- 3 marks
  1) article pres. (mid-nov)
  2) exam ??
  3) homework
- Seminar "Frontiers of parameterized complexity"
  Edouard: 1/10 17h

Kernelization

A mathematical framework for the analysis of pre-processing algorithms.

Definition: a kernel for a problem Q is an algorithm which takes as input an instance x of Q with parameter value k, and outputs an instance y of Q with parameter k' such that:

- runs in polynomial time $(|x| + k)$
- $k' \leq k$
- x is a positive instance $\Leftrightarrow$ y is positive
- $|y| + k' \leq f(k)$ for some computable function f

f is the **size** of the kernel

if f is a polynomial => Polynomial kernel.

Lemma: Q admits a kernel $\Rightarrow$ Q is FPT.

PF: apply any brute force on the reduced instance.

Lemma: Q is FPT $\Rightarrow$ Q has a kernel.

Proof: suppose we have an FPT algorithm $\mathcal{A}$ running in $f(k) \cdot n^c$ for some function f some constant c. Let (x, k) be an instance.

- $|x| \leq f(k)$ then we already have a kernel.
- $|x| > f(k)$. Apply algorithm $\mathcal{A}$ it runs in time $f(k) \cdot |x|^c < |x|^{c+1}$ which is polynomial $\Rightarrow$ output a stupid positive or negative instance of constant time

problem: cannot compute $f(k)$ in polynomial time!

instead: run $\mathcal{A}$ $|x|^{c+1}$ first steps.

- if $\mathcal{A}$ is finished $\Rightarrow$ done.
- otherwise: $f(k) |x|^c > |x|^{c+1}$
  $\rightarrow |x| < f(k)$. we have a kernel.

Theorem: Q is FPT $\Leftrightarrow$ Q has a kernel.

usually: when we have a problem Q:

1) is it FPT?

if yes: 2) is there a polynomial kernel?

later: tools to answer negatively 1) and 2)

solve the problem in polynomial kernel $\neq$ polynomial time.

Previously: a kernel for Vertex Cover with $O(k^2)$ vertices.

Generalize this result.

d-Hitting Set

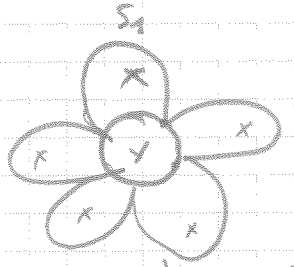
input: a family of sets $\mathcal{S}$ over a universe $\mathcal{U}$ of finite size, $\forall S \in \mathcal{S}$ $S \subseteq \mathcal{U}$ $|S| \leq d$, $k \in \mathbb{N}$

question: find $X \subseteq \mathcal{U}$ $|X| \leq k$ such that $\forall S \in \mathcal{S}$ $S \cap X \neq \emptyset$

Remark: 2-HS $\Leftrightarrow$ Vertex Cover

Goal: kernel for d-HS with $O(k^d)$ elements.

Definition: a sunflower (with core Y) is a collection of sets $S_1, \dots, S_q$ such that $\forall i, j$ $i \neq j$ $S_i \cap S_j = Y$ — q petals

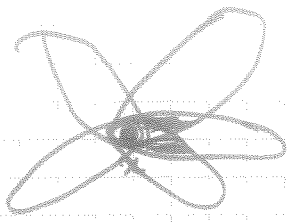


$S_i \neq \emptyset$ $\forall i$

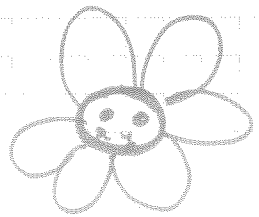
$S_i \setminus Y \neq \emptyset$ $\forall i$

Remark: Y can be empty (a collection of pairwise disjoint sets is a sunflower)

idea:



$k+1$ petals of sunflower with core $\{x\}$ $\Rightarrow$ must take k in the solution.



either x or y must be part of any solution.

Lemma (Sunflower Lemma): [Erdős and Rado] Let $\mathcal{S}$ be a collection of sets of size d over some universe $\mathcal{U}$. If $|\mathcal{S}| > d!(k-1)^d$, $k \in \mathbb{N}$ then there is a sunflower with k petals which can be found in polynomial time

Proof: by induction on d: $d=1$ is obvious assume $d \geq 2$. Build a (maximal) collection of pairwise disjoints sets M. if $|M| \geq k$ ☺

$|M| < k$. $|\mathcal{S}| > d!(k-1)^d$.

