

Sparse graphs with bounded induced cycle packing number have logarithmic treewidth

Marthe Bonamy^{*1}, Édouard Bonnet^{†2}, Hugues Déprés², Louis Esperet^{‡3}, Colin Geniet², Claire Hilaire¹, Stéphan Thomassé², and Alexandra Wesolek^{§4}

¹*CNRS, LaBRI, Université de Bordeaux, Bordeaux, France.*

²*Univ Lyon, CNRS, ENS de Lyon, Université Claude Bernard Lyon 1, LIP UMR5668, France.*

³*CNRS, G-SCOP, Univ. Grenoble Alpes, Grenoble, France.*

⁴*Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada.*

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Abstract

A graph is O_k -free if it does not contain k pairwise vertex-disjoint and non-adjacent cycles. We show that MAXIMUM INDEPENDENT SET and 3-COLORING in O_k -free graphs can be solved in quasi-polynomial time. As a main technical result, we establish that "sparse" (here, not containing large complete bipartite graphs as subgraphs) O_k -free graphs have treewidth (even, feedback vertex set number) at most logarithmic in the number of vertices. This is proven sharp as there is an infinite family of O_2 -free graphs without $K_{3,3}$ -subgraph and whose treewidth is (at least) logarithmic.

Other consequences include that most of the central NP-complete problems (such as MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, MINIMUM COLORING) can be solved in polynomial time in sparse O_k -free graphs, and that deciding the O_k -freeness of sparse graphs is polynomial time solvable.

1 Introduction

A natural approach to tackle NP-hard graph problems is to consider them on restricted classes. A simple example is the case of forests, that is, graphs *without cycles*, on which most hard problems become tractable. The celebrated Courcelle's theorem [Cou90] generalizes that phenomenon to graphs of bounded treewidth and problems expressible in monadic second-order logic.

For the particular yet central MAXIMUM INDEPENDENT SET (MIS, for short), the mere absence of *odd* cycles makes the problem solvable in polynomial time. Denoting by $\text{ocp}(G)$

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(for odd cycle packing) the maximum cardinality of a collection of vertex-disjoint odd cycles in G , the classical result that MIS is polytime solvable in bipartite graphs corresponds to the $\text{ocp}(G) = 0$ case. Artmann et al. [AWZ17] extended the tractability of MIS to graphs G satisfying $\text{ocp}(G) \leq 1$. One could think that such graphs are close to being bipartite, in the sense that the removal of a few vertices destroys all odd cycles. This is not necessarily true: Adding to an $n \times n$ grid the edges between $(1, i)$ and $(n, n + 1 - i)$, for every $i = 1, \dots, n$, yields a graph G with $\text{ocp}(G) = 1$ such that no removal of less than n vertices make G bipartite; also see the *Escher wall* in [Ree99].

It was believed that Artmann et al.'s result could even be lifted to graphs with bounded odd cycle packing number. Conforti et al. [CFH⁺20] proved it on graphs further assumed to have bounded genus, and Fiorini et al. [FJWY21] confirmed that conjecture for graphs G satisfying $\text{ocp}(G) = O(1)$, in general. A polynomial time approximation scheme (PTAS), due to Bock et al. [BFMR14], was known for MIS in the (much) more general case of n -vertex graphs G such that $\text{ocp}(G) = o(n/\log n)$.

Similarly let us denote by $\text{cp}(G)$, $\text{icp}(G)$, $\text{iocp}(G)$ the maximum cardinality of a collection of vertex-disjoint cycles in G that are unconstrained, non-adjacent, and non-adjacent and of odd length, respectively (for cycle packing, induced cycle packing, and induced odd cycle packing). The Erdős-Pósa theorem [EP65] states that graphs G with $\text{cp}(G) = k$ admit a feedback vertex set (i.e., a subset of vertices whose removal yields a forest) of size $O(k \log k)$, hence treewidth $O(k \log k)$. Thus, graphs with bounded cycle packing number allow polynomial time algorithms for a wide range of problems.

