Large Finite Point Sets Have 4 Collinear Points or a 6-Clique

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Abstract

We prove that every finite point set of size at least $10^{11055931}$ has four collinear points or six points that pairwise see each other. This resolves the first open case of the big-line-big-clique conjecture of Kára, Pór, and Wood.

1 Introduction

Given a point set $P \subseteq \mathbb{R}^2$, two distinct points $p, p' \in P$ *see* each other or are *visible* (implicitly, with respect to P) if the only points of P on the line segment $\overline{pp'}$ are p and p'. If instead there is a point $q \in P \setminus \{p, p'\}$ that lies on $\overline{pp'}$, we say that q *blocks* the pair p, p'. The visibility graph of P, denoted by $\mathcal{V}(P)$, has vertex set P and an edge between two distinct points of P whenever they see each other in P. We denote by χ and ω the chromatic number and the clique number, respectively.

Kára, Pór, and Wood conjectured that every sufficiently large finite planar point set has many collinear points or many pairwise visible points [7]. This is commonly referred to as the big-line-big-clique conjecture.

Conjecture 1 (Big-line-big-clique [7])**.** *For all positive integers k, ℓ, there is an integer $n := n(k, \ell)$ such that every finite point set $P \subseteq \mathbb{R}^2$ of size at least n has ℓ collinear points or $\omega(\mathcal{V}(P)) \geqslant k$.*

The big-line-big-clique conjecture fails for infinite point sets: there is an infinite point set without four collinear points whose visibility graph is triangle-free [10]. A simple projection argument shows that the conjecture is equivalent if stated in $\mathbb{R}^d$ for any dimension $d \geqslant 2$ [7].

The conjecture is trivial for $\ell = 3$ (thus $\ell \leqslant 3$) and every k because if no three points are collinear, then $\mathcal{V}(P)$ is a complete graph. The authors of Conjecture 1 showed the case $k \leqslant 4$ and every ℓ [7]. The case $k = 5$ and $\ell = 4$ was established [2]. Then Conjecture 1 was proved for $k = 5$ and every ℓ [1], and later the bound in ℓ was improved to $\Theta(\ell^2)$ [3]. Prior to our work, the conjecture was open for all the other pairs (k, ℓ).

In this note, we show the first open case: $k = 6$ and $\ell = 4$. This marks the first resolution of an open case since 2009, when the arXiv version of [1] appeared.

Theorem 1. *Every finite point set P of size at least $10^{11055931}$ contains 4 collinear points or $\omega(\mathcal{V}(P)) \geqslant 6$.*

We rely on the following weakening of Theorem 1, due to Hujter and Kisfaludi-Bak.

Theorem 2 ([6])**.** *Every finite point set P of size at least 2311 contains 4 collinear points or $\chi(\mathcal{V}(P)) \geqslant 6$.*

Visibility graphs of point sets are not χ-bounded: they can have arbitrarily large chromatic number and clique number 6 [8]. This rules out a general approach to Conjecture 1 solely based on showing its chromatic weakening (where ω is replaced by χ).

Proof outline. Let $P \subseteq \mathbb{R}^2$ be any finite point set without four collinear points. We set $G := \mathcal{V}(P)$, and further assume that $\omega(G) < 6$. We upper-bound $n := |P|$ to prove the contrapositive of Theorem 1.

We first totally order P: $p_1, p_2, \dots, p_n \in P$ such that every interval $p_i, p_{i+1}, \dots, p_j$ induces the same visibility graph as in G. (The order is the left-to-right order on an injective projection of P onto a line.) By the result of Hujter and Kisfaludi-Bak (Theorem 2), any sufficiently long interval I cannot be made 5-partite (i.e., of chromatic number at most 5) by removing less than a constant fraction of I. Otherwise, a still large remaining subinterval of I contradicts Theorem 2.

As a consequence of a vertex-removal version of the Erdős–Simonovits stability theorem (Lemma 2), this implies that $G[I] = \mathcal{V}(I)$ has at most $(\frac{2}{5} - \varepsilon_1)|I|^2$ edges for some constant $\varepsilon_1 > 0$. Passing to the complement $B := \overline{G}$, we get that $B[I]$ has at least $c_1|I|^2 - \mathcal{O}(|I|)$ edges for some constant $c_1 > \frac{1}{10}$. The exact value $\frac{1}{10}$ and the strict inequality will matter.