Let $X = \bigcup_{S \in M} S$. Since M is maximal, each $S \in \mathcal{S}$ intersects X. $\Rightarrow$ there must be $u \in X$ which belongs to $\frac{|\mathcal{S}|}{|X|}$ sets

$$\frac{|\mathcal{S}|}{|X|} > \frac{d!(k-1)^d}{d(k-1)} = (d-1)!(k-1)^{d-1} \quad (|X| \le d(k-1))$$

$$> (d-1)!\,(k-1)^{d-1}$$

~~Consider the collection $\mathcal{S}$~~

Let A be the sets containing u.

Consider ~~$\mathcal{S}' = \{A \setminus \{u\}\}$~~

$$\mathcal{S}' = \{S \setminus \{u\} : S \in A\}$$

apply the induction hypothesis on $\mathcal{S}'$ and get a sunflower with k petals $S_1, \dots, S_k$. Observe that $S_1 \cup \{u\}, \dots, S_k \cup \{u\}$ is a sunflower of $\mathcal{S}$ with k petals.

Exercise: turn it into an algorithm

$O(k^d)$ kernel for d-HS:

"obvious reduction rules" (isolated element...)

Reduction Rule: if $|\mathcal{S}| > d!\,k^d$, then find a sunflower with core Y and $k+1$ petals $S_1, \dots, S_{k+1}$ (remove S_i $\forall i$)

- ~~$S_i' = S_i \setminus Y$ for every $i \in \{1 \dots k+1\}$~~ (but not the elements other S doesn't change.)
- add Y to the collection.

Exercise: check that this rule is safe. (the new instance is positive $\Leftrightarrow$ the former is)

if this rule cannot apply, then we have at most $d \times d!\,k^d$ sets and at most $d^2 d!\,k^d$ elements.

Theorem

d-HS has a kernel with $O(k^d)$ elements and sets

Currently: the "best" kernel for d-HS has $O(k^d)$ sets but $O(k^{d-1})$ elements.

and, under some complexity assumptions, there cannot be a kernel with $O(k^{d-\varepsilon})$ sets for any $\varepsilon > 0$

Open problem: is there a kernel with ~~$O(k^{d-\varepsilon})$~~ $O(k^{d-1-\varepsilon})$ elements for some $\varepsilon > 0$?

(for instance: it would be possible to have a kernel with $O(k)$ elements and $O(k^d)$ sets)

Vertex Cover: a $O(k)$-vertices kernel ($3k$ vertices)

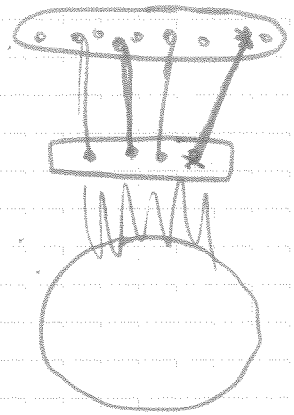
CROWN DECOMPOSITIONS.

Definition: a crown decomposition of a graph G is a triple (C, H, R) $C, H, R \subseteq V(G)$

$C \cap H = \emptyset$ $C \cap R = \emptyset$ $H \cap R = \emptyset$.

such that:

- $C \neq \emptyset$
- C is an independent set
- no edge between C and R
- There is a matching between C and H saturating H
  - ↳ each vertex of H is the endpoint of an edge of the matching.



Konig's theorem: in a bipartite graph, the size of a minimum vertex cover equals the size of a maximum matching, and both of them can be found in polynomial time.

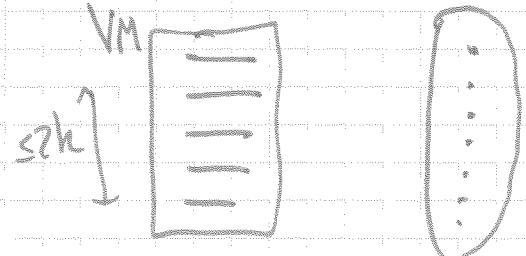
Crown lemma: Let G be a graph with no isolated vertices, $k \in \mathbb{N}$. Assume G has at least $3k+1$ vertices. then either:

- G has a matching of size $k+1$
- G has crown decomposition

and they can be found in P-time.