However, graphs with bounded feedback vertex set are very restricted. This is a motivation to consider the larger classes for which solely the induced variants icp and iocp are bounded. Graphs with $\text{iocp} \leq 1$ have their significance since they contain all the complements of disk graphs [BGK⁺18] and all the complements of unit ball graphs [BBB⁺18]. Concretely, the existence of a polynomial time algorithm for MIS on graphs with $\text{iocp} \leq 1$ —an intriguing open question—would solve the long-standing open problems of whether MAXIMUM CLIQUE is in P for disk graphs and unit ball graphs. Currently only efficient PTASes are known [BBB⁺21], even when only assuming that $\text{iocp} \leq 1$ and that the solution size is a positive fraction of the total number of vertices [DP20]. Let us mention that recognizing the class of graphs G satisfying $\text{iocp}(G) \leq 1$ is NP-complete [GKPT12].

Summarizing the introduction so far, graphs with bounded cp , ocp , iocp have been studied in close connection with solving MIS (or a broader class of problems), respectively forming the Erdős-Pósa theory, establishing a far-reaching generalization of total unimodularity, and improving the approximation algorithms for MAXIMUM CLIQUE on some geometric intersection graph classes. Relatively less attention has been given to icp . In this paper, we start filling this gap. We initiate the study of graphs with bounded induced cycle packing, focusing on the sparse regime.

Two vertex-disjoint cycles C and C' in a graph G are *independent* or *non-adjacent* if there is no edge between C and C' . *Independent cycles* are simply vertex-disjoint cycles that are pairwise independent. A graph is O_k -free if it does not contain k independent cycles. Note that a graph is O_k -free if and only if it does not contain k independent *induced* cycles. Therefore, G is O_k -free is equivalent to $\text{icp}(G) < k$. These graphs can equivalently be defined in terms of forbidden induced subdivisions. Letting T_k be the disjoint union of k triangles, a graph is O_k -free if and only if it does not contain an induced subdivision of T_k .

Even the class of O_2 -free graphs is rather complex. Observe indeed that complements of graphs without $K_{3,3}$ subgraph are O_2 -free. As MIS remains NP-hard in graphs with girth at least 5 (hence without $K_{3,3}$ subgraph) [Ale82], MAXIMUM CLIQUE is NP-hard in O_2 -free graphs.

Nonetheless MIS could be tractable in O_k -free graphs, as is the case in graphs of bounded ocp :

Conjecture 1.1. *MAXIMUM INDEPENDENT SET is solvable in polynomial time in O_k -free graphs.*

As far as we can tell, MIS could even be tractable in graphs with bounded iocp . This would constitute a surprising and formidable generalization of Conjecture 1.1 and of the same result for bounded ocp [FJWY21].

We say that a graph G is t -biclique-free if G does not contain $K_{t,t}$ as a subgraph (not necessarily induced). A *feedback vertex set* is a set of vertices whose removal yields a forest. Our main technical contribution is the following.

Theorem 1.2. *Any t -biclique-free O_k -free graph on n vertices has a feedback vertex set of size $O_{t,k}(\log n)$.*

Since a graph with a feedback vertex set of size k has treewidth at most $k + 1$, this implies a corresponding result on treewidth.

Corollary 1.3. *Every t -biclique-free O_k -free graph on n vertices has treewidth $O_{t,k}(\log n)$.*

Corollary 1.3 implies that a number of fundamental problems, such as MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, MINIMUM COLORING, can be solved in polynomial time in "sparse" O_k -free graphs. In particular, this proves Conjecture 1.1 in the sparse regime.

We come short of proving Conjecture 1.1 in general, but not by much. We obtain a quasi-polynomial time algorithm for MIS in general O_k -free graphs, excluding that this problem is NP-complete without any complexity-theoretic collapse (and making it quite likely that the conjecture indeed holds).

Theorem 1.4. *For every positive integer k , MAXIMUM INDEPENDENT SET can be solved in quasi-polynomial time $n^{O(k^2 \log n)}$ in n -vertex O_k -free graphs.*

This is in sharp contrast with what is deemed possible in general graphs. Indeed, any exact algorithm for MIS requires time $2^{\Omega(n)}$ unless the Exponential Time Hypothesis¹ (ETH) fails [IPZ01].

It should be noted that Conjecture 1.1 is a special case of an intriguing and very general question by Dallard, Milanič, and Štorgel [DMŠ21] of whether there are planar graphs H for which MIS is NP-complete on graphs excluding H as an induced minor. In the same paper, the authors show that MIS is in fact polytime solvable when H is W_4 (the 4-vertex cycle with a fifth universal vertex), or K_5^- (the 5-vertex clique minus an edge), or $K_{2,t}$ (an induced biclique with two vertices on one side). Gartland et al. [GLP⁺21] (at least partially) answered that question when H is a path, or even a cycle, by presenting in that case a quasi-polynomial algorithm for MIS. As we will mention again later, Korhonen [Kor22] showed that bounded-degree graphs excluding a fixed planar graph H as an induced minor have bounded treewidth, thereby fully settling the question of Dallard et al. when the degree is bounded. He also derived an algorithm running in time $2^{O(n/\log^{1/6} n)} = 2^{o(n)}$, in the general (non bounded-degree) case.

