2k-kernel for Vertex Cover

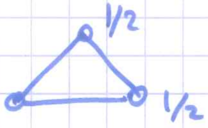
ILP: variable x_v for every $v \in V$

$\min \sum_{v \in V} x_v$

s.t. $x_u + x_v \geq 1$ $\quad uv \in E$

$x_v \in \{0, 1\}$ $\quad \forall v \in V$

Relaxation: replace $x_v \in \{0,1\}$ by $0 \leq x_v \leq 1$ => Linear Program (LP)



OPT(LP) $\leq$ OPT(ILP)

→ Run the LP → solution (fractional) $\{x_v\}_{v \in V}$

$V_0 = \{v : x_v < 1/2\}$

$V_{1/2} = \{v : x_v = 1/2\}$

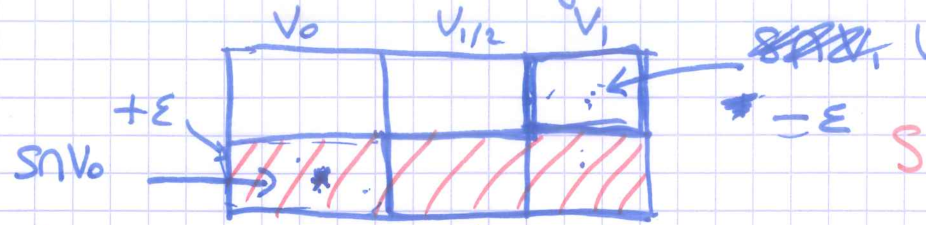
$V_1 = \{v : x_v > 1/2\}$

Nemhauser-Trotter Lemma:

there is an optimal Vertex cover S of G such that ~~S_1~~

$$V_1 \subseteq S \subseteq V_1 \cup V_{1/2}$$

Proof: Consider an minimum VC S: assume it doesn't satisfy the statement



$$S' = (S \setminus V_0) \cup (V_1 \setminus S)$$

Observe that S' is also a vertex cover of the graph: any vertex in $V_0 \cap S$ must be adjacent only to vertices in V_1.

We want: $\boxed{|V_0 \cap S| \geq |V_1 \setminus S|}$

suppose $|V_0 \cap S| < |V_1 \setminus S|$ by contradiction

$$\varepsilon = \min \{ |1/2 - x_v| : x_v \in V_0 \cup V_1 \}$$

define $y_v = \begin{cases} x_v - \varepsilon & \forall v \in V_1 \setminus S \\ x_v + \varepsilon & \forall v \in S \cap V_0 \\ x_v & \text{otherwise} \end{cases}$

we have $\sum_{v \in V} y_v < \sum_{v \in V} x_v$.

prove: $\{y_v\}_v$ is a feasible solution of the LP

look at edges uv where $u \in V_1 \setminus S$

- if $v \in S \cap V_0$: $y_v + y_v = x_v + x_u$
- if $v \in V_{1/2}$: $\underbrace{y_v}_{\geq 1/2} + \underbrace{y_u}_{\geq 1/2} \geq 1$.

- if $v \in V_1$: $\underbrace{y_v}_{\geq 1/2} + \underbrace{y_u}_{\geq 1/2} \geq 1$

feasible solution with cost strictly smaller than $\{x_v\}_{v \in V}$ $\Rightarrow$ impossible. ∎

How to use NT-lemma ?

$\rightarrow$ V_0 are useless $\Rightarrow$ remove them

$\rightarrow$ V_1 must be in solution $\Rightarrow$ remove them and decrease k by $|V_1|$

$\Rightarrow$ return $\underbrace{G[V_{1/2}]}$, $k - |V_1|$

$x_v = 1/2 \ \forall v \in V_{1/2}$ and $\sum_{v \in V} x_v \leq k$

$\Rightarrow$ $|V_{1/2}| \leq 2k$

NT-lemma can also be used to prove that VC admits a 2-approximation.

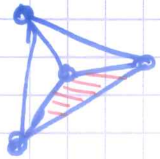
$\Rightarrow$ return $V_{1/2} \cup V_1$ as the solution.

Open problem : is there a $\underbrace{(2-\varepsilon)k}_{\rightarrow \text{vertices}}$ kernel for Vertex Cover ? for some $\varepsilon > 0$

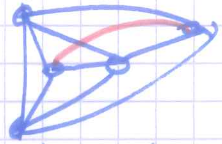
We only know that a kernel with $O(k^{2-\varepsilon})$ edges is ~~impossible~~ unlikely

Planar graphs: the "easy part": kernels

planar graph G = there is ~~...~~ a drawing of G in the plane s.t. no edge cross.