Next, we introduce the following potential, intending to show conflicting upper and lower bounds on it in terms of n,

$$W := \sum_{p_i p_j \in E(B), i<j} \frac{1}{j-i},$$

summing the reciprocal of the positive *stretch* for every nonvisible pair $p_i, p_j \in P$. Observe that $\sum_{1 \leqslant i < j \leqslant n} \frac{1}{j-i} = n \ln n + \mathcal{O}(n)$. Now, because P has no four collinear points, we can show that every unit of potential from a nonvisible pair p_i, p_j forces four units of potential from *visible* pairs; those five units are only used for p_i, p_j and no other nonvisible pair. Therefore, $W \leqslant \frac{1}{5}\, n \ln n + \mathcal{O}(n)$.

We now wish to leverage the lower bound on $|E(B[I])|$ for every sufficiently large interval I to get a lower bound on W. Partitioning $[n]$ into intervals of length m, one gets $\left(\frac{n}{m} - \mathcal{O}(1)\right)\left(c_1 m^2 - \mathcal{O}(m)\right) = c_1 nm - \mathcal{O}(n + m^2)$ edges within these intervals. Note that

$$A_m := \sum_{p_i p_j \in E(B),\ j-i \in [m-1]} \left(1 - \frac{j-i}{m}\right)$$

is the expected number of edges of B within the intervals, thus in particular, $A_m \geqslant c_1 nm - \mathcal{O}(n + m^2)$. It can be shown that $\sum_{m=j-i+1}^{n} \frac{2}{m^2-1}\left(1 - \frac{j-i}{m}\right) \leqslant \frac{1}{j-i}$. By summing this inequality over every nonvisible pair p_i, p_j and swapping the summation order, we get

$$W \geqslant \sum_{m=2}^{n} \frac{2}{m^2-1} A_m.$$

With the preceding lower bound on A_m, this gives $W \geqslant 2c_1 n \sum_{m=2}^{n} \frac{1}{m} - \mathcal{O}(n) = 2c_1 n \ln n - \mathcal{O}(n)$. We conclude that n cannot be arbitrarily large from the upper bound on W of the previous paragraph, since $2c_1 > \frac{1}{5}$. The section organization follows the paragraphs of the outline.

2 Intervals and lower bound on distance to 5-colorability

We use $[n]$ as a shorthand for $\{1, \dots, n\}$. We use the standard graph-theoretic notation. For any graph H, $V(H)$ and $E(H)$ denote the vertex set and edge set, respectively, of H. If $S \subseteq V(H)$, then $H[S]$ denotes the subgraph of H induced by S, and $H - S := H[V(H) \setminus S]$.

Fix a non-vertical line L such that the orthogonal projection π of P onto L gives $|P|$ distinct points. Let $p_1, \dots, p_n$ enumerate the points of P by the left-to-right order of their projections $\pi(p_1), \dots, \pi(p_n)$. We now call any set of consecutive points $\{p_i, p_{i+1}, \dots, p_j\}$, with $i, j \in [n]$ and $i \leqslant j$, an *interval* of P.

Observation 1. *For every interval I of P, it holds that $\mathcal{V}(I) = G[I]$.*

Proof. Two points of I that are visible in P are a fortiori visible in I. Conversely, if two points p_i, p_j of I are blocked by a point $q \in P$, then the projection $\pi(q)$ lies on the line segment $\pi(p_i)\pi(p_j)$. Thus $q \in I$, and p_i and p_j do not see each other in I either. $\square$

Let $n_{\mathrm{HKB}} := 2310$, where the subscript stands for the authors of Theorem 2. The latter theorem can be rephrased as follows: every finite point set with no four collinear points and with a 5-colorable visibility graph has size at most n_{HKB}. We set

$$\varepsilon_0 := \frac{1}{2(n_{\mathrm{HKB}}+1)}.$$

We show that, due to Theorem 2, the visibility graph of every sufficiently large interval of P cannot be made 5-colorable by removing less than an ε_0 fraction of its vertices.