Theorem 1.4 now adds a quasi-polynomial time algorithm when H is the disjoint union of triangles. This is an orthogonal generalization of the trivial case when H is a triangle (hence the graphs are forests) to that of Gartland et al. We increase the number of triangles, while the latter authors increase the length of the cycle. Our proofs are very different, yet they share

¹asserting that solving n -variable 3-SAT requires time $2^{\Omega(n)}$.

a common feature, that of measuring the progress of the usual branching on a vertex by the remaining amount of relevant (semi-)induced subgraphs.

A natural related problem is the complexity of deciding O_k -freeness. Surprisingly this problem is open already when $k = 2$.

Conjecture 1.5. *Deciding if a graph has two independent cycles is in P .*

Again, a simple consequence of Corollary 1.3 is that one can test whether a t -biclique-free graph is O_k -free in polynomial time. Le [Le17] proposed an interesting approach towards solving Conjecture 1.5.

Conjecture 1.6 (Le [Le17]). *There is a constant c such that every O_2 -free n -vertex graph has at most n^c distinct induced paths.*

This would in particular imply that the number of induced cycles is polynomial (since this number cannot be more than n times larger than the number of induced paths). If Conjecture 1.6 is true, enumerating the induced cycles and testing pairwise independence resolves Conjecture 1.5. Theorem 1.2 yields a quasi-polynomial upper bound of $n^{O(\log n)}$ on the number of induced paths in O_k -free t -biclique-free graphs. Indeed any induced path P in an n -vertex graph G with a feedback vertex set X of size $O(\log n)$ can be described by its vertex set: $X' = X \cap V(P)$ and the vertices along $O(\log n)$ paths in the forest $G - X$; hence at most $2^{O(\log n)} \cdot (n^2)^{O(\log n)} = n^{O(\log n)}$ descriptions. In a recent work, Nguyen, Scott, and Seymour [NSS22] used Theorem 1.2 to obtain a polynomial upper bound on the number of induced paths in O_k -free t -biclique-free graphs, thereby proving the sparse case of Le's conjecture.

Our work is also related to an ongoing project devoted to unraveling an *induced* version of the grid minor theorem of Robertson and Seymour [RS86]. This theorem implies that every graph not containing a subdivision of a $k \times k$ wall as a subgraph has treewidth at most $f(k)$, for some function f . This result had a deep impact in algorithmic graph theory since, as we previously mentioned, many natural problems are tractable in graphs of bounded treewidth.

Now, what are the forbidden *induced* subgraphs in graphs of small treewidth? It is clear that large cliques, bicliques, subdivided walls, and line graphs of subdivided walls shall be excluded. It was actually suggested that in t -biclique-free graphs, the absence of induced subdivisions of large walls and their line graphs might imply bounded treewidth, but counterexamples were found [ST21, Dav22, Tro22]. However, Korhonen recently showed that this absence suffices within bounded-degree graphs [Kor22]. Abrishami et al. [AAC⁺22] proved that a vertex with at least two neighbors on a hole (i.e., an induced cycle of length at least four) is also necessary in a counterexample. Echoing our main result, it was proven that (triangle, θ^2)-free graphs have logarithmic treewidth [ACHS22].

As we will see, the class of 3-biclique-free O_2 -free graphs have unbounded treewidth. Since these graphs do not contain as an induced subgraph a subdivision of a large wall or its line graph, they constitute yet another family of counterexamples. To our knowledge, this also yields the first example of hereditary classes with unbounded treewidth in which every graph has a feedback vertex set of size logarithmic in its number of vertices.

We leave as an open question whether t -biclique-free O_k -free graphs have bounded *twin-width*, that is, if there is a function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that their twin-width is at most $f(t, k)$, and refer the reader to [BKTW22] for a definition of twin-width.

²A θ is made of three paths each on at least two edges between the same pair of vertices.

Organization of the paper. In Section 2, we prove that Theorem 1.2 and Corollary 1.3 are tight already for $k = 2$ and $t = 3$. Section 3 solves MIS in O_k -free graphs in quasi-polynomial time, among other algorithmic applications of Corollary 1.3.

