CR 07: Parameterized and Exact Algorithms
Homework 1

Date: 05/10/2020, to send us by 19/10/2020
Total Marks: 20

- The 2 exercises are independent from each other. You may tackle them in any order.
- For each question, you may consider the statements of the previous questions as true even if you could not prove them.

1 Maximum Independent Set in restricted graph classes (10 marks)

We recall that an independent set in a graph is a set of pairwise non-adjacent vertices. We study the following classical problem:

MAXIMUM INDEPENDENT SET	**Parameter:** k
Input: A graph G and a positive integer k.	
Question: Is there an independent set $S \subseteq V(G)$ such that $\lvert S\rvert \geqslant k$?	

For a given (fixed) graph H, we say that a graph G is H-free if G does not contain H as an induced subgraph.

Q.1) A *triangle* is a clique of size 3. Prove that a triangle-free graph G on n vertices always contains an independent set of size at least $\lfloor\sqrt{n}\rfloor$, and such an independent set can be constructed in polynomial time. To do this, consider the two following cases: either G admits a vertex of degree at least $\lfloor\sqrt{n}\rfloor$, or every vertex has degree smaller than $\lfloor\sqrt{n}\rfloor$. 1 mark

Q.2) Deduce from the previous question that MAXIMUM INDEPENDENT SET admits a kernel with $O(k^2)$ vertices if its input graph is triangle-free. 0.5 marks

Q.3) By induction, generalize the result of Question **Q.1** to the case of K_t-free graphs, for every (constant) integer $t \geqslant 3$, that is graphs with no clique on t vertices. Deduce that in that case, MAXIMUM INDEPENDENT SET admits a kernel with $O(k^{t-1})$ vertices. 1.5 marks

We now consider the case of diamond-free graphs. A *diamond* is a graph on four vertices having all possible edges but one. The goal is to obtain a kernel with $O(k^3)$ vertices for MAXIMUM INDEPENDENT SET in diamond-free graphs.

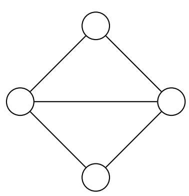

Figure 1: A diamond

Q.4) Prove that a diamond-free graph G on n vertices always contains either a clique or an independent set of size at least $\lfloor n^{1/3} \rfloor$, and this set can be constructed in polynomial time.
To do so, consider the two following cases: either G admits a vertex of degree at least $\lfloor n^{2/3} \rfloor$, or every vertex has degree smaller than $\lfloor n^{2/3} \rfloor$. For the first case, observe that the neighborhood of any vertex in a diamond-free graph induces a P_3-free graphs, where P_3 is the path on three vertices (how does a P_3-free graph look like?). 2 marks

In the following, we assume that (G, k) is an instance of Maximum Independent Set, and that G is diamond-free.

Q.5) Prove that if the algorithm of question **Q.4** applied to G returns an independent set, there is a direct kernel of the desired size. 1 mark

We now assume that the algorithm of question **Q.4** returns a clique C of G (of size at least $\lfloor n^{1/3} \rfloor$). We furthermore assume that C is maximal, that is, for every $v \in V(G) \setminus C$, $C \cup \{v\}$ is *not* a clique. Let N be the set of vertices of $V(G) \setminus C$ having at least one neighbor in C.

Q.6) Prove that every vertex in N has exactly one neighbor in C. 1 mark

Consider the following reduction rule:
Reduction Rule: *if* $|C| \geqslant k + 1$*, remove any vertex from* C*.*

Q.7) Prove that this reduction rule is safe. To do so, prove that if the instance is positive and $|C| \geqslant k + 1$, then for every $c \in C$, there is a solution avoiding c. 2 marks

Q.8) Prove that the previous reduction rule yields a kernel of the desired size. 1 mark

Q.9) Open question: find a kernel with $O(k^{3-\varepsilon})$ vertices for some $\varepsilon > 0$:-) ∞ marks

2 FPT and exacts algorithms for Activation Level (10 marks)

A *k-level assignment* of a simple (no parallel edges, no self-loops) undirected graph is a surjective mapping $L : V(G) \to \{1, 2, \dots, k\}$ such that for every vertex $v \in V(G)$, the neighborhood of v, $N_G(v)$, contains for every $i \in \{1, 2, \dots, L(v) - 1\}$ at least one vertex w such that $L(w) = i$. The integer $L(v)$ is called the *level* of vertex v. We see 1 as the lowest level, and k as the highest one. Informally, in a k-level assignment, every vertex is adjacent to at least one vertex of each lower level.

Activation Level	**Parameter:** k
Input: A graph G and a positive integer k.	
Question: Does G admit a k-level assignment?	

Q.1) Give a graph on k vertices admitting a k-level assignment, and argue that this is (up to isomorphism) the only such graph on at most k vertices. 1 mark

A *partial k-level assignment* is the same as a *k-level assignment* except not all the vertices are given a level. Formally it is a surjective mapping $L : S \to \{1, 2, \dots, k\}$, with $S \subseteq V(G)$, such that for every vertex $v \in S$ and for every $i \in \{1, 2, \dots, L(v) - 1\}$, there is a vertex $w \in N_G(v) \cap S$ such that $L(w) = i$. We say that the vertices of S *are given a level*, but not the vertices of $V(G) \setminus S$.

Q.2) Show that if a graph admits a partial k-level assignment, then it admits a k-level assignment. 1 mark

Q.3) Show that if a graph admits a k-level assignment, then it admits a partial k-level assignment where at most 2^{k-1} vertices are given a level. 2 marks

Q.4) Detail a randomized Monte-Carlo algorithm with one-sided error, solving ACTIVATION LEVEL on n-vertex graphs in time $k^{2^k} n^{O(1)}$. Justify the running time and the correctness of your algorithm. 3 marks

Q.5) Using dynamic programming, show how to solve ACTIVATION LEVEL on n-vertex graphs in time $3^n n^{O(1)}$. You may first observe that ACTIVATION LEVEL has an equivalent formulation in terms of vertex-partitioning and dominating sets. 3 marks