Lemma 1. *For every interval I of P with $|I| \geqslant 2n_{\mathrm{HKB}}$, and every $Z \subseteq I$,*

$$\chi(G[I]-Z) \leqslant 5 \ \textit{ implies that } \ |Z| \geqslant \varepsilon_0 |I|.$$

Proof. Indeed, assume that $\chi(G[I]-Z) \leqslant 5$. Let $z := |Z|$. The set Z cuts the interval I in at most $z+1$ subintervals. More precisely, $I \setminus Z$ is the disjoint union of at most $z+1$ intervals $I_1, \dots, I_h$ of P.

For every $j \in [h]$, the interval I_j has no four collinear points, by assumption on P. Furthermore $\mathcal{V}(I_j) = G[I_j]$ by Observation 1. Therefore,

$$\chi(\mathcal{V}(I_j)) = \chi(G[I_j]) \leqslant \chi(G[I]-Z) \leqslant 5,$$

where the first inequality holds since $G[I_j]$ is an induced subgraph of $G[I]-Z$. (Note that it is irrelevant that the latter graph is not necessarily a visibility graph.)

The previous paragraph implies, by Theorem 2, that $|I_j| \leqslant n_{\mathrm{HKB}}$. Consequently,

$$|I| - z \leqslant h n_{\mathrm{HKB}} \leqslant (z+1) n_{\mathrm{HKB}}.$$

Thus,

$$z \geqslant \frac{|I| - n_{\mathrm{HKB}}}{n_{\mathrm{HKB}}+1} \geqslant \frac{|I|}{2} \cdot \frac{1}{n_{\mathrm{HKB}}+1} = \varepsilon_0 |I|,$$

where the second inequality holds because $|I| \geqslant 2n_{\mathrm{HKB}}$. □

3 Turán stability and edge density

For any graph H, we use $e(H)$ as a shorthand for $|E(H)|$, i.e., the number of edges of H. We denote by $\mathrm{ex}(m, K_6)$ the maximum number of edges of a K_6-free (i.e., without a 6-vertex clique) m-vertex graph. We need a vertex-removal version of the Erdős–Simonovits stability theorem [5, 4, 11]. The proof of [9, Lemma 2.3] gives the following explicit form for $r = 5$.

Lemma 2 ([9, Lemma 2.3] for $r = 5$)**.** *For every $\varepsilon \in (0, 1/3750)$, every K_6-free graph H*

$$\textit{with } m \geqslant 20 \textit{ vertices and more than } \mathrm{ex}(m, K_6) - \varepsilon m^2 \textit{ edges}$$

admits a set Z of fewer than $3500\varepsilon m$ vertices such that $\chi(H-Z) \leqslant 5$.

We get the following useful consequence from the previous two lemmas.

Lemma 3. *Set $\varepsilon_1 := \varepsilon_0/3500$. For every interval I of P with $m := |I| \geqslant 2n_{\mathrm{HKB}}$,*

$$e(G[I]) \leqslant \mathrm{ex}(m, K_6) - \varepsilon_1 m^2.$$

Proof. We apply Lemma 2 with $\varepsilon := \varepsilon_1 < 1/3750$ and $H := \mathcal{V}(I) = G[I]$. The conclusion that there is some set $Z \subseteq V(H)$ such that $|Z| < \varepsilon_0 m$ and $\chi(H-Z) \leqslant 5$ is ruled out by Lemma 1. Since H is K_6-free (as an induced subgraph of G) and $m = |V(H)| \geqslant 20$, we have $e(H) \leqslant \mathrm{ex}(m, K_6) - \varepsilon_1 m^2$. □

We now translate Lemma 3 into a lower bound for $B := \overline{G}$, the complement of G.