K_4 

K_5 is not planar

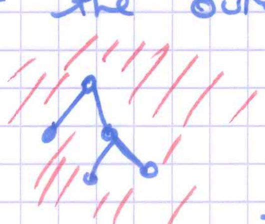


Planar graph G : n : number of vertices

m : ——— edges

f : ——— faces ← connected region

⚠ do not forget the outer face



Euler's formula: for any connected planar graph:

$$n - m + f = 2$$

Proof: true for trees ($f = 1$)

if not a tree: take a cycle remove an edge: ~~...~~ $f - m$ stays the same

at the end → tree.

Properties of planar graphs

• sparse: linear number of edges.

any edge belongs to ≤ 2 faces:

$$\sum_{\text{face } a} \#\text{edges}(a) \leq 2m$$

every face has ≥ 3 edges

$\Rightarrow 2m \geq 3f$

$\Rightarrow m \leq 3n - 6$ by Euler's formula

corollary: any planar graph contains a vertex of degree ≤ 5

d-degenerate: always contain a vertex of degree $\leq d$

$\Rightarrow$ every planar graph has a proper 6-coloring

• Four Color theorem: any planar graph has a proper 4-coloring.

Max Independent Set problem in planar graphs

MIS remains NP-hard in planar graphs (of max degree 4)

MIS is unlikely to be FPT in general. But it is FPT in planar graphs (kernel with 4k vertices)

$\Rightarrow$ any planar graph has an independent set of size $\geq \frac{n}{4}$

kernel: if $k \leq n/4 \Rightarrow$ answer YES

otherwise: $n < 4k$.

---

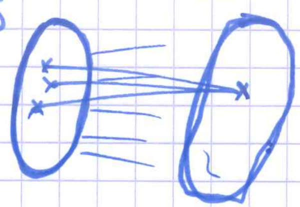
using Euler's formula:

Lemma: in a planar bipartite graph:

$m \leq 2n - 4$

Corollary: : ~~planar bipartite graph~~

set S $V \setminus S$ no isolated vertices.



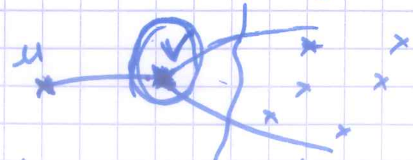
if $\forall v \in V \setminus S$: $d_S(v) \geq 3$, then $|V \setminus S| \leq 2|S| - 4$

Connected Vertex Cover

input: graph G, $k \in \mathbb{N}$

quest: is there a vertex cover $S \subseteq V(G)$ $|S| \leq k$ such that $G[S]$ is connected

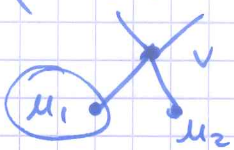
in ~~planar~~ general graphs: no polynomial kernel.



Planar graphs: 4k-vertices kernel for CVC

Proof: RR0: remove isolated vertices

RR1: if $\exists$ v with two pendant vertices u_1 u_2 $(d(u_1) = d(u_2) = 1 \quad N(u_1) = N(u_2) = \{v\})$



$\Rightarrow$ remove one of them (u_2)

$\Rightarrow$ every vertex has at most one neighbor of degree 1.

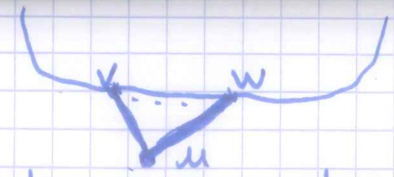
let u be a degree 2 vertex

1) u is a cut-vertex : if remove u: graph gets disconnected

2) u is not a cut-vertex

if 1): ~~we have~~ u must be part of any solution $\Rightarrow$ don't do anything.

if 2)



we must take ≥ 2 vertices among $\{u, v, w\}$

it is always better to take v and w

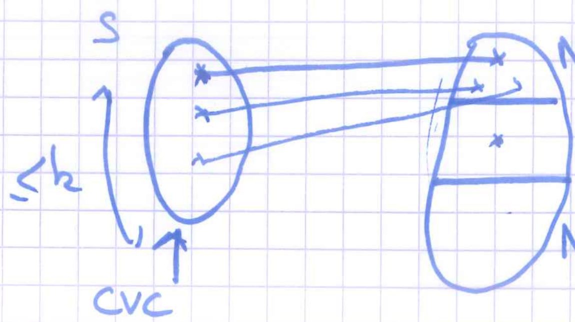
we have to "remember" u

<u>RR2</u>: if $\exists$ u which is not a cut vertex an has degree 2 (neighbors v w), remove u, create u_1, u_2: attach u_1 to v and u_2 to w

$\Rightarrow$ by doing so, we "remember" that we have to take both v and w in the solution.

Suppose that none of RR0, RR1, RR2 apply. Suppose (G, k) is a positive instance:

V / S : ind. set.



