Lecture 8. Treewidth

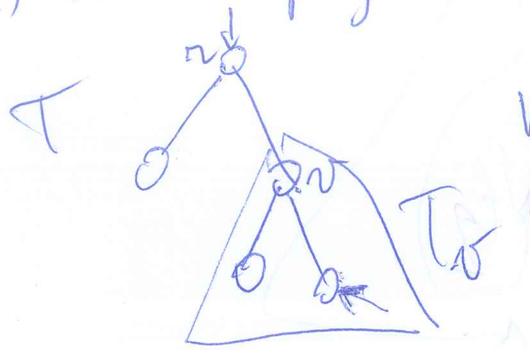
Weighted Max Independent Set

$w : V(G) \to \mathbb{R}^+$

Find S ind. set in G with max weight

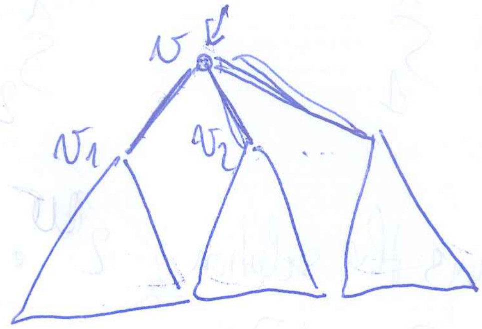
i.e. maximizing $w(S) := \sum_{v \in S} w(v)$

Q) Find a polynomial-time algorithm for WMIS on trees.



$WMIS[T_v] = \dots$

T_v the subtree of T rooted at v



$A[T_v] =$ weight of Max Ind set in T_v

$B[T_v] =$ ____________________ not containing v in T_v

v is a leaf: $A[T_v] = w(v)$, $B[T_v] = 0$

$\boxed{O(n)\text{-time}}$

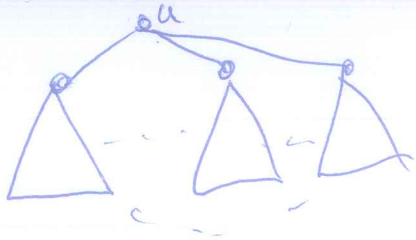
v is an internal node: $B[T_v] = \sum_{i=1}^{s} A[T_{w_i}]$

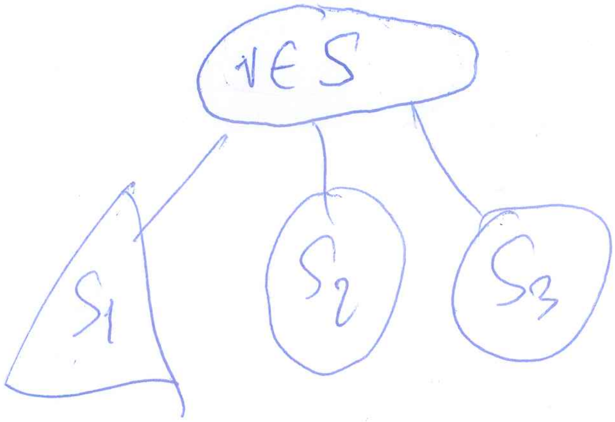
v has children $w_1, \dots, w_s$

$A[T_v] = \max \{ B[T_v], w(v) + \sum_{i=1}^{s} B[T_{w_i}] \}$

Crucial elements:

- $\boxed{\text{"independent" of the subinstances}}$
- not too many states A, B



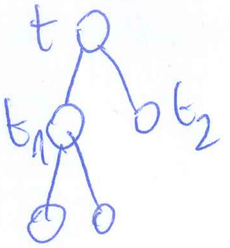


$G - S$ get disconnected pieces

$S \cap S_i$ could be non-empty

Treewidth

Tree decomposition of G: pair (T, β) $\quad \beta : V(T) \to 2^{V(G)}$



$\forall v \in V(G)$ (i) the $\beta(t)$, $t \in V(T)$ containing v, T_v it's a connected subtree ✓

