1 **Integer Linear Programming**
ILP for FPT

ILP-feasibility

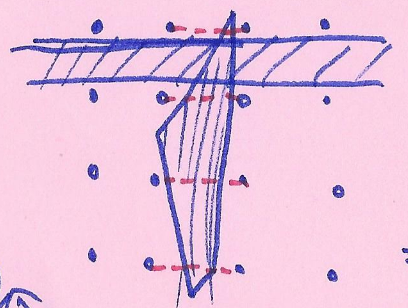
input: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$
q? is there $x \in \mathbb{Z}^n$ such that $Ax \leq b$?

size of an instance $I = O(nm \log A + m \log B)$

Theorem: [Lenstra '83, Kannan / Frank and Tardos '87]
ILP feasibility can be solved in time
$O(n^{2.5n + o(n)} \cdot |I|)$

$\Rightarrow$ ILP feasibility is FPT parameterized by the number of variables.

Idea of proof: what does an ILP with no feasible ~~sol.~~ look like?



$\Rightarrow$ the polytope must be "flat" in some direction

$\Rightarrow$ partition into a small number of intersections with hyperplanes

in each subproblem (intersections) dimension --

Examples

numbers problems parameterized by number of numbers

SUBSET SUM

input: multiset of integers S', $s \in \mathbb{N}$ (repetitions of elements)
q?: is there $A \subseteq S'$ such that $\sum_{a \in A} a = s$

- NP-h. used a lot in scheduling
- W[1]-h param by $|A|$
- FPT param. by ~~[illegible]~~ $\|S'\|$ (number of distinct numbers in S'.)

input: $s_1, \dots, s_k$ distinct numbers
$m_1, \dots, m_k$: their multiplicity

x_i: how many time do we have to pick s_i in the solution?

ILP:

$$\begin{cases} \sum_{i=1}^{k} x_i s_i = s \\ 0 \leq x_i \leq m_i \end{cases}$$

Closest string

input: $s_1 \dots s_k$: strings of length L over some finite alphabet Σ
$d \in \mathbb{N}$

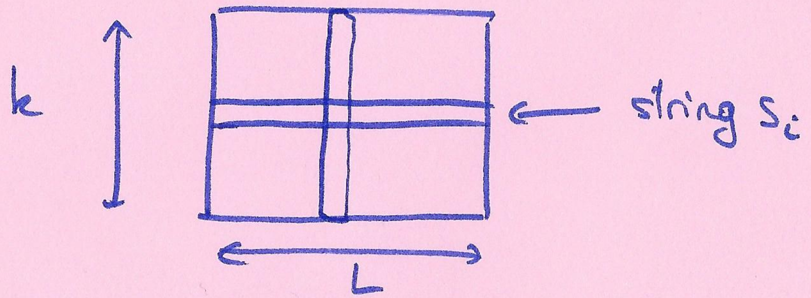
q? is there a string s of length L such that $\underbrace{Ham(s, s_i)}_{}$ $\leq d$

↳ number of differences letter by letter.

- NP-h even if $|\Sigma| = 2$
- FPT param. by k, L or d independently

Here: FPT param. by k

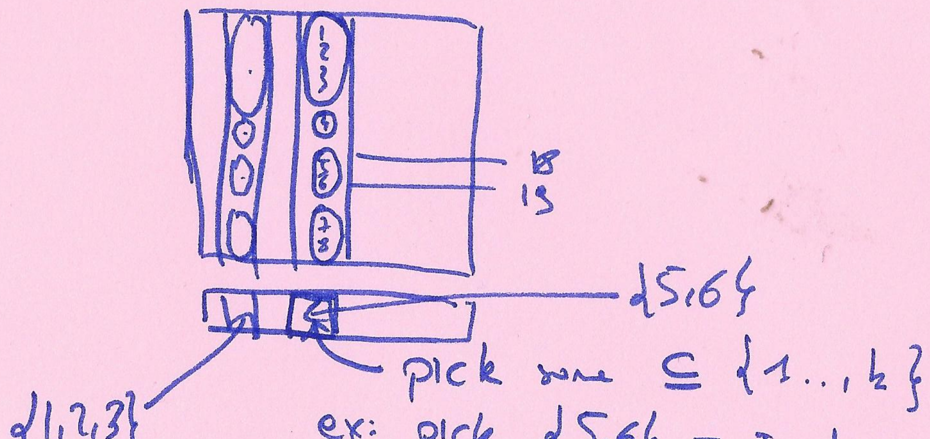
Proof: instance = $k \times L$ matrix



obs: given a column, we can change the elements wrt some permutation $\pi : \Sigma \to \Sigma$

$\Rightarrow$ we can assume that in each column, the symbols ~~are~~ $\in \{1, \dots, k\}$

$\Rightarrow$ we can only remember the partition of $\{1 \dots k\}$ $\Leftrightarrow$ which rows share the same value?



ex: pick $\{5, 6\}$ $\equiv$ I choose the symbol shared by strings 18 and 19 for this column.