The proof of our main structural result, Theorem 1.2, spans from Section 4 to Section 8. After some preliminary results (Section 4), we show in Section 5 that it suffices to prove Theorem 1.2 when the graph G has a simple structure: a cycle C , its neighborhood N (an independent set), and the remaining vertices R (inducing a forest). Instead of directly exhibiting a logarithmic-size feedback vertex set, we rather prove that every such graph contains a vertex of degree linear in the so-called “cycle rank” (or first Betti number) of the graph. For sparse O_k -free graphs, the cycle rank is at most linear in the number of vertices and decreases by a constant fraction when deleting a vertex of linear degree. We then derive the desired theorem by induction, using as a base case that if the cycle rank is small, we only need to remove a small number of vertices to obtain a tree. To obtain the existence of a linear-degree vertex in this simplified setting, we argue in Section 6 that we may focus on the case where the forest $G[R]$ contains only paths or only large “well-behaving” subdivided stars. In Section 7, we discuss how the O_k -freeness restricts the adjacencies between these stars/paths and N . Finally, in Section 8, we argue that the restrictions yield a simple enough picture, and derive our main result.

2 Sparse O_2 -free graphs with unbounded treewidth

In this section, we show the following.

Theorem 2.1. *For every natural k , there is an O_2 -free 3-biclique-free $(2^k + k - 1)$ -vertex graph with treewidth at least $k - 1$.*

In particular, for infinitely many values of n , there is an O_2 -free 3-biclique-free n -vertex graph with treewidth at least $\log n - 2$.

Construction of G_k . To build G_k , we first define a word w_k of length $2^k - 1$ on the alphabet $[k]$. We set $w_1 = 1$, and for every integer $i > 1$, $w_i = i w_{i-1}[1] i w_{i-1}[2] i \dots i w_{i-1}[2^{i-1} - 2] i w_{i-1}[2^{i-1} - 1] i$. It is worth noting that equivalently $w_i = \text{incr}(w_{i-1}) 1 \text{incr}(w_{i-1})$, where incr adds 1 to every letter of the word. Let Π_k be the $(2^k - 1)$ -path where the ℓ -th vertex of the path (say, from left to right) is denoted by $\Pi_k[\ell]$.

The graph G_k is obtained by adding to Π_k an independent set of k vertices v_1, v_2, \dots, v_k , and linking by an edge every pair $v_i, \Pi_k[\ell]$ such that $i \in [k]$ and $w_k[\ell] = i$.

Observe that we can also define the graph G_k directly, rather than iteratively: it is the union of a path $u_1, \dots, u_{2^k - 1}$ and an independent set $\{v_0, \dots, v_{k-1}\}$, with an edge between v_i and u_j if and only if i is the 2-order of j (the maximum k such that 2^k divides j).

See Fig. 1 for an illustration.

G_k is O_2 -free and 3-biclique-free. The absence of $K_{3,3}$ (even $K_{2,3}$) as a subgraph is easy to check. At least one vertex of the $K_{3,3}$ has to be some v_i , for $i \in [k]$. It forces that its three neighbors x, y, z are in Π_k . In turn, this implies that a common neighbor of x, y, z (other than v_i) is some $v_{i'} \neq v_i$; a contradiction since distinct vertices of the independent set have disjoint neighborhoods.

We now show that G_k is O_2 -free. Assume towards a contradiction that $G_k[C_1 \cup C_2]$ is isomorphic to the disjoint union of two cycles $G_k[C_1]$ and $G_k[C_2]$. As C_1 and C_2 each induce

