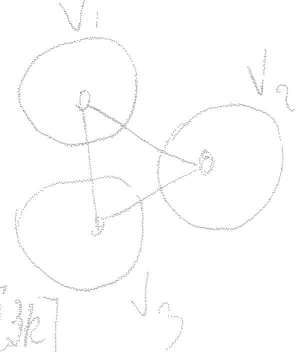
Lecture 5.

Color Coding.

Triangle Packing  $k^{O(k)} n^{O(1)}$ time

$2^{O(k)} n^{O(1)}$ time?

$$\frac{k!}{(3k)^{3k}}$$



Again "good strings" are multicolored on $[3k]$

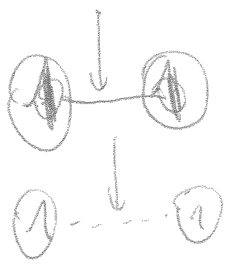
$$\frac{(3k)!}{(3k)^{3k}} \geq e^{-3k}$$

$2^{O(k)}$ runs

DP.

Color u.a.r on $[3k]$.

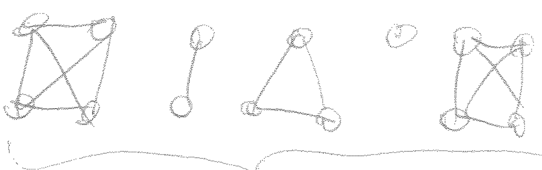
Chromatic coding. Variant of Color Coding for problems where we want to edit edges.



Color s.t. every monochromatic "edge" cannot be part of the solution.

d-Cluster Edition : Edit $\leq k$ edges s.t. G edited is a d-Cluster

Cluster : disjoint union of cliques



at most d cliques

d-Cluster

$2^{O(\sqrt{k})} n^{O(1)}$

Prop. G on k edges, color u.a.r $q := \left\lceil \sqrt{8k} \right\rceil$ colors

$$P(G \text{ is properly colored}) \geq 2^{-\sqrt{\frac{k}{2}}} \longrightarrow 2^{O(\sqrt{k})} \text{ runs}$$

Proof. v_1 with minimum degree

$G - v_1 \to v_2$ with min degree

$G_0 = G$ $\quad G_1 = G_0 - v_1$

$(v_1, v_2, v_3, \dots, v_n$ $\qquad G_0 = G, G_1, G_2, \dots, G_{n-1}, \underset{\emptyset}{G_n}$

$G_2 = G_1 - v_2$

$$d_i = \text{min degree}(G_i) = d_{G_i}(v_{i+1})$$

Claim. $\boxed{d_i \leq \sqrt{2k}}$

$d_i \leq |V(G_i)|$

$2k = 2|E(G)| \geq 2|E(G_i)| \geq d_i |V(G_i)| \geq d_i^2$

Color u.a.r with q colors starting from $v_n, v_{n-1}, \dots$

W_i = event that G_i is properly colored

$P(W_n) = 1$

Q) $P(W_i \mid W_{i+1}) = ?$ $\quad v_{i-1}$ in $G[v_{i-1}, v_i, \dots v_n]$

$\sqrt{2k}$ colors that should be avoided

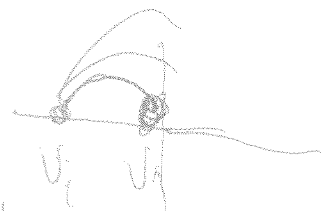
$$P(W_i \mid W_{i+1}) \geq \frac{q - \cancel{\sqrt{2k}}\, d_i}{q} = 1 - \frac{\cancel{\sqrt{2k}}\, d_i}{q}$$

$P(W_0) = P(W_m) \cdot P(W_{m-1} | W_m) \cdot \cdot P(W_0 | W_1)$

$$= \prod_{i=1}^{m} \left(1 - \frac{d_i}{q}\right) \qquad 1 - x \le 2^{-2x} \quad \forall x \in [0, \tfrac{1}{2}]$$

$$\ge \prod_{i=1}^{m} 2^{-\frac{2d_i}{q}} = 2^{-\frac{2}{q}\sum_{i=1}^{m} d_i} = 2^{-\frac{2k}{q}} \ge 2^{-\sqrt{\frac{k}{2}}}$$