Corollary 1. *Set* $c_1 := \frac{1}{10} + \varepsilon_1$*. For every interval* I *of* P *with* $m := |I| \geqslant 2n_{\mathrm{HKB}}$*,*

$$e(B[I]) \geqslant c_1 m^2 - \frac{m}{2}.$$

Proof. By Lemma 3,

$$e(G[I]) \leqslant \mathrm{ex}(m, K_6) - \varepsilon_1 m^2 \leqslant \left(1 - \frac{1}{5}\right)\frac{m^2}{2} - \varepsilon_1 m^2 = \frac{2}{5}m^2 - \varepsilon_1 m^2,$$

where the second inequality is by Turán's theorem [12]. Thus,

$$e(B[I]) \geqslant \binom{m}{2} - \frac{2}{5}m^2 + \varepsilon_1 m^2 = \frac{m^2}{2} - \frac{2}{5}m^2 + \varepsilon_1 m^2 - \frac{m}{2} = \left(\frac{1}{10} + \varepsilon_1\right)m^2 - \frac{m}{2}. \qquad \square$$

The constant $\frac{1}{10}$ and the strict inequality $c_1 > \frac{1}{10}$ will turn out to be crucial.

4 Harmonic upper bound

We define the weight of a pair of points $p_i, p_j \in P$, with $i < j$, by $w(p_i, p_j) := \frac{1}{j-i}$. Note that every weight is positive. We introduce

$$W := \sum_{\substack{p_i p_j \in E(B)\\ i<j}} w(p_i, p_j) = \sum_{\substack{p_i p_j \in E(B)\\ i<j}} \frac{1}{j-i}.$$

Thus, W is the sum of weights of pairs of points that do not see each other in P. The rest of the proof consists of showing conflicting upper and lower bounds on W, resulting in an upper bound on $n = |P|$.

In this section, we show the following upper bound on W. The proof hinges on the absence of four collinear points in P.

Lemma 4. $W \leqslant \frac{1}{5} n \ln n + \frac{n}{5}$.

Proof. Every nonvisible pair $p_i, p_j \in P$ with $i < j$ admits a *unique* blocker $p_k \in P$, for otherwise there would be four points of P on a single line. We denote by $k(i,j)$ the index of this unique blocker. Furthermore, we have $i < k(i,j) < j$. We charge the weight of (p_i, p_j) to the weights of the three pairs $(p_i, p_{k(i,j)})$, $(p_{k(i,j)}, p_j)$, and (p_i, p_j). Note that these three pairs cannot be charged by another nonvisible pair: this would again imply the existence of four collinear points in P.

We set $a := k(i,j) - i$ and $b := j - k(i,j)$. Thus, $a + b = j - i$. Therefore,

$$w(p_i, p_{k(i,j)}) + w(p_{k(i,j)}, p_j) + w(p_i, p_j) = \frac{1}{a} + \frac{1}{b} + \frac{1}{a+b}.$$

Since $(a-b)^2 \geqslant 0$, we have $a^2 + b^2 \geqslant 2ab$, so $\frac{a}{b} + \frac{b}{a} \geqslant 2$, as $a, b > 0$. This implies that

$$\frac{a+b}{b} + \frac{a+b}{a} \geqslant 4, \quad \text{thus} \quad \frac{1}{a} + \frac{1}{b} \geqslant \frac{4}{a+b}.$$

Therefore,

$$w(p_i, p_{k(i,j)}) + w(p_{k(i,j)}, p_j) + w(p_i, p_j) \geqslant \frac{4}{a+b} + \frac{1}{a+b} = \frac{5}{a+b} = 5\, w(p_i, p_j).$$

We conclude that

$$W = \sum_{\substack{p_i p_j \in E(B)\\ i<j}} w(p_i, p_j) \leqslant \frac{1}{5} \sum_{\substack{p_i p_j \in E(B)\\ i<j}} \left(w(p_i, p_{k(i,j)}) + w(p_{k(i,j)}, p_j) + w(p_i, p_j)\right) \leqslant \frac{1}{5} \sum_{1 \leqslant i < j \leqslant n} \frac{1}{j-i},$$

where the second inequality holds because the weights $w(p_i, p_{k(i,j)}), w(p_{k(i,j)}, p_j), w(p_i, p_j)$ are used only once, by the argument in the first paragraph. Finally,

$$\sum_{1 \leqslant i < j \leqslant n} \frac{1}{j-i} \leqslant n \sum_{1 \leqslant i \leqslant n-1} \frac{1}{i} \leqslant n \ln n + n, \quad \text{and} \quad W \leqslant \frac{1}{5} n \ln n + \frac{n}{5}. \qquad \square$$

Again, the exact coefficient $\frac{1}{5}$ of $n \ln n$ (and the fact that it is twice $\frac{1}{10}$) will be crucial.