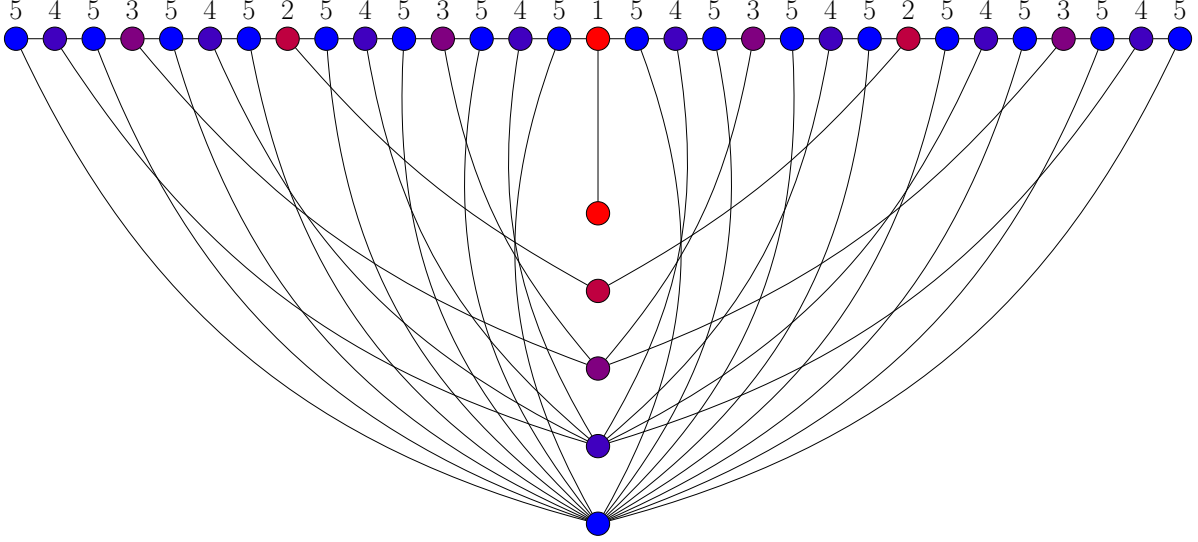


Figure 1: The graph G_k for $k = 5$: an O_2 -free graph without $K_{3,3}$ subgraph, $k + 2^k - 1$ vertices, and treewidth at least $k - 1$.

a cycle, they each have to intersect $\{v_1, \dots, v_k\}$. Assume without loss of generality that C_1 contains v_i , and C_2 is disjoint from $\{v_i, v_{i+1}, \dots, v_k\}$. Consider a subpath S of C_2 with both endpoints in $\{v_1, \dots, v_k\}$, thus in $\{v_1, \dots, v_{i-1}\}$, and all the other vertices of S form a set $S' \subseteq V(\Pi_k)$. It can be that the endpoints are in fact the same vertex $v_{i'}$, and in that case S is the entire C_2 .

Let $v_{i'}, v_{i''}$ be the two (possibly equal) endpoints. Observe that S' is a subpath of Π_k whose two endpoints have label $i', i'' < i$. In particular there is a vertex labeled i somewhere along S' . This makes an edge between $v_i \in C_1$ and C_2 , which is a contradiction.

G_k has treewidth at least $k - 1$. We build a K_k -minor as follows. We set $V_1 = \{v_1\} \cup R_1$ where $R_1 \subseteq V(\Pi_k)$ consists of the unique neighbor x_1 of v_1 together with all the vertices of Π_k to the right of x_1 . Then, for every $i \in [2, k]$, we iteratively define V_i as $\{v_i\} \cup R_i$ where R_i is made of x_i the unique neighbor of v_i in $V(\Pi_k) \setminus \bigcup_{1 \leq j \leq i-1} V_j$ and all the vertices of $V(\Pi_k) \setminus \bigcup_{1 \leq j \leq i-1} V_j$ to the right of x_i .

By construction, it is clear that the V_i 's are disjoint and that $G_k[V_i]$ is a path for every $i \in [k]$, hence is connected. It can also be observed that there is an edge between R_i and v_j with $j > i$. Thus the V_i 's are the branch sets of a K_k -minor. Therefore $\text{tw}(G_k) \geq \text{tw}(K_k) = k - 1$. (It is easy to see that the treewidth of G_k is at most k , since $\{v_2, \dots, v_k\}$ is a feedback vertex set.) The *twin-width* of G_k , however, can be shown to be at most a constant independent of k .

3 Algorithmic applications

This section presents algorithms on O_k -free graphs based on our main result, specifically using the treewidth bound.

Corollary 1.3. *Every t -biclique-free O_k -free graph on n vertices has treewidth $O_{t,k}(\log n)$.*

Single-exponential parameterized $O(1)$ -approximation algorithms exist for treewidth. Already in 1995, Robertson and Seymour [RS95] present a $2^{O(\text{tw})}n^2$ -time algorithm yielding a tree-decomposition of width $4(\text{tw} + 1)$ for any input n -vertex graph of treewidth tw . Run

on n -vertex graphs of logarithmic treewidth, this algorithm outputs tree-decompositions of width $O(\log n)$ in polynomial time. We thus obtain the following.