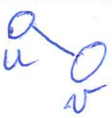
$\beta(t) := B_t$

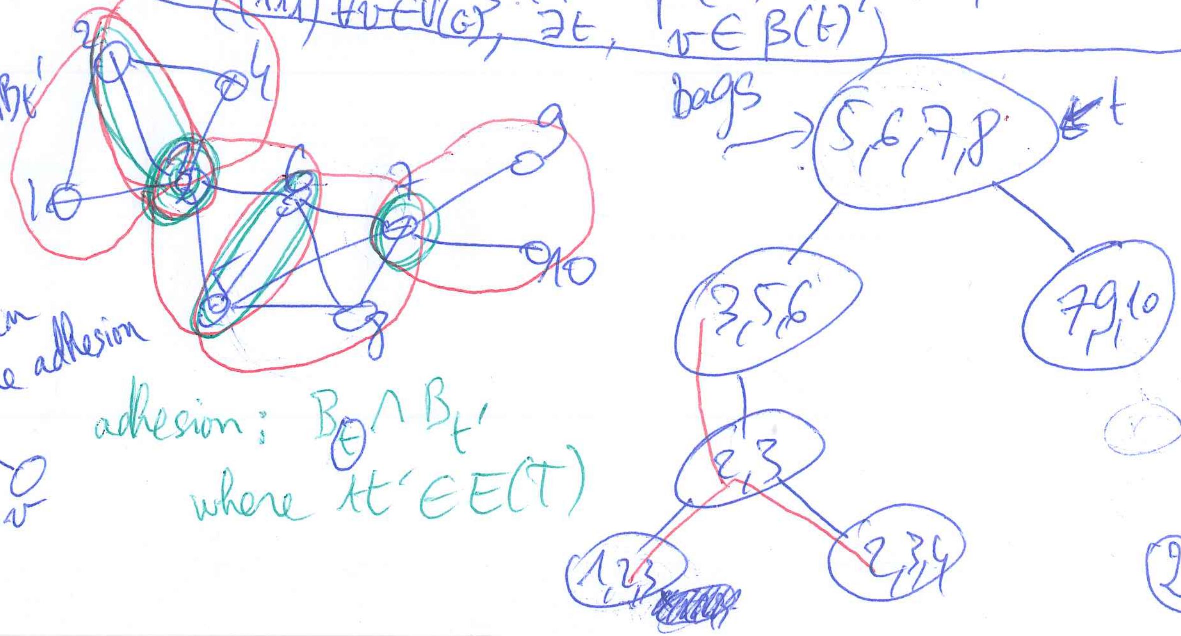
(ii) $\forall uv \in E(G)$, $\exists t \in V(T)$, $\beta(t) \supseteq \{u, v\}$ ✓

((iii) $\forall v \in V(G)$, $\exists t$, $v \in \beta(t)$)

$v \in B_t$ but not $B_t \cap B_{t'}$

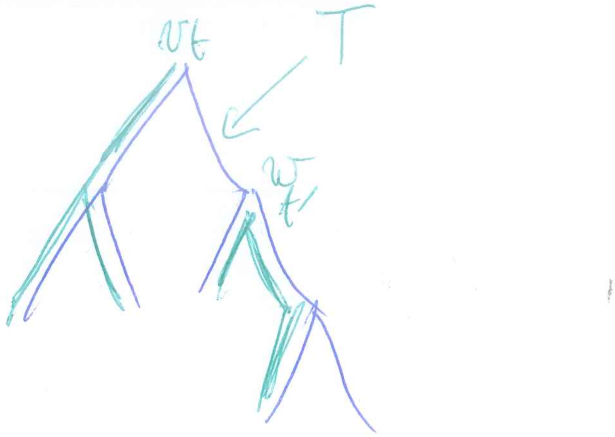
$v \in B_t$ not in the adhesion





adhesion: $B_t \cap B_{t'}$ where $tt' \in E(T)$

2



width of a tree decomposition : $\max_{t \in V(T)} |B_t| \underbrace{-1}_{\text{for trees to have treewidth 1}}$

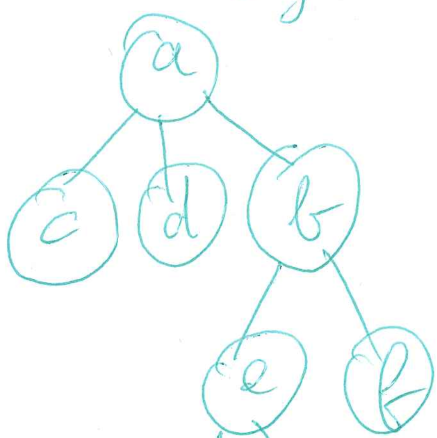
treewidth of G : is the minimum width of a tree decomposition of G.

denoted by $tw(G)$.