5 Harmonic lower bound

We say that a pair $p_i, p_j \in P$ has *stretch* d if $|i-j| = d$. We first need the following preparatory lemma, which determines the coefficients λ_m for which $\sum_{m>d} \lambda_m \left(1-\frac{d}{m}\right)$ is equal to $\frac{1}{d}$, i.e., to the weight of a pair p_i, p_j of stretch d.

Lemma 5. *For every positive integer d,*

$$\sum_{m=d+1}^{+\infty} \frac{2}{m^2-1}\left(1-\frac{d}{m}\right) = \frac{1}{d}.$$

Proof. First note that $\frac{2}{m^2-1} = \frac{1}{m-1} - \frac{1}{m+1}$. Thus,

$$\sum_{m=d+1}^{+\infty} \frac{2}{m^2-1}\left(1-\frac{d}{m}\right) = \sum_{m=d+1}^{+\infty}\left(\frac{1}{m-1}-\frac{1}{m+1}\right) - d\left(\sum_{m=d+1}^{+\infty}\frac{1}{m(m-1)} - \sum_{m=d+1}^{+\infty}\frac{1}{m(m+1)}\right).$$

By telescoping,

$$\sum_{m=d+1}^{+\infty}\left(\frac{1}{m-1}-\frac{1}{m+1}\right) = \frac{1}{d} + \frac{1}{d+1}.$$

Moreover, $\frac{1}{m(m-1)} = \frac{1}{m-1} - \frac{1}{m}$, so

$$\sum_{m=d+1}^{+\infty}\frac{1}{m(m-1)} = \sum_{m=d+1}^{+\infty}\left(\frac{1}{m-1}-\frac{1}{m}\right) = \frac{1}{d} \quad \text{and} \quad \sum_{m=d+1}^{+\infty}\frac{1}{m(m+1)} = \frac{1}{d+1}.$$

Therefore,

$$\sum_{m=d+1}^{+\infty} \frac{2}{m^2-1}\left(1-\frac{d}{m}\right) = \frac{1}{d} + \frac{1}{d+1} - \frac{d}{d} + \frac{d}{d+1} = \frac{1}{d}.$$ □

We can now show the following lower bound on W by leveraging Corollary 1. We set the constant $m_0 := 2n_{\mathrm{HKB}} = 4620$.

Lemma 6. $W \geqslant 2c_1 n \ln n - 3c_1 \ln(4m_0)\, n$.

Proof. We have

$$W = \sum_{\substack{p_ip_j\in E(B)\\ i<j}} \frac{1}{j-i} = \sum_{d\geqslant 1}\sum_{\substack{p_ip_j\in E(B)\\ j-i=d}} \frac{1}{d}.$$

By Lemma 5, and since every term of the series is positive, we have

$$\frac{1}{d} \geqslant \sum_{m=d+1}^{n} \frac{2}{m^2-1}\left(1-\frac{d}{m}\right).$$

Thus,

$$W \geqslant \sum_{d\geqslant 1}\sum_{\substack{p_ip_j\in E(B)\\ j-i=d}}\sum_{m=d+1}^{n} \frac{2}{m^2-1}\left(1-\frac{d}{m}\right) = \sum_{\substack{p_ip_j\in E(B)\\ i<j}}\sum_{m=j-i+1}^{n} \frac{2}{m^2-1}\left(1-\frac{j-i}{m}\right).$$

Interchanging the order of summation, we get

$$W \geqslant \sum_{m=2}^{n}\sum_{\substack{p_ip_j\in E(B)\\ j-i\in[m-1]}} \frac{2}{m^2-1}\left(1-\frac{j-i}{m}\right) = \sum_{m=2}^{n}\frac{2}{m^2-1}\sum_{\substack{p_ip_j\in E(B)\\ j-i\in[m-1]}}\left(1-\frac{j-i}{m}\right).$$

Thus

$$W \geqslant \sum_{m=2}^{n} \frac{2}{m^2-1} A_m, \tag{1}$$

by setting

$$A_m := \sum_{\substack{p_i p_j \in E(B) \\ j-i \in [m-1]}} \left(1 - \frac{j-i}{m}\right).$$

Our goal is now to lower-bound A_m in some regime of m.