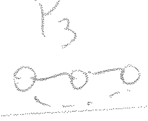
u.a.r color $V(G)$ with $[q]$

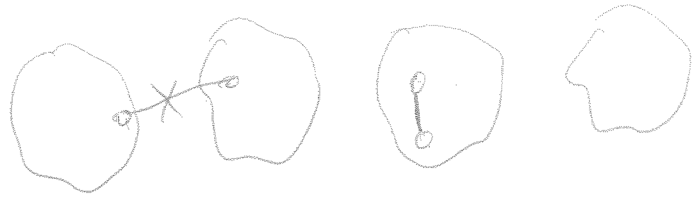


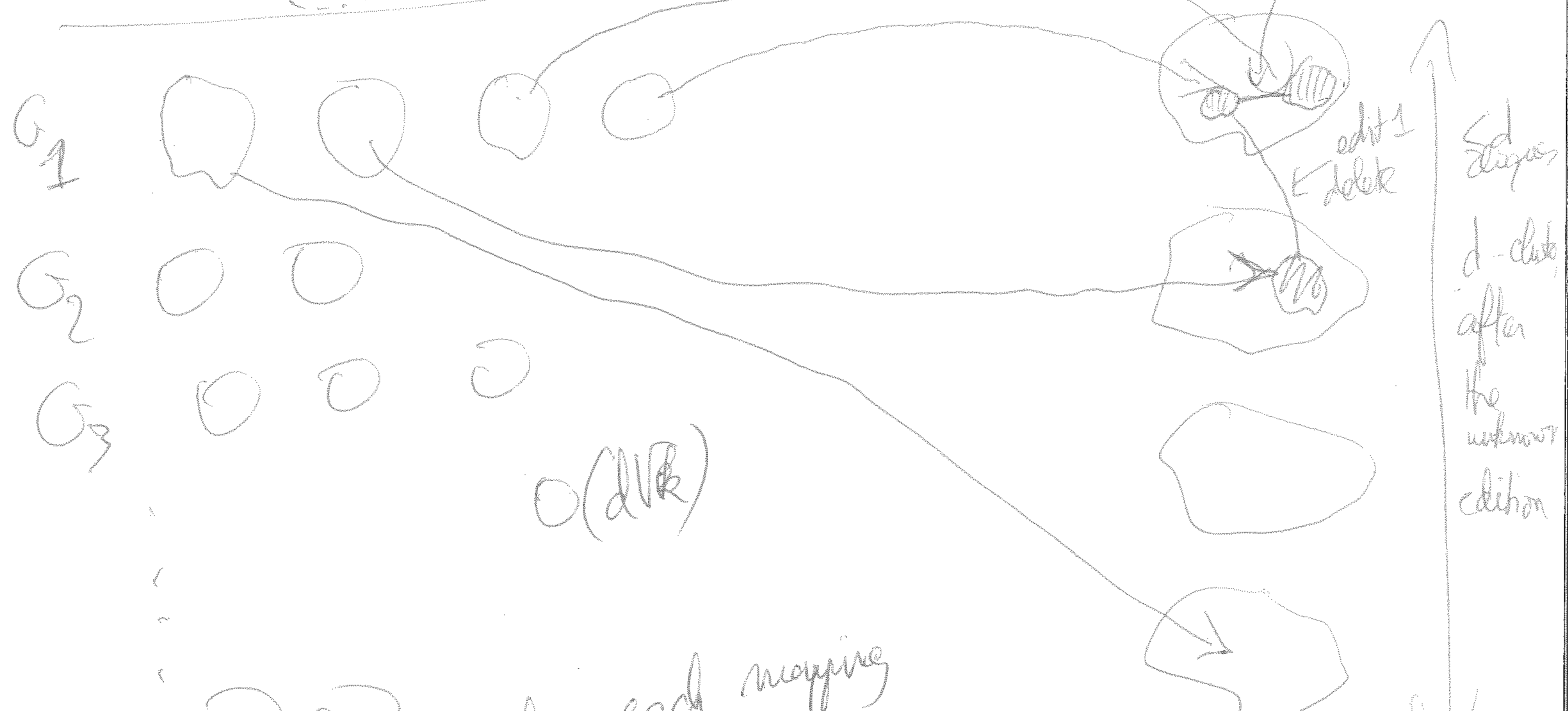
"Good coloring" no monochromatic pair is edited $i < j$

V_i = color class i "k editions"

$\forall i$ $G_i = G[V_i] = ?$ d-cluster G







$O(d\sqrt{k})$

for each mapping $d^{O(d\sqrt{k})}$

$q = O(\sqrt{k})$

$2^{O(\sqrt{k})} \times 2^{O(\sqrt{k})} = 2^{O(\sqrt{k})}$

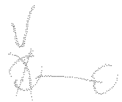
Exact algorithms Best running in size of the solution