Q) treewidth of a tree?

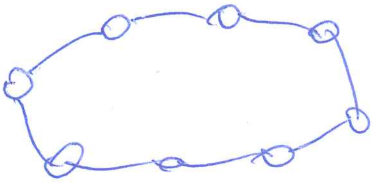
T $V(T), E(T)$

→ tree decomposition one bag for each edge

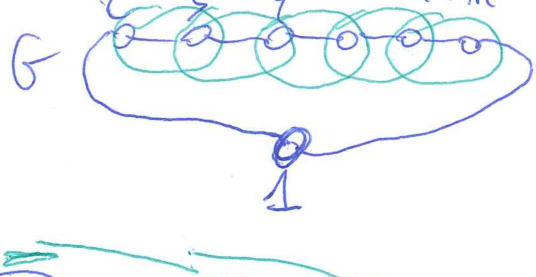


cycles?

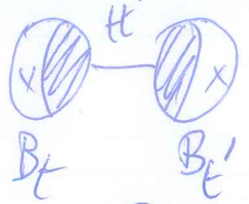
C_n



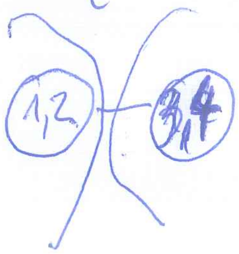
bags of max size 3.

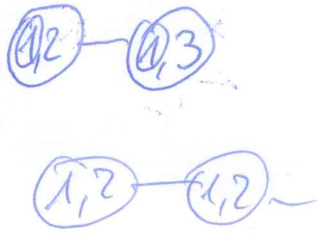


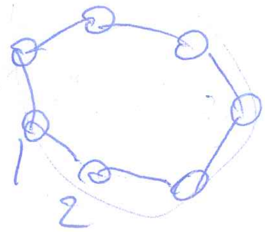
(1,2,3) — (1,3,4) — (1,4,5) — . . . — (1,n-1,n) → $tw(G) \le 2$.



adhesion disconnecting the graph





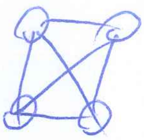


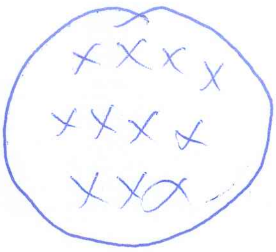
$tw(C_4) \geq 2$ otherwise cut-vertex

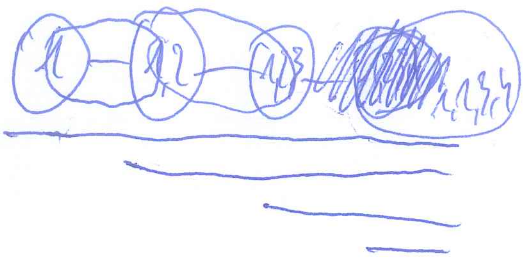
↳ $tw(C_n) = 2$

Cliques ?

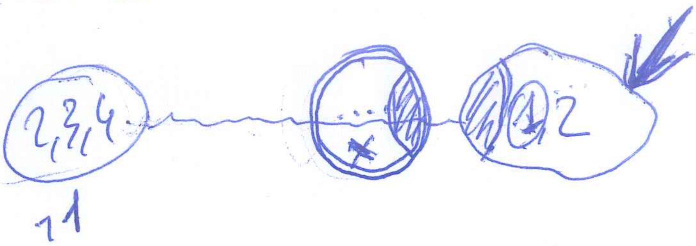
$tw(K_n)$?