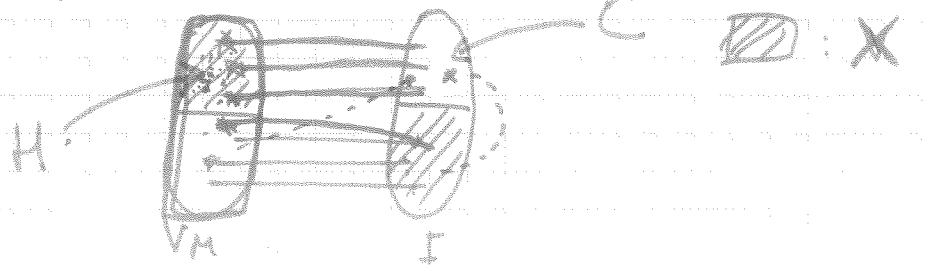
Proof: Build M a maximal matching in G. If $|M| \geq k+1$ ☺

- $|M| \leq k$. Let V_M be the vertices in the edges of M $|V_M| \leq 2k$ and $V(G) \setminus V_M$ is an independent set



Find a min. vertex cover X and a maximum matching M' between V_M and I

$|X| = |M'| \leq k$



Claim: $X \cap V_M$ is non empty:

if $X \subseteq I$ then $X = I$ (because if there is $w \in I \setminus X$ it must be isolated) but then G would have at most $3k$ vertices. (impossible $(|V_M| \leq 2k$ and $|X| \leq k)$

CROWN Decomposition:

$H = X \cap V_M \qquad C = I \setminus X$

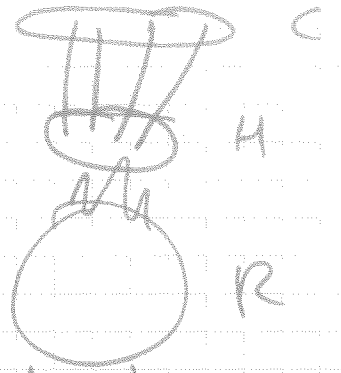
$R = V(G) \setminus (H \cup C)$

Remark: no edge between R and C because I is an Indep. set and X a vertex cover.

it can be constructed in polynomial time.

Reduction Rule 1: remove isolated vertices.

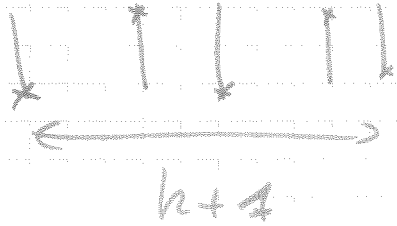
Reduction Rule 2: if there is a crown decomposition C, H, R, then remove H and decrease k by |H|.



RR2 is safe because any solution will have to take at least |H| vertices in order to cover all edges in C ∪ H

By the crown lemma, if no halting RR applies and RR1 and RR2 don't apply, then G has at most $3k$ vertices $\Rightarrow$ kernel with $3k$ vertices.

(if the crown lemma outputs a matching of size $k+1$:



$\Rightarrow$ no solution)

Even better: kernel with $2k$ vertices for Vertex Cover

[2001, Jianer Chen, I. Kanj, W. Jia]

- Integer Linear Programming Formulation for Vertex Cover: variable x_v for each $v \in V$. ($x_v = 1$ means take v in the solution)

$$\text{minimize} \sum_{v \in V} x_v$$

subj. to:

$$x_u + x_v \geq 1 \quad \forall uv \in E$$

$$x_v \in \{0,1\} \quad \forall v \in V$$

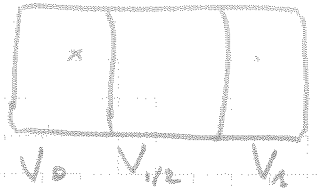
Integer relaxation: replace $x_v \in \{0,1\}$ by $0 \leq x_v \leq 1$ $\Rightarrow$ Linear Program

Solving a linear Program can be done in polynomial time. $\Rightarrow$ ~~a~~ fractional solution $\{x_v\}_{v \in V}$.

$$V_0 = \{v \in V \quad x_v < 1/2\}$$

$$V_{1/2} = \{v \in V \quad x_v = 1/2\}$$

$$V_1 = \{v \in V \quad x_v > 1/2\}$$



Nemhauser - Trotter Lemma: ~~For every~~ There is a minimum vertex cover S of G, we have

$$V_1 \subseteq S \subseteq V_{1/2} \cup V_1$$

Proof: Friday.