Type = partition of $\{1 \dots, k\}$

number of types $\leq k!$

$\Rightarrow$ see the instance as the multiplicity of each type

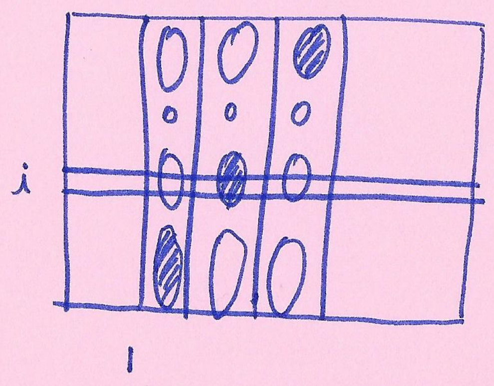
for a type P $\xrightarrow{}$ partition of $\{1..,k\}$ and $C \in P$ $\longrightarrow$ $\subseteq \{1..k\}$

variable $x_{P,C}$ $\equiv$ number columns of type P where I choose C as the solution

ILP:

$\forall$ type P $\quad \sum_{C \in P} x_{P,C} \leq \underline{\#\text{ columns with type } P}$

$\forall i \in \{1..,k\}$ $\quad \sum_{\text{type } P} \sum_{\substack{C \in P \\ i \notin C}} x_{PC} \leq \underline{d}$



exercise: convince yourself that it solves the problem

number of variables $\leq k \times k!$

#: number of partitions of $\{1..k\}$

$\equiv$ Bell number $B_k = O(k!)$

Cuts and separators

Some def: input: graph G

- $X, Y \subseteq V(G)$ $X \cap Y = \emptyset$
  $X \neq \emptyset$ $Y \neq \emptyset$

- (X,Y) - cut is a set of edges whose removal disconnects X from Y (there is no longer any X-Y path)

  ⚠ minimal versus minimum

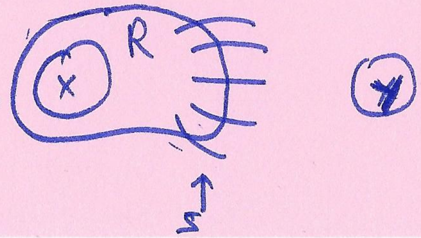
  minimal → inclusion-wise.

  minimum → cardinality

- $R \subseteq V(G)$ $\Delta(R)$ = set of edges with exactly one endpoint in R

  $$d(R) = |\Delta(R)|$$

Remark: if $S \subseteq E(G)$ is a minimal (X,Y)- cut, ~~then~~ and R is the set of vertices reachable from X in $G \setminus S$, then $S = \Delta(R)$



$\Rightarrow$ 1-to-1 correspondence between (X,Y)-cut S and ~~$R \subseteq \Delta$~~ the set R reachable from X.

Menger's theorem + Ford-Fulkerson	there is a $O(k(n+m))$ algorithm which either:

→ find a minimum (X,Y)-cut R and a collection of $\Delta(R)$ disjoint X,Y - path

→ conclude that there ~~the~~ is no (X,Y)-cut of $\boxed{\text{size}} \leq k$

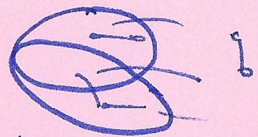
size → number of edges.

$$c^*(G,X,Y) = \text{size of a min } (X,Y)\text{-cut in } G.$$

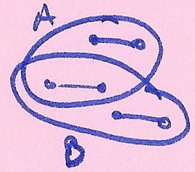
Lemma 1 $d(.)$ is submodular

$f : 2^V \to \mathbb{R}$ is submodular if $\forall A, B \subseteq V$

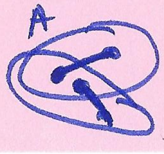
$$f(A)+f(B) \geq f(A \cup B) + f(A \cap B)$$



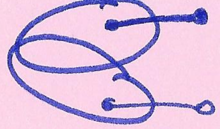
Proof distinguish the type of edges:

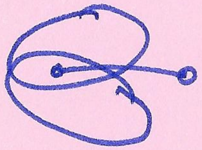
1) 

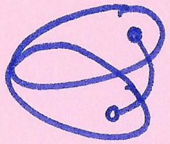
→ contribution: 0 in both sides

2) 

each such edge: contribute 1 in both sides.