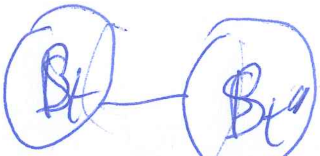




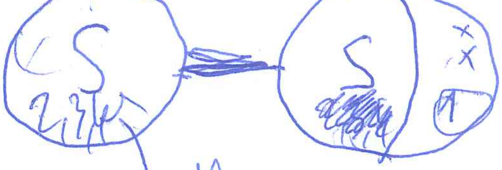


$tw(K_n) \leq n-1$





$B_t \subseteq B_{t'}$



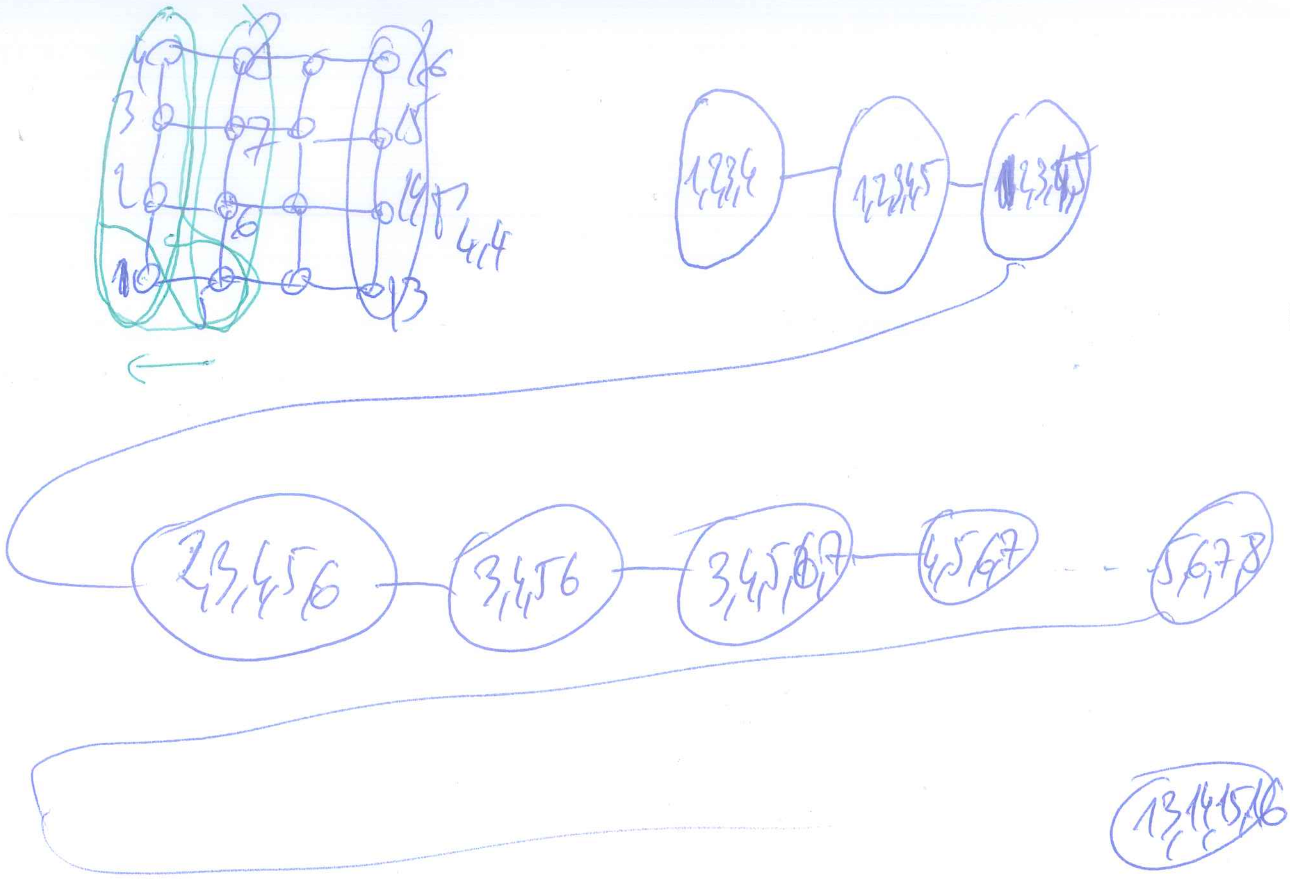
adhesion

$|S| \leq n-2$

$tw(K_n) \geq n-1 \rightarrow tw(K_n) = n-1$

Q) grids?