Consider the m ways of partitioning the integers into intervals of size m, intersected with $[n]$. We say that an interval partition *captures* an edge e if both endpoints of e are in the same interval of the partition. Drawing one of the m partitions uniformly at random, the probability that an edge of B of stretch d is captured is $\max(1-\frac{d}{m}, 0)$. Thus A_m is the expected number of edges of B captured by the partition.

Fix any $m \geqslant m_0 = 2n_{\mathrm{HKB}}$ with $m \leqslant n' := \lfloor \frac{n}{4} \rfloor$. We assume that $m_0 \leqslant n'$, since otherwise the lemma statement trivially holds from the nonnegativity of W. For each partition, there are at least $\frac{n}{m} - 2$ intervals of size m fully contained in $[n]$. Thus, by Corollary 1, *any* partition captures at least

$$\left(\frac{n}{m} - 2\right)\left(c_1 m^2 - \frac{m}{2}\right).$$

In particular,

$$A_m \geqslant \left(\frac{n}{m} - 2\right)\left(c_1 m^2 - \frac{m}{2}\right) \geqslant c_1 mn - 2c_1 m^2 - \frac{n}{2}.$$

Together with Eq. (1), this implies that

$$W \geqslant \sum_{m=2}^{n} \frac{2}{m^2-1} A_m \geqslant \sum_{m=m_0}^{n'} \frac{2}{m^2-1} A_m \geqslant \sum_{m=m_0}^{n'} \frac{2}{m^2-1} \left(c_1 mn - 2c_1 m^2 - \frac{n}{2}\right).$$

Therefore,

$$W \geqslant 2c_1 n \sum_{m=m_0}^{n'} \frac{m}{m^2-1} - 4c_1 \sum_{m=m_0}^{n'} \frac{m^2}{m^2-1} - n \sum_{m=m_0}^{n'} \frac{1}{m^2-1}. \tag{2}$$

We observe that

$$\sum_{m=m_0}^{n'} \frac{m}{m^2-1} \geqslant \ln \frac{n'+1}{m_0} \geqslant \ln \frac{n}{4m_0}.$$

Moreover,

$$\sum_{m=m_0}^{n'} \frac{m^2}{m^2-1} \leqslant 2n' \leqslant \frac{n}{2} \quad \text{and} \quad \sum_{m=m_0}^{n'} \frac{1}{m^2-1} \leqslant \frac{3}{4}.$$

We therefore conclude from Eq. (2) that

$$W \geqslant 2c_1 n \ln n - \left(2c_1 \ln(4m_0) + 2c_1 + \frac{3}{4}\right) n \geqslant 2c_1 n \ln n - 3c_1 \ln(4m_0) n. \qquad \square$$

We can now finish the proof of the main result.

Proof of Theorem 1. Lemmas 4 and 6 imply that

$$2c_1 n \ln n - 3c_1 \ln(4m_0) n \leqslant \frac{1}{5} n \ln n + \frac{n}{5}.$$

Therefore,

$$\left(2c_1 - \frac{1}{5}\right) \ln n \leqslant 3c_1 \ln(4m_0) + \frac{1}{5}.$$

Since the constant $2c_1 - \frac{1}{5}$ is positive (recall that $c_1 > \frac{1}{10}$),

$$\ln n \leqslant \frac{3c_1 \ln(4m_0) + \frac{1}{5}}{2c_1 - \frac{1}{5}}, \text{ thus } n \leqslant \exp\left(\frac{3c_1 \ln(4m_0) + \frac{1}{5}}{2c_1 - \frac{1}{5}}\right) < 10^{11055931}. \qquad \square$$

AI disclosure. After one fruitless attempt and the now customary generic encouragement, a relatively detailed proof of Theorem 1 was provided by GPT-5.6 Sol Pro after pondering for 222 minutes. The author's contributions were limited to checking the proof, simplifying some parts, and eventually writing the paper in a way that would give the author (and hopefully other human readers) a more pleasant reading experience.

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