Lecture 8.

Inclusion - Exclusion

$$\left|\bigcap_{i\in[n]} A_i\right| = \sum_{X\subseteq[n]} (-1)^{|X|} \left|\bigcap_{i\in X} U \setminus A_i\right|$$

HAMILTONIAN CYCLE

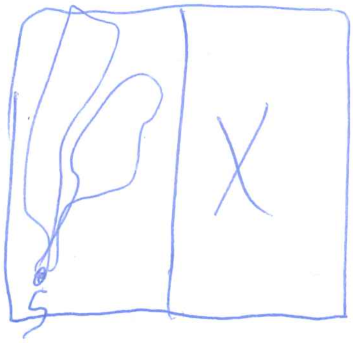
closed = last vertex is also s

U = set of all closed n-walks starting at a fixed s

A_v = subset of U but visiting v at least once

$\left|\bigcap_{v\in V(G)} A_v\right| = \#$ Ham cycles

$\left|\bigcap_{v\in X} U\setminus A_i\right|$ = closed n-walks starting at s avoiding X



graph G $\longrightarrow$ Adjacency matrix A_G of G

Compute A_G^n

$$\begin{array}{c|c} & j \quad s \\ \hline i & \#\text{walks } i\text{-}j \\ s & \quad\quad \boxed{\cdot} \end{array}$$

$\lceil \log n \rceil$ matrix multiplications [k-]Coloring: $f: V(G) \to [k]$ s.t. every pair of adjacent vertices $u, v \in V(G)$ satisfies $f(u) \neq f(v)$.

Q) Solve k-Coloring using DP. Think partition into k ind. sets

$$T[X] = \text{min \# colors necessary to color } G[X]$$

$X \subseteq V(G)$

$\forall v \in V(G)$, $T[\{v\}] = 1$

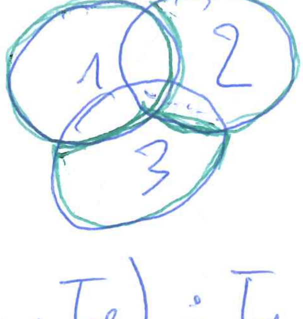
$$T[X] = 1 + \min_{\substack{S \subseteq X \\ S \text{ ind. set in } G}} T[X \setminus S]$$

What's the complexity of this algorithm

$$O^*\left(\sum_{k=0}^{n} \binom{n}{k} 2^k\right) = O^*(3^n)$$

Let's try incl-excl.

k-coloring = covering $V(G)$ with k-ind.-sets.



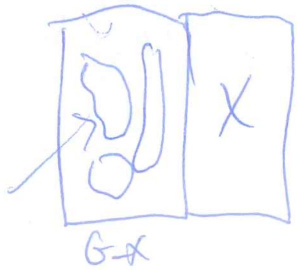
$$U = \{(I_1, \dots, I_k) : I_1, \dots, I_k \text{ ind. sets in } G\}$$

$v \in V(G)$

Q) A_v = subset of U s.t. $v \in I_1 \cup I_2 \cup \ldots \cup I_k$. $\left|\bigcap_{v \in V(G)} A_v\right| > 0 \Rightarrow \exists k$-coloring

$(I_1, \ldots, I_k)$

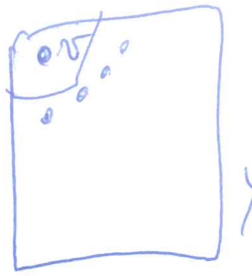
$$\left|\bigcap_{v \in X} U \setminus A_v\right|$$



$$\big(\#IS(\underbrace{G\_X}_{\text{vertex set of } G-X})\big)^k$$

How to compute $\#IS(Y) \longrightarrow$ DP

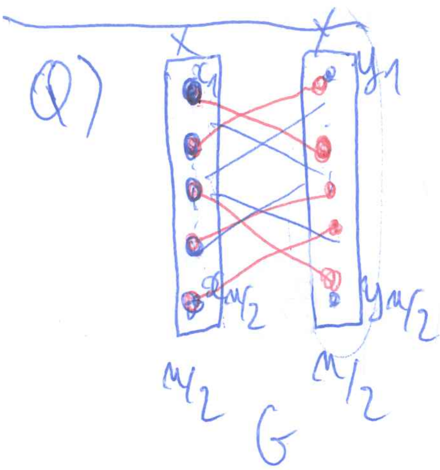
$$\#IS(Y) = \#IS[Y - \{v\}] + \#IS[Y \setminus N[v]]$$