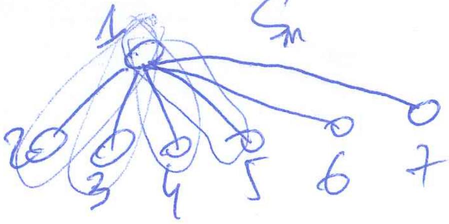
$\Gamma_{n,n}$



$tw(\Gamma_{n,n}) \leq n.$

Path decomposition, path width : same with T is a path

pathwidth

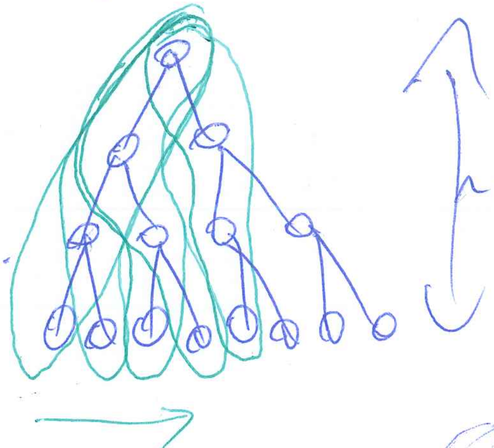


S_n

$pw(S_n) = tw(S_n) = 1$

pathwidth?

G.



$\log n$

$pw(G) \leq \lceil \log n \rceil$

$pw \leq tw \log n$

Nice path decomposition: $(S) - (S')$ $S\setminus S'$ is a singleton or $S'\setminus S$

$\emptyset - v - v,w - w - w,z - z - \emptyset$

$S - S' \rightarrow S - S\setminus\{a_1\} - S\setminus\{a_1,a_2\} \cdots I - I\cup b_1 \cdots I\cup\{b_1,\dots,b_r\} = S'$

$S\cap S' = I$

$A = S\setminus I$ $a_1,\dots,a_s$

$B = S'\setminus I$ $b_1,\dots,b_r$

Nice tree decomposition:

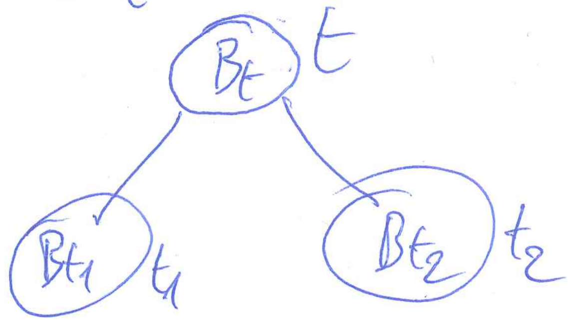
- leaf bags: $\emptyset$
- root bag $\emptyset$

binary tree

one child:

- t, t' introduce nodes $B_t = B_{t'}\cup\{v\}$
- t, t' forget node $B_t = B_{t'}\setminus\{v\}$, $v\in B_{t'}$

two children –

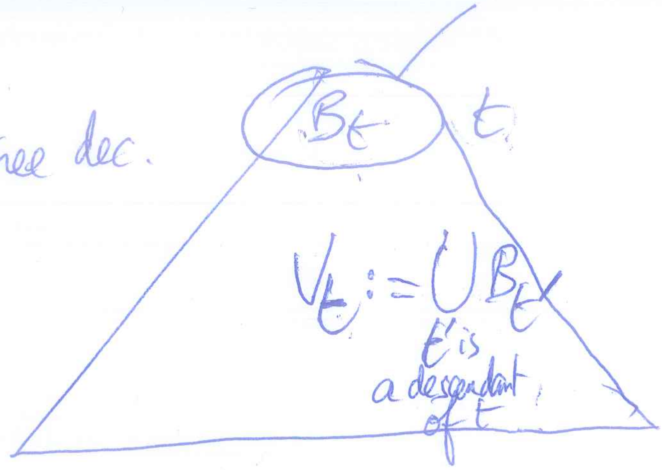


joint node $B_t = B_{t_1} = B_{t_2}$

Exercise for home: Turn any tree decomposition into a nice tree decomposition in polytime

Going back to WMIS

Dynamic programming over nice tree dec.