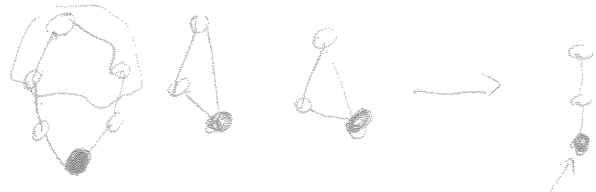
Measure & Conquer. NOT an algorithmic technique

analysis running time

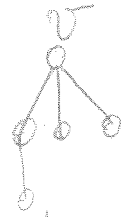
Max Independent Set $\to 3^{n/3}$ maximal IS 1.443^n

↳ 1.38^n

(R1) If v of degree ≤ 1  $\to$ include v to our solution

max degree $\to \Delta(G) \leq 2$ 

δ

$\Delta(G) \geq 3$  3 neighbors

taking v in sol — discarding v

$G - N[v]$ has $n-4$ vertices

$G - v$ has $n-1$ vertices

branching vector is $(1,4)$.

$$T(n) \leq T(n-1) + T(n-4) + O(1)$$

$$B_n = B_{n-1} + B_{n-4}$$

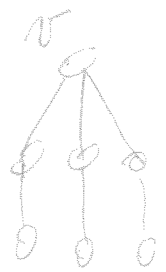
largest positive root of

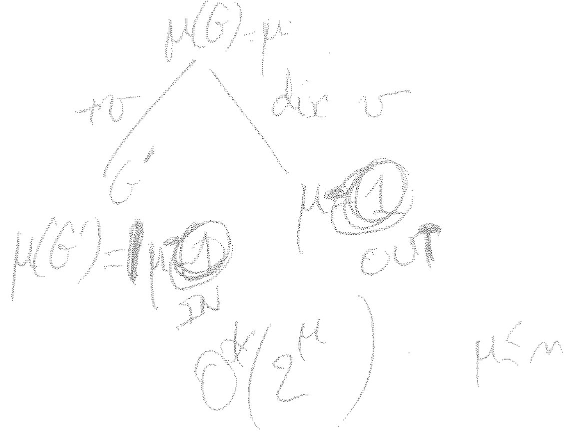
$$c^n = c^{n-1} + c^{n-4} \to c^4 - c^3 - 1 = 0$$

$$O^*(1.38\ldots^n)$$

Measure & Conquer: Find a twisted measure capturing reduction rules might happen after branching.

$\mu(G) := n_{\geq 3}$ = # vertices of G with at least 3 neighbours

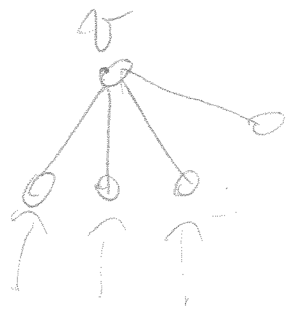


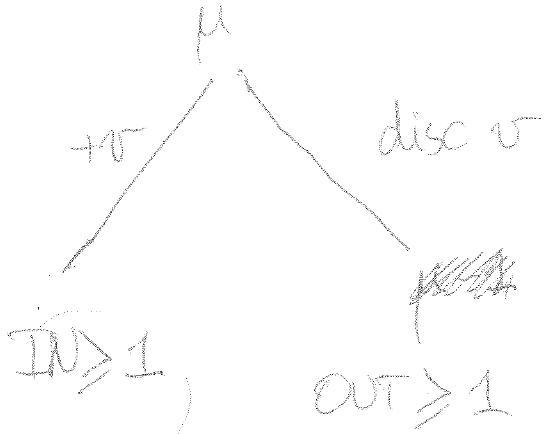


NOT WORKING

$\mu(G) := \frac{1}{2} n_2 + n_{\geq 3}$

vertices of degree = 2





Q) Claim: $\boxed{IN + OUT \geq 2 + \underbrace{d(v)}_{\geq 3}} \geq 6$

$(1,4)$

$(2,3)$

$\lambda(1,4) \to O^*(\lambda^n)$

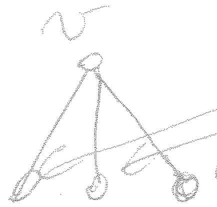
$\lambda^5 - \lambda^4 - 1 = 0$

$\lambda^5 - \lambda^4 - 1 = 0$

1.32

What about $\Delta(G) = 3$?