3)  1 in both sides

4)  2 in both sides

5)  2 to the left

0 to the right

$$d(A) + d(B) \geq d(A \cup B) + d(A \cap B)$$

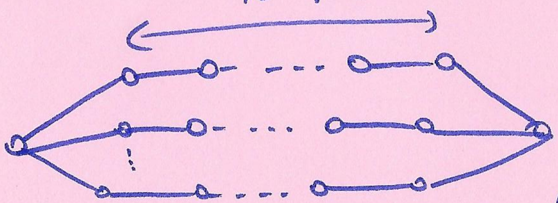
<u>Lemma 2</u>: if A and B $\subseteq V$ are ~~minimal~~ (X,Y)-cuts, then so are $A \cup B$ and $A \cap B$.

<u>Proof</u> exercise.

<u>Remark</u>: a graph may have a large number of (X,Y)-cuts

$\approx n/k$

k



$\Omega\left(\left(\frac{n}{k}\right)^k\right)$

minimum (X,Y)-cuts.

<u>Multiway Cut</u>

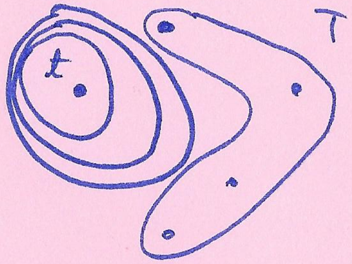
<u>input</u>: graph G, $T \subseteq V(G)$ terminals

$k \in \mathbb{N}$

<u>q?</u> is there $S \subseteq E(G)$ $|S| \leq k$ such that in $G \setminus S$, each $t \in T$ is in a different connected component ?

- $|T| = 2 \longrightarrow$ polynomial (s,t)-cut
- NP-h for every fixed $|T| \geq 3$
- trivial $n^{O(k)}$ algorithm.

<u>Here</u>: Multiway cut is FPT (k)

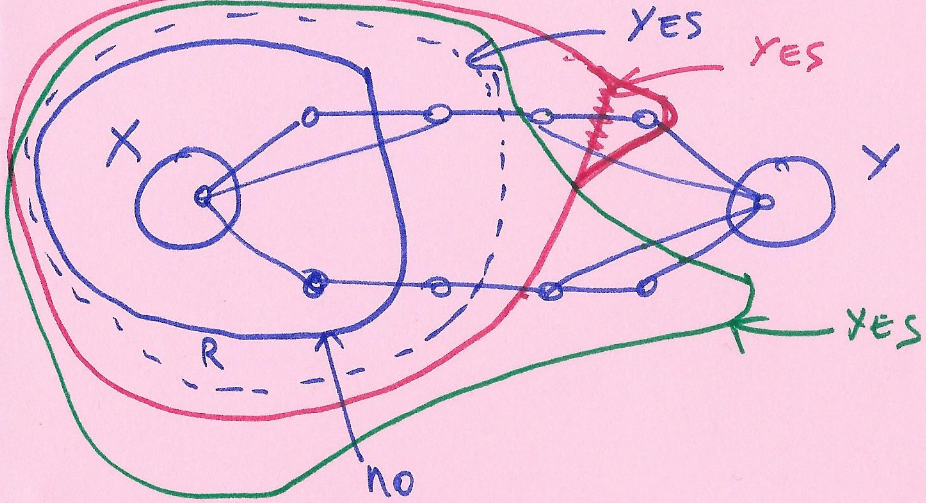


$\approx$ enumerate all $(\{t\}, T \setminus \{t\})$-cuts

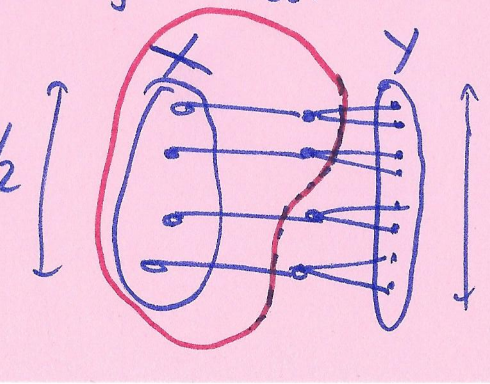
Def: a (X,Y)-cut $S = \Delta(R)$ is **important** if there is no (X,Y)-cut $S' = \Delta(R')$ such that

- $|S'| \leq |S|$
- $R \subsetneq R'$

idea: "push" cuts far from X



Remark: there might be many important sep. of a given size:



