2^k)$

- no PK unless $NP \subseteq coNP/poly$.

compose 2 instances over same alphabet $\{1,\dots,k\}$

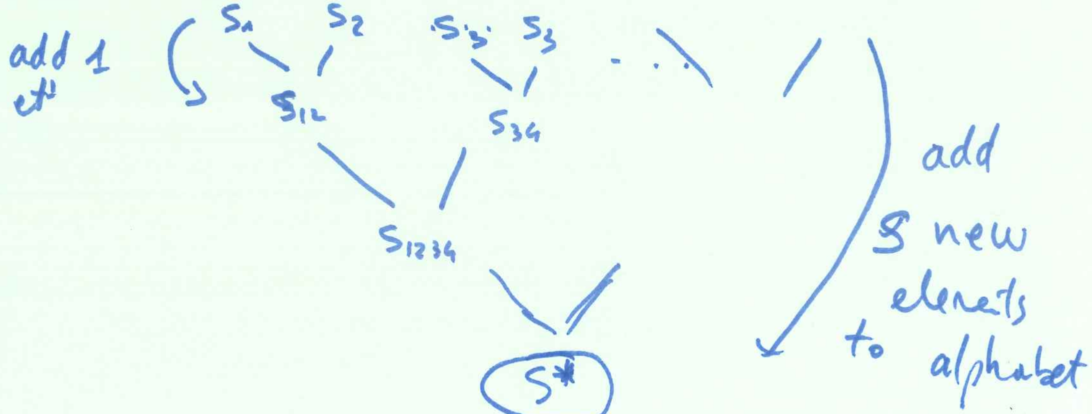
s_1 $\quad s_2$

k' depend on $\log t$

$(k+1)\,\boxed{\quad s_1 \quad}\,(k+1)\,\boxed{\quad s_2 \quad}\,(k+1) =: s^*$

s^* has $(k+1)$-disjoint factor dec. iff s_1 or s_2 has k-disj... dec.

$s_1 \dots s_t$ $\qquad t = 2^s$ for some s



→ defined on alphabet of size $k + \log t$

Transferring kernel lower bounds

* for proving W[1]-h: PPT
(parameter preserving transformations)

$$(x,k) \xrightarrow{\;Q_1 \longrightarrow Q_2\;} (x',k') \quad \text{runs in FPT time}$$

$$k' \le f(k)$$

Q_2 is FPT $\Rightarrow$ Q_1 is FPT

* polynomial PPT:
  * run in polynomial-time
  * f has to be a polynomial.

- $Q_1 \overset{\text{polynomial PPT}}{\rightsquigarrow} Q_2$
- Q_2 has a PK $\Rightarrow$ Q_1 has a PK.
  ($Q_1, Q_2 \in NP$)

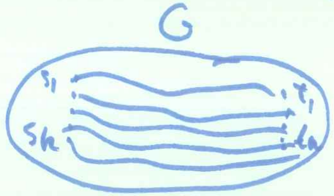
Disjoint Path

input: graph G, $k \in \mathbb{N}$

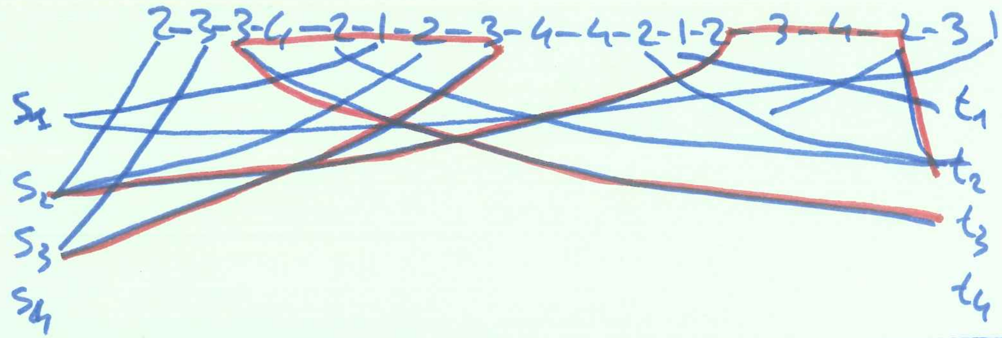
$s_1, \dots, s_k \in V(G)$, $t_1, \dots, t_k \in V(G)$ $\Big\} \neq$

question: $\forall i \in [k]$ path P_i between s_i and t_i

P_i ~~disjo~~ vertex-disjoint from P_j $\forall i \neq j$



- Disjoint Path is FPT (Robertson Seymour)
- PK? reduction from Disjoint factors



thm: SAT param. by number of variables has no PK unless $NP \subseteq coNP/poly$.

(good starting point for reductions)

Refined kernel lower bounds

for some problems $\longrightarrow$ $O(k^c)$ kernel.

improve constant c?

Def weak-cross-compositions from L to Q : param. problem.

input: $x_1, \dots, x_t$ (equivalent) $\longmapsto (y, k)$

- run in P-time (in $\sum |x_i|$)
- $(y,k) \in Q \iff \exists i : x_i \in L$
- $\exists\, d > 1 : \quad k \le p(\max_i |x_i|) + t^{1/d}$

thm: weak cross-comp. from L NP-hard to Q ↳ for some d, then Q has no kernel of size $O(k^{d-\varepsilon})$ ↳ bitwize. $\forall \varepsilon > 0$

unless $NP \subseteq coNP/poly$.