Q)



the neighbors of v degree ≤ 3

$IN \geq 1 + \frac{3}{2}$ $OUT \geq 1 + \frac{3}{2}$

$\lambda\left(\frac{5}{2}, \frac{5}{2}\right) \longrightarrow 1.31^{\mu}$

$\mu \leq n$

$\frac{1}{2} n_2 + n_3$ $\sum w_i n_i$

$w_0 = w_1 = 0$

$w_2 = 0.5966$

$w_3 = 0.9286$

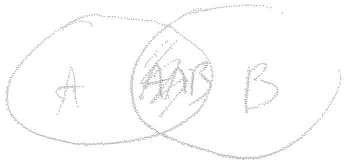
$w_{4+} = 1$

$\longrightarrow$ analyse branching

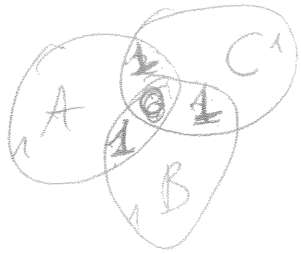
1.2905^n

Exact algorithms

Inclusion - Exclusion



$|A \cup B| = |A| + |B| - |A \cap B|$



$|A \cup B \cup C| = |A| + |B| + |C|$
$- |A \cap B| - |A \cap C| - |B \cap C|$
$+ |A \cap B \cap C|$

Inclusion - Exclusion, Union Version

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq X \subseteq [n]} (-1)^{|X|+1} \left|\bigcap_{j \in X} A_j\right|$$

x is in : $A_{i_1}, A_{i_2}, \dots, A_{i_p}$

$$\sum_{\emptyset \neq X \subseteq \{i_1, \dots, i_p\}} (-1)^{|X|+1} = \sum_{j=1}^{p} \binom{p}{j} (-1)^{j+1}$$

$$= (-1) \underbrace{\sum_{j=1}^{p} \binom{p}{j} (-1)^{j} 1^{p-j}}_{(-1+1)^p = 0} = (-1)(0-1) = 1$$

0

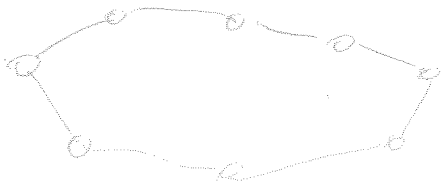
$$\boxed{\left|\bigcap A_i\right| = \sum_{X\subseteq[n]} (-1)^{|X|} \left|\bigcap_{i\in X} \overline{A_i}\right|}$$

total universe

$\bigcap A_i$

$\overline{A_i} = U \setminus A_i$

HAMILTONIAN CYCLE :



↳ DP : $O^*(2^n)$-time but the space

cycle going through every exactly once

A polyspace alg. HAM CYCLE $O^*(2^n)$

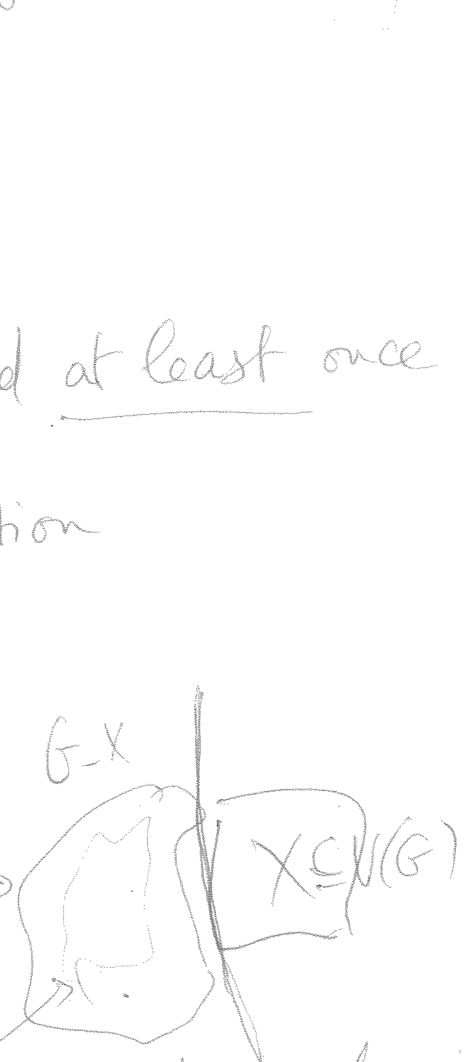
universe U : closed walks rooted at $s \in V(G)$ of size n.

$\forall v \in V(G)$

↳ $A_v \subseteq U$: but v is visited at least once

$$\underbrace{\left|\bigcap_{v\in V(G)} A_v\right|}_{2^n \text{ terms}} > 0 \rightarrow \exists \text{ solution}$$

$\left|\bigcap_{v\in X} U\setminus A_v\right|$



how to compute in polynomial time