$B_t \subseteq V_t \subseteq V(G)$

G, w and $T, (B_t)_{t \in V(T)}$ a given tree decomposition of width w. WMIS in time $2^w n^{O(1)}$.

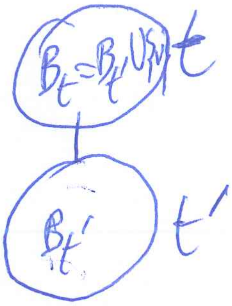
$IS[t, S \subseteq B_t]$ = value of a max ind. set $\hat{S}$ in $G[V_t]$, $\hat{S} \cap B_t = S$.

$t \in V(T)$

two cases $v \in S$, $v \notin S$

t leaf node: $IS[t, \emptyset] = 0$

t introduce node:



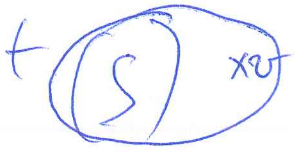
$$IS[t, S] = \begin{cases} IS[t', S] & v \notin S \\ IS[t', S \setminus \{v\}] + w(v) & v \in S \end{cases}$$

(?)

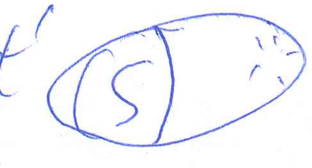
ind. set

Correctness $IS[t, S] = IS[t', S]$ $v \notin S$

$v \notin S$



$\hat{S}$ ind. set in $G[V_t]$, $\hat{S} \cap B_t = S$



$\hat{S}'$ ind. set in $G[V_{t'}]$, $\hat{S}' \cap B_{t'} = S$

$G[V_t] - v$

S ind. set $v \in S$ $\quad IS[t,S] = IS[t', S \setminus \{v\}] + w(v)$

- $IS[t,S] \leq IS[t', S \setminus \{v\}] + w(v)$

$\hat{S}$ ind. set in $G[V_t]$ with $\hat{S} \cap B_t = S$

$\hat{S}' := \hat{S} \setminus \{v\}$

$B_{t'} \cap \hat{S}' = S \setminus \{v\}$

$$w(\hat{S}') = w(\hat{S}) - w(v)$$

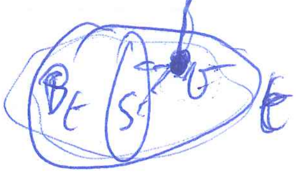
$$IS[t', S \setminus \{v\}] \geq IS[t,v] - w(v).$$

- $IS[t,S] \geq IS[t', S \setminus \{v\}] + w(v)$

$\hat{S}'$ ind. set intersecting $B_{t'}$ at precisely $S \setminus \{v\}$

$\hat{S} := \hat{S}' \cup \{v\}$ $\quad \hat{S} \cap B_t = \underbrace{\hat{S}' \cap B_{t'}}_{S \setminus \{v\}} \cup \{v\} = S.$ $\quad \in B_t \cap \hat{S}$

$\hat{S}$ ind. set?

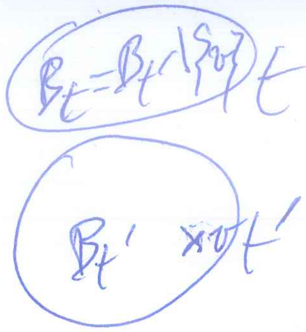


$w(\hat{S}) = w(\hat{S}') + w(v)$

no edge between v and $\hat{S} \setminus S$?