> Vertex cover : G (n vertices), k
>
> - kernel with $O(k)$ vertices : $O(k^2)$ bits
> - kernel with n vertices $\to$ $O(n^2)$ bits

theorem: Vertex Cover has no kernel with $O(n^{2-\varepsilon})$ bits $\forall \varepsilon > 0$ (unless ...)

Cor: no kernel with $O(k^{2-\varepsilon})$ edges $\forall \varepsilon > 0$

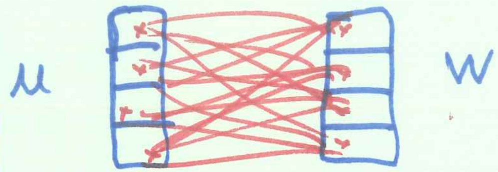
unless $NP \subseteq coNP/poly$

Proof: weak cross-composition with $d = 2$ from Multicolored biclique.

input: bipartite graph (U, W), $k \in \mathbb{N}$

$$U = U_1 \cup \dots \cup U_k$$
$$W = W_1 \cup \dots \cup W_k$$

question: find a $k \times k$ biclique



$U^1, \dots, U^t$

$W^1, \dots, W^t$

$U^i = U^i_1 \cup \dots \cup U^i_k$

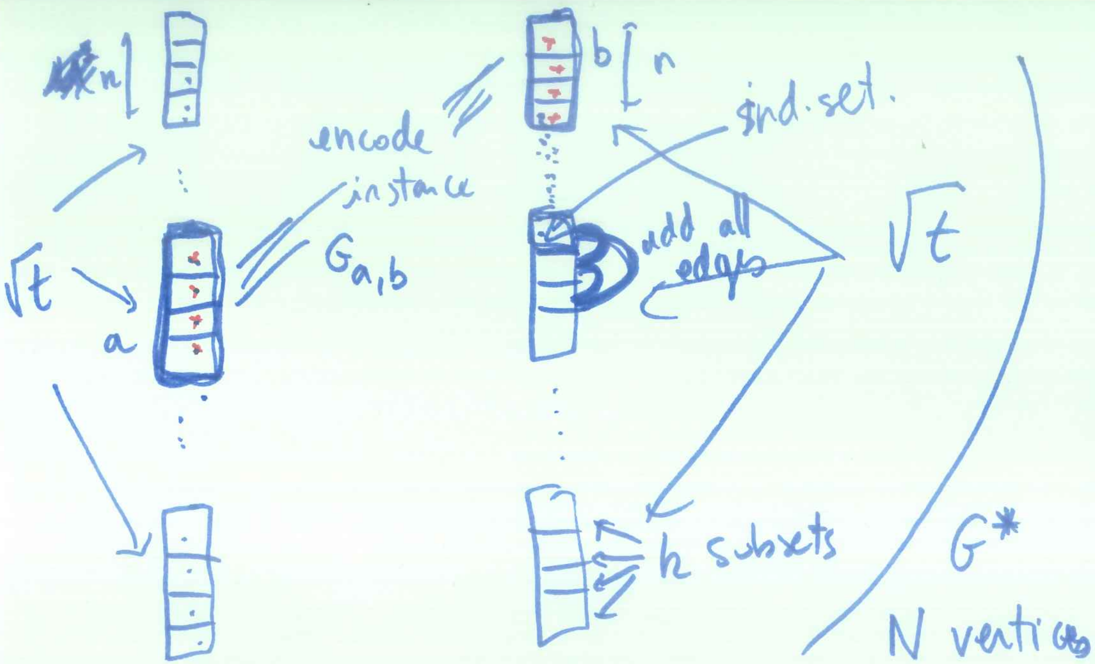
$W^i = W^i_1 \cup \dots \cup W^i_k$

$$\left.\begin{array}{l} |U^i_r| = |U^j_r| =: n \\ |W^i_r| = |W^j_r| =: n \end{array}\right\} \text{equiv. relation}$$

Assume that t is a square

$\sqrt{t} = s \in \mathbb{N}$.

$G_{(a,b)} : 1 \le a, b \le \sqrt{t}$



total number of vertices is a polynomial in n, k, ~~log t~~ $\sqrt{t}$

$\overline{G^*}$ has a vertex cover of size $\leq N - 2k$

~~$\Leftrightarrow$ G^* has~~

$\Leftrightarrow$ $\overline{G^*}$ has an indep. set of size $\geq 2k$

$\Leftrightarrow$ G^* has a clique of size $\geq 2k$

$\Leftrightarrow$ (W^i, W^i) has a $k \times k$ biclique.

G has a VC of size $\leq k$ $\Leftrightarrow$ G has an indep. set of size $\geq n-k$

$\Updownarrow$

. . . $\Leftrightarrow$ $\overline{G}$ has a clique of size $\geq n-k$

$\Rightarrow$ weak cross-composition with $d=2$

<u>VC</u>: no $O(k^{2-\varepsilon})$ bitsize kernel.

kernel with 2k vertices $\rightarrow$ open for $(2-\varepsilon)$

no $O(n^{2-\varepsilon})$ kernel (bitsize) $\forall \varepsilon > 0$.

can be generalized to d-Hitting set:

$\rightarrow$ kernel with $O(k^{d-1})$ ~~vertices~~ elements and $O(k^d)$ sets

<u>thm</u>: d-HS has no kernel with $O(k^{d-\varepsilon})$ bits unless $NP \subseteq coNP/poly$.