2^n values in $\boxed{O^*(2^n)}$

overall running time

Q)



Find a formula giving # perfect matchings of G

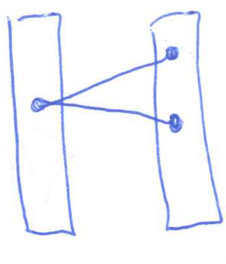
U = ~~sets of $\frac{n}{2}$ edges~~ $\forall x_i \longrightarrow y_{f(i)}$ f needs not to be bijective

A_v = subset of U where v is touched by an edge

$v \in Y$

$$\left|\bigcap_{v \in Y} A_v\right| = \#\text{perfect matchings}$$

$2^{n/2}$

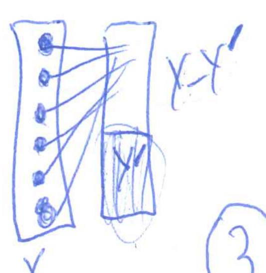


We need to compute 2^n values

$$\left|\bigcap_{v \in Y'} U \setminus A_v\right|$$

$O^*(2^{n/2})$

Ryser's formula



- k-Coloring $O^*(2^n)$.

~~k~~-colorings? $X \subseteq V(G)$ is X k-Colorable?

Zeta and Möbius transforms

$f : 2^{V(G)} \longrightarrow \mathbb{Z}$ (any ring $\mathcal{R}$)

$f(X)$ outputs a "value" linked to X.

Zeta transform: $(\zeta f)(X) = \sum_{Y \subseteq X} f(Y)$

Möbius tranform:
$$(\mu f)(X) = \sum_{Y \subseteq X} (-1)^{|X \setminus Y|} f(Y)$$

Convenient involution: $(\sigma f)(X) = (-1)^{|X|} f(X)$ $\boxed{\zeta\mu = \mu\zeta = id}$

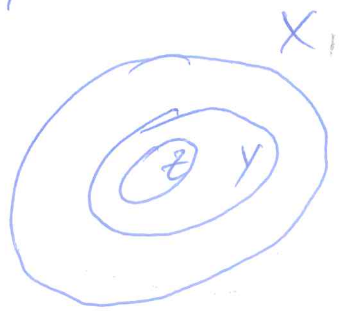
Prop: $\zeta = \sigma\mu\sigma$, $\mu = \sigma\zeta\sigma$.

$\forall f, \forall X$
$$(\sigma\mu\sigma f)(X) = \sigma\mu((-1)^{|X|} f(X))$$
$$= \sigma\Big(\sum_{Y \subseteq X} (-1)^{|X \setminus Y|} (-1)^{|Y|} f(Y)\Big)$$
$$= \sigma\Big(\sum_{Y \subseteq X} (-1)^{|X|} f(Y)\Big) = \sum_{Y \subseteq X} f(Y) = (\zeta f)(X)$$

$$\sigma\zeta\sigma = \underbrace{\sigma\sigma}_{id}\mu\underbrace{\sigma\sigma}_{id} = \mu$$ $\square$

<u>Theorem</u>: (Inversion Formula) $\zeta\mu = \mu\zeta = \mathrm{id}$.

