CR 07: Parameterized and Exact Algorithms
Homework 2

Date: 27/10/2020, to send us by 15/11/2020
Total Marks: 20

- The 2 exercises are independent from each other. You may tackle them in any order.
- For each question, you may consider the statements of the previous questions as true even if you could not prove them.

Several questions ask for a reduction of some sort (an FPT reduction or an OR-composition). We encourage you to draw figures for these questions. A figure together with a concise formal description of it provide the necessary redundancy to error-correct potential typos. Also, it is a pedagogic approach if you want to be easily read. If your figure is good, your reader will understand the global strategy by looking at it, and check the details by reading the textual description. More generally, every time you think an idea is better conveyed by a picture than by words, we encourage you to draw.

1 k-Dominating Set on some restricted classes of graphs (17 marks)

In this section, the graphs are undirected, unweighted, and simple (no self-loops, no multiple edges). We recall the definition of the k-Dominating Set problem.

k-Dominating Set	**Parameter:** k
Input: A graph G and a positive integer k.	
Question: Is there a set $S \subseteq V(G)$ such that $\|S\| \leqslant k$ and for all $v \in V(G) \setminus S$, the vertex v has at least one neighbor in S?	

We admit that k-Dominating Set is W[2]-hard; you can use this fact without a proof. We recall that if there is an FPT reduction from k-Dominating Set to a problem Π, then Π is also W[2]-hard (hence, unlikely to admit an FPT algorithm). By k-Dominating Set *restricted to a class of graphs* X, we mean the same problem where the input graph G is assumed to belong to X.

We also recall that the *degree* $d(v)$ of a vertex v is its number of neighbors. The *degree* Δ of a graph G is the maximum of $d(v)$ taken over all $v \in V(G)$.

Q.1) Let Δ be a fixed positive integer. Prove that k-Dominating Set restricted to graphs of maximum degree Δ admits a linear kernel. 1 mark

Q.2) Give an FPT algorithm running in time $(\Delta+1)^k|V(G)|^{O(1)}$ for k-Dominating Set when restricted to graphs G of maximum degree Δ. Briefly justify its correctness and running time. 1.5 marks

The *distance* $d(u,v)$ between two vertices u and v is the number of edges in a shortest path from u to v. The *diameter* of a graph is the shortest distance between two farthest vertices, that is, $\max_{u,v \in V(G)} d(u,v)$. For instance, the diameter of a clique is 1, and the diameter of a path on n vertices is $n-1$. A *split graph* is a graph G whose vertex set $V(G)$ can be bipartitioned into a clique A and an independent set B (see figure 1 for an example).

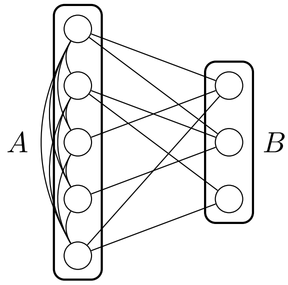

Figure 1: Example of a split graph with bipartition (A, B).

Q.3) Show that k-Dominating Set restricted to split graphs is W[2]-hard (i.e., as hard as on the class of all graphs). Deduce that k-Dominating Set restricted to graphs with diameter at most 3 is W[2]-hard. 2 marks

We will now try to strengthen the previous result to the more restricted class of split graphs with diameter at most 2. The next two questions will help us achieve that goal.

Q.4) Characterize "diameter at most 2" for split graphs, by an equivalent property on the neighborhoods of vertices of the independent set B. 1 mark

Q.5) Show that the multicolored version of k-Dominating Set, where the vertex set $V(G)$ is partitioned into $V_1 \uplus V_2 \uplus \ldots \uplus V_k$ and one looks for a dominating set intersecting each V_i exactly once, is also W[2]-hard. 2 marks

Q.6) Using the previous three questions, prove that k-Dominating Set restricted to split graphs with diameter at most 2 is W[2]-hard. 3 marks

A *triangle* is a clique on three vertices, that is, x, y, z such that xy, xz, and yz are edges. A graph is said *triangle-free* if it does not contain any triangle.

Q.7) Is k-Dominating Set restricted to triangle-free graphs FPT or W[2]-hard? Justify your answer. 2 marks

We now consider an alternative parameterization of k-Dominating Set restricted to split graphs: the size of the clique, that is, $|A|$.

Q.8) Prove that k-Dominating Set restricted to split graphs and parameterized by the size of the clique, $|A|$, is FPT. 0.5 marks

We will now show that a polynomial kernel for the problem of question **Q.8** is unlikely.

Q.9) Draw a split graph G where the clique A and the independent set B have $2n$ vertices each, and A is partitioned into $A_1 \uplus A_2$ with $|A_1| = |A_2| = n$ such that A_1 and A_2 are the only dominating sets of G of size at most n. 1 mark

The previous question gives a useful gadget to tackle the following.

Q.10) Prove that k-Dominating Set restricted to split graphs and parameterized by $|A|$ has no polynomial kernel, unless NP $\subseteq$ coNP/poly. To do so, you may perform an OR-cross-composition from the same problem: k-Dominating Set restricted to split graphs. You may use the fact that the clique of your output instance can be of size $n_A + O(n_B \cdot \log_2 t)$ (where t is the number of input instances, n_A is the maximum size of the cliques among the input instances, and n_B is the maximum size of the independent sets among the input instances), and use $O(n_B \cdot \log_2 t)$ of these vertices to encode (using gadgets from the previous question) the binary representation of every integer $i \in \{1, 2, \ldots, t\}$.

3 marks

2 k-Coloring parameterized by treewidth (3 marks)

We recall that the k-Coloring problem asks, given a graph, for a partition of its vertex set into k (some possibly empty) independent sets.

Q.1) Given a graph G with a nice tree decomposition of width t (and polynomially many nodes), present an algorithm solving k-Coloring in time $k^{t+1}|V(G)|^{O(1)}$. Only justify the correctness for the case of an introduce node (i.e., a node whose only child holds the same bag minus one vertex). 3 marks