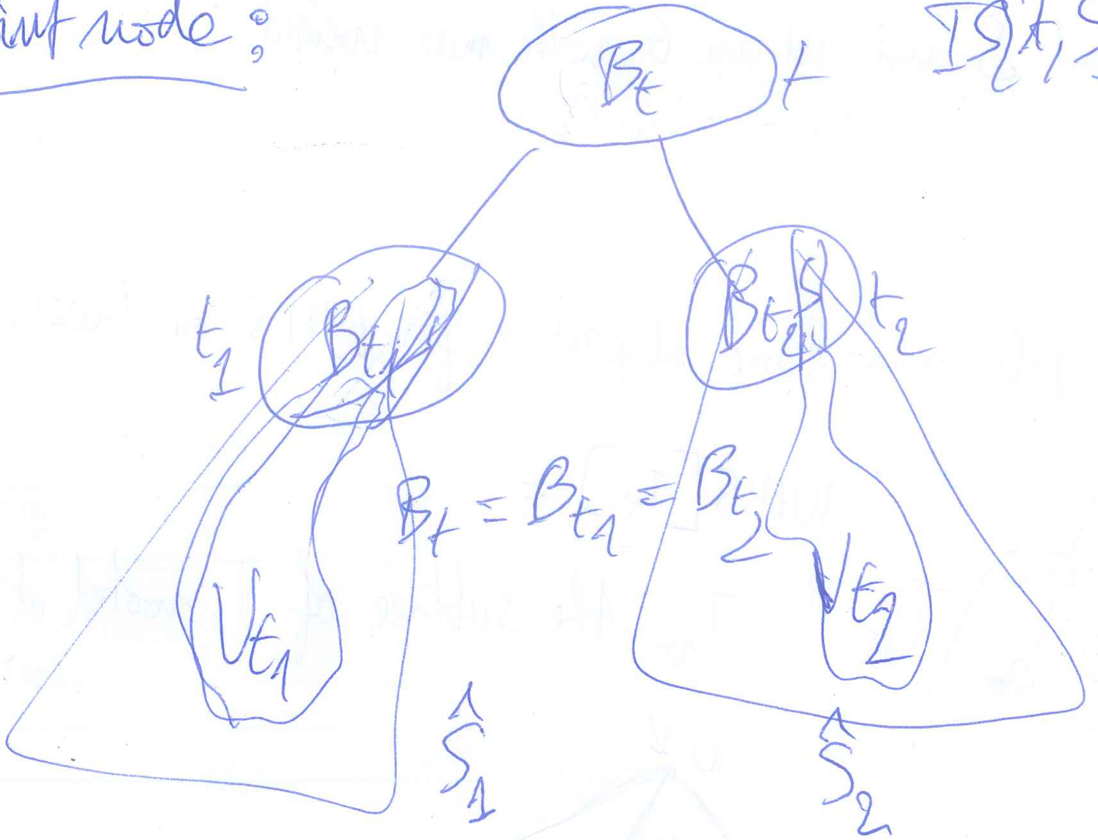
$\hat{S} \setminus B_t$

forget node:



$IS[t, S] = \max\{ IS[t', S], IS[t', S \cup \{v\}] \}$

joint node:



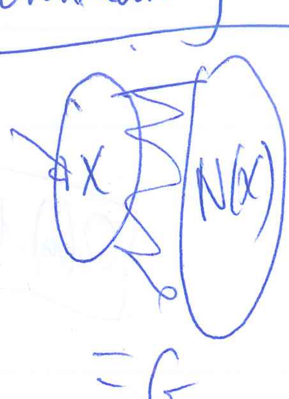
$IS[t, S] = IS[t_1, S] + IS[t_2, S] - w(S)$

$2^{w} \cdot |T|$

root contains the solution

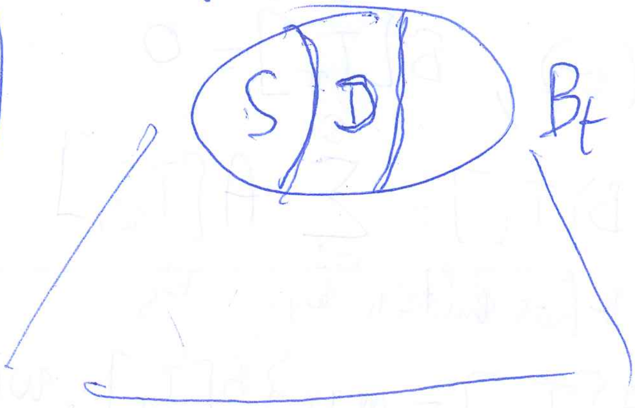
$IS[r, \phi]$ = max ind. set of $G[V_r]$
$V_r = V(G)$

algorithm
Dominating Set



$4^{w} \cdot O(1)$ time algorithm?
n

when given nice tree decomposition of width w



- algorithms for tree dec.
- brambles
- ...