Pf: $\forall f, X$ $(\mu\zeta f)(X) = (\sigma\zeta\sigma\zeta f)(X) = (-1)^{|X|}\sum_{Y\subseteq X}(\sigma\zeta f)(Y)$



$$= (-1)^{|X|}\sum_{Y\subseteq X}(-1)^{|Y|}\sum_{Z\subseteq Y} f(Z)$$

$$= (-1)^{|X|}\sum_{Z\subseteq X} f(Z)\sum_{\substack{Y\subseteq X\\ Y\supseteq Z}}(-1)^{|Y|}$$

$$= f(X) + (-1)^{|X|}\sum_{Z\subsetneq X} f(Z)\underbrace{\sum_{\substack{Y\subseteq X\\ Y\supseteq Z}}(-1)^{|Y|}}_{=0\,?} = f(X)$$

$$Z\subsetneq X \;\equiv\; Z\subseteq X,\ Z\neq X$$

$v \in X\setminus Z$

$Y^{\not\ni v} \longleftrightarrow Y\cup\{v\}$

$$\sum \underbrace{(-1)^{|Y|} + (-1)^{|Y\cup\{v\}|}}_{=0}$$

$$\zeta\mu = \sigma\mu\sigma\sigma\zeta\sigma = \sigma\mu\zeta\sigma = \sigma\sigma = \mathrm{id}. \;\square$$

$f(X)$ = # covers of X by ind. sets $(I_1, \ldots, I_k)$, i.e. $\bigcup_{j=1}^{k} I_j = X$

$$\zeta f(X) = \sum_{Y \subseteq X} f(Y)$$

$(I_1, \ldots, I_k)$ included in X

i.e. $\bigcup_{j=1}^{k} I_j \subseteq X$

$$\| $$

$\# IS(X)^k$ already computed.

All 2^n values of ζf can be computed in time $O^*(2^n)$

Q? How to retrieve $f(V(G))$?

$$(\mu \zeta f)(V(G)) = \sum_{Y \subseteq V(G)} \zeta f(Y) \rightarrow O^*(2^n)\text{-time}$$

~~Ask to~~ We can compute 2^n values of μf, ζf in time $O^*(2^n)$.

Pf: $\zeta f(X)$. $f(x_1, \ldots, x_n)$ $\in \{0,1\}^n$ $\begin{cases} x_i = 1 & \text{if } i \in X \\ x_i = 0 & \text{if } i \notin X \end{cases}$

$$\zeta f(x_1, \ldots, x_n) = \sum_{\substack{y_1, \ldots, y_n \\ \in \{0,1\}}} [y_1 \le x_1 \wedge y_2 \le x_2 \wedge \ldots \wedge y_n \le x_n] f(y_1, \ldots, y_n)$$

iverson bracket

$$\zeta_j(x_1,\dots,x_n) = \sum_{y_1,\dots,y_j \in \{0,1\}} [y_1 \le x_1 \wedge \dots \wedge y_j \le x_j] f(y_1,\dots,y_j,\underbrace{x_{j+1},\dots,x_n}_{\text{fixed}})$$

$$\zeta_0(x_1,\dots,x_n) = f(x_1,\dots,x_n) \quad \checkmark \quad \text{initialization}$$

$$\boxed{\zeta_n(x_1,\dots,x_n) = \zeta f(x_1,\dots,x_n)} \longrightarrow \text{what we want eventually.}$$

DP.

$$\zeta_j(x_1,\dots,x_j,x_{j+1},\dots,x_n) = \begin{cases} \zeta_{j-1}(x_1,\dots,x_n) & x_j = 0 \\ \zeta_{j-1}(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_n) + \zeta_{j-1}(x_1,\dots,x_{j-1},1,x_{j+1},\dots,x_n) & x_j = 1 \end{cases}$$

$$n \cdot 2^n = O^*(2^n)$$

From the 2^n values of $f \longrightarrow 2^n$ values of ζf in $O^*(2^n)$

μf? $\quad \mu = \sigma\zeta\sigma \longrightarrow 2^n$ values of μf in $O^*(2^n)$ $\square$

Convolution and cover product

Convolution: $(f * g)(X) = \sum_{\substack{A \cup B = X \\ A \cap B = \emptyset}} f(A) g(B) \longrightarrow O^*(3^n)$