N_1: only 1 neighbor in S

N_2: only 2 neighbors in S

N_3: ≥ 3 neighbors in S

By RR 1: $|N_1| \leq k$

By RR2: N_2 does not contain a cut vertex

$\Rightarrow N_2 = \emptyset$

<u>By corollary</u>: $|N_3| \leq 2k - 4$

$$\Rightarrow |V(G)| \leq |S| + |N_1| + |N_3| \leq 4k.$$

5

$\boxed{\text{Turing kernels}}$

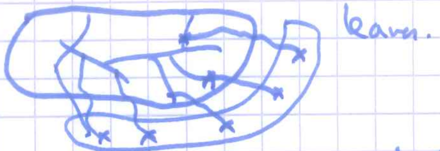
Max leaf subtree

input: Graph G, $k \in \mathbb{N}$

question: find a subtree with at least k leaves.

- Min Spanning tree is P-time solvable.
- Finding a Spanning tree with $\geq k$ leaves: NP-h.

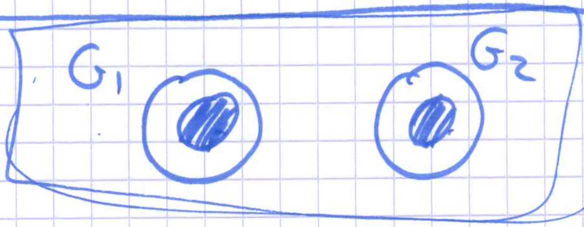
c. DS $\leq n-k$



- Finding a Spanning tree with $\leq k$ leaves: NP-h.
  ($k=2 \Leftrightarrow$ Hamiltonian Path)

Surprisingly:

- if G is connected, then MLST has a kernel with $O(k)$ vertices.
- in the general case: MLST has no PK (unless ...)



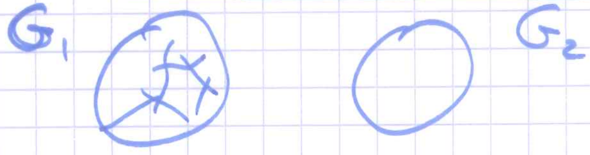
intuition "no PK for MLST"

suppose we have a kernel with k^c vertices for some constant c.

Remark: G_1, G_2 $k \in \mathbb{N}$.

disjoint union $G_1 \uplus G_2 =: G'$

G' has a ST with $\geq k$ leaves iff G_1 or G_2 has ST with $\geq k$ leaves.



Take $G_1, \dots, G_{2c+1}$ graphs, $k \in \mathbb{N}$.

$$G' := \biguplus_{i=1}^{2c+1} G_i$$

Apply the kernel on $\binom{G'}{k} \longrightarrow G^*, k^*$

G^* has a ST with $\geq k^*$ leaves

$\Leftrightarrow$ G' has a ST with $\geq k$ leaves

$\Leftrightarrow$ $\exists i$: G_i has a ST with k leaves.

$|V(G^*)| \leq k^c \rightsquigarrow$ we need $O(k^{2c})$ bits to encode G^*

$\Rightarrow$ we must have "lost" the information about one instance $\approx$ we have decided one instance

. MLST is unlikely to have a PK in general (disconnected graphs)

• MLST has a PK if the input graph is connected.

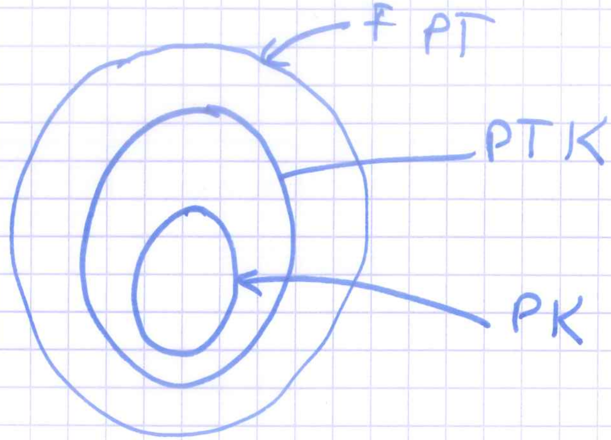
What we can do if G is disconnected:

conn. comp. $C_1, \ldots, C_q$

$\Rightarrow$ run the kernel on $G[C_i]$ $\forall i$

$C_1^* \ldots C_2^* \ldots C_q^*$ $\quad \sum_i |C_i^*| \leq ??$

Definition: a problem Q has a Turing kernel of size f, for some computable function f if there is an algorithm which decides whether an instance (x, k) is positive in polynomial time (in $|x| + k$) when given access to an oracle ~~thats~~ which decides the problem in a single step for every instance (y, k') such that $|y| + k' \leq f(k)$



* tools for "proving" $\notin$ FPT ✓
* tools for "proving" $\notin$ PK ✓

?? tools for proving $\notin$ PTK ??