$2^{k/2}$ important sep. of size $\leq k$

using these important separators in Multiway cut

given G, $T \subseteq V(G)$, $k \in \mathbb{N}$

~~if the~~

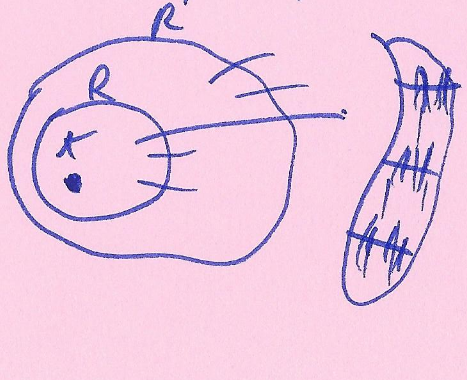
Lemma: if there is a solution S of size $\leq k$ then, $\forall t \in T$, there is a solution of size $\leq k$ which includes an important $(\{t\}, T \setminus \{t\})$-cut

$\to$ if not the case, ~~replace~~ consider an important sep $\Delta(R')$, replace $S = \Delta(R)$ by

$\cancel{(S \setminus \Delta(R)) \cup \Delta(R')}$

Algorithm: pick $t \in T$, enumerate all important $(t, T \setminus t)$-separators, recurse.

$\Rightarrow$ we need to bound the number of important separators.

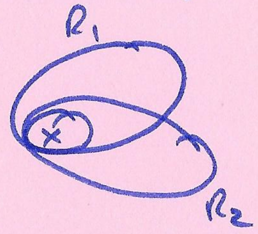


Lemma: given G, X, Y, k (..)
G has at most $O(4^k)$ important separators of size at most k

Proof:

Claim: there is a unique important (X,Y)-cut of minimum size $c^*(G,X,Y)$

proof: suppose there are two: R_1, R_2
apply submodularity:



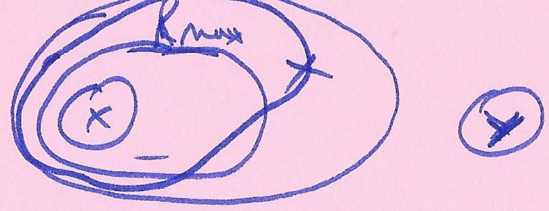
$$\underbrace{d(R_1)+d(R_2)}_{c^*} \geq \underbrace{d(R_1 \cup R_2)}_{} + \underbrace{d(R_1 \cap R_2)}_{\geq c^*}$$

~~$\Rightarrow d(R_1 \cap R_2) \leq c^*$~~

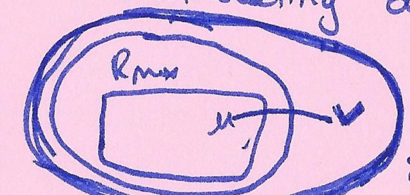
~~but~~ $d(R_1 \cup R_2) \leq c^*$

but $R_1, R_2 \subseteq R_1 \cup R_2$, contradicts the fact that R_1 and R_2 are important.

Let R_{max} this unique important (X,Y)-cut.
by definition: R_{max} is contained in every other important separator



enumerating all important (X,Y)-separators
$\Longleftrightarrow$ enumerating all important (R_{max}, Y)-separators



consider an edge $uv \in \Delta(R_{max})$
2 cases: either uv is in an important separator or not.
instance $I = (G, X, Y, k)$

- branch 1: $(G \setminus uv, X, Y, k-1)$
- branch 2: $(G, X \cup \{v\}, Y, k)$

Correctness: in branch 1: if S is an import. sep. $\leq k$ containing uv, then $S \setminus uv$ is an imp. sep $\leq k-1$ in $G \setminus uv$

$\mu(I) = 2k - c^*(G, X, Y)$

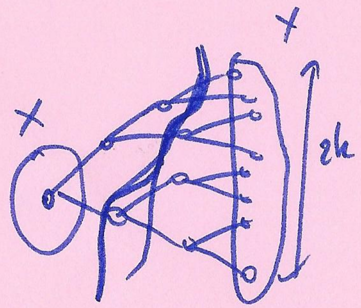
- branch 1: $\mu(I') = 2(k-1) - \overbrace{c^*(G \setminus uv, X, Y)}^{\geq c^*(G, X, Y)}$
- branch 2: $\mu(I') = 2k - \underbrace{c^*(G, X \cup \{v\}, Y)}_{> c^*(G, X, Y)}$

$\Rightarrow$ in both cases, $\mu(I') < \mu(I)$.

$\mu(I) \leq 2k$

$\Rightarrow$ enumerate all important sep. there are at most $O(2^{2k-c^*}) = O(4^k)$ of them.

this bound is tight:



$\Rightarrow$ any subtree with k leaves is an important separator of size $\leq k$

$\Rightarrow$ number of subtrees with k leaves

catalan number $C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1} \geq \frac{4^k}{poly(k)}$

multiway cut is solved in $O^*(4^{k^2})$

(can run in $O^*(4^k)$)