Cover product: $(f *_c g)(X) = \sum_{A \cup B = X} f(A) g(B) \longrightarrow O^*(4^n)$

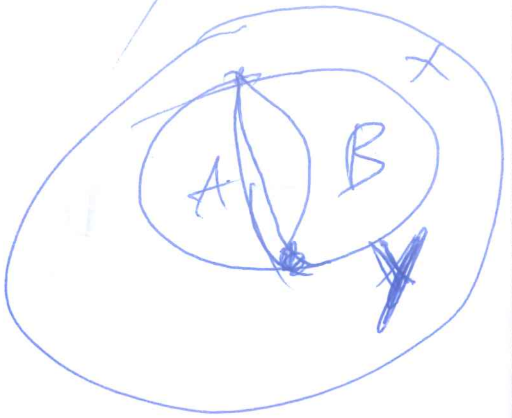
$\hookrightarrow O^*(2^n)$-time to compute $*, *_c$

$(f \cdot g)(X) = f(X) \cdot g(X)$ pointwise product.

<u>Th</u>: (Fast computing of $*_c$)

$$\zeta(f *_c g) = (\zeta f) \cdot (\zeta g)$$

Pf: $\mu(\zeta(f *_c g))(X) = \zeta\left(\sum_{A \cup B = X} f(A)\,g(B)\right)$



$$= \sum_{Y \subseteq X} \sum_{A \cup B = Y} f(A)\,g(B)$$

$$= \sum_{A \cup B \subseteq X} f(A)\,g(B)$$

$$= \left(\sum_{A \subseteq X} f(A)\right) \cdot \left(\sum_{B \subseteq X} g(B)\right) \neq \mu(\underline{\zeta f}) \cdot (\underline{\zeta g})(X)$$

$\hookrightarrow \zeta f *_c g$

apply μ

<u>Th</u>: (Fast subset convolution)

$$(f * g)(X) = \sum_{\substack{A \cup B = X \\ A \cap B = \emptyset}} f(A)\,g(B)$$

$f_i(A) = \begin{cases} f(A) & \text{if } |A| = i \\ 0 & \text{otherwise} \end{cases}$

$f_i(A) = [|A| = i]\, f(A)$

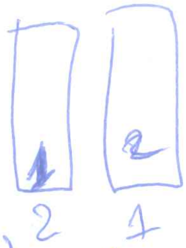
$$= \sum_{i=0}^{|X|} \sum_{A \cup B = X} f_i(A)\, g_{|X|-i}(B) = \sum_{i=0}^{|X|} (f_i *_c g_{|X|-i})(X)$$

8

back to k-Coloring

$f(X) = [X \text{ is an independent set}]$

Q) What is $(f*f)(X) = \sum_{\substack{A\cup B=X\\ A\cap B=\emptyset}} \underbrace{f(A)\,f(B)}_{=1}$ iff A and B are ind. sets in G



$$\underbrace{(f*f*\dots*f)}_{k \text{ times}}(X) = \sum_{A_1 \uplus A_2 \uplus \dots \uplus A_k = X} f(A_1)\,f(A_2)\dots f(A_k)$$

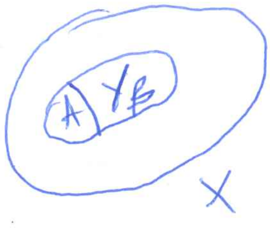
↑ disjoint union

Compute # k-colorings not only in G but every $G[X]$ in total time $O^*(2^n)$.

<ins>Exercise</ins>: Compute all 2^n values of

$$(f *_p g)(X) = \sum_{\substack{A\subseteq X\\ B\subseteq X\\ A\cap B=\emptyset}} f(A)\,g(B)$$

in $O^*(2^n)$-time?



. <ins>List Coloring</ins> G, $L(v)$ = available colors $\forall v \in V(G)$

color of v should be in $L(v)$

Check that what worked for k-Coloring also works for List k-Coloring