Corollary 3.1. *MAXIMUM INDEPENDENT SET, HAMILTONIAN CYCLE, MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, MINIMUM FEEDBACK VERTEX SET, and MINIMUM COLORING can be solved in polynomial time in t -biclique-free O_k -free graphs.*

Proof. Algorithms running in time $2^{O(\text{tw})}n^{O(1)}$ exist for all these problems but for MINIMUM COLORING. They are based on dynamic programming over a tree-decomposition, which by Corollary 1.3 has logarithmic width and by [RS95] can be computed in polynomial time. For MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, and q -COLORING (for a fixed integer q) see for instance the textbook [CFK⁺15, Chapter 7.3]. For HAMILTONIAN CYCLE and MINIMUM FEEDBACK VERTEX SET, deterministic parameterized single-exponential algorithms require the so-called rank-based approach; see [CFK⁺15, Chapter 11.2].

By Corollary 4.6, t -biclique-free O_k -free graphs have bounded chromatic number. Thus a polynomial time algorithm for MINIMUM COLORING is implied by the one for q -COLORING. \square

In a scaled-down refinement of Courcelle’s theorem [Cou90], Pilipczuk showed that any problem expressible in Existential Counting Modal Logic (ECML) admits a single-exponential fixed-parameter algorithm in treewidth [Pil11]. In particular:

Theorem 3.2 ([Pil11]). *ECML model checking can be solved in polynomial time on any class with logarithmic treewidth.*

In a nutshell, this logic allows existential quantifications over vertex and edge sets followed by a counting modal formula that should be satisfied from every vertex v . Counting modal formulas enrich quantifier-free Boolean formulas with $\diamond^S\varphi$, whose semantics is that the current vertex v has a number of neighbors satisfying φ in the ultimately periodic set S of non-negative integers. Another consequence of Corollary 1.3 (and Theorem 3.2) is that testing if a graph is O_k -free can be done in polynomial time among sparse graphs, further indicating that the general case could be tractable.

Corollary 3.3. *For any fixed k and t , deciding whether a t -biclique-free graph is O_k -free can be done in polynomial time.*

Proof. One can observe that O_k -freeness is definable in ECML. Indeed, one can write

$$\varphi = \exists X_1 \exists X_2 \dots \exists X_k \left(\bigwedge_{1 \leq i \leq k} X_i \rightarrow \diamond^{\{2\}} X_i \right) \wedge \left(\bigwedge_{1 \leq i < j \leq k} \neg(X_i \wedge X_j) \wedge (X_i \rightarrow \diamond^{\{0\}} X_j) \right).$$

Formula φ asserts that there are k sets of vertices X_1, X_2, \dots, X_k such that every vertex has exactly two neighbors in X_i if it is itself in X_i , the sets are pairwise disjoint, and every vertex has no neighbor in X_j if it is in some distinct X_i (with $i < j$). Thus G is O_k -free if and only if φ does not hold in G . \square

We now show the main algorithmic consequence of our structural result. This holds for any (possibly dense) O_k -free graph, and uses the sparse case (Corollary 3.1) at the basis of an induction on the size of a largest collection of independent 4-vertex cycles. It should be noted that this result (as well as the previous result on MIS above) also works for the weighted version of the problem, with minor modifications.

Theorem 1.4. *For every positive integer k , MAXIMUM INDEPENDENT SET can be solved in quasi-polynomial time $n^{O(k^2 \log n)}$ in n -vertex O_k -free graphs.*

Proof. Let G be our n -vertex O_k -free input. Let q be the maximum integer such that G admits q independent 4-vertex cycles (the cycles themselves need not be induced). Clearly $q < k$. We show the theorem by induction on q , namely that MIS can be solved in time $n^{c(q+1)^2 \log n}$ for some constant c (later specified). If $q = 0$, G is 2-biclique-free, so we can even solve MIS in polynomial time, by Corollary 3.1. We now assume that $q \geq 1$, $n \geq 4$, and that the case $q - 1$ of the induction has been established.

Let C be a 4-vertex cycle part of a $4q$ -vertex subset consisting of q independent 4-vertex cycles. Let \mathcal{S} be the set of all $4q$ -vertex subsets consisting of q independent 4-vertex cycles in the current graph (at this point, G), and $s = |\mathcal{S}|$. Thus $1 \leq s \leq n^{4q}$. By assumption, the closed neighborhood of C , $N[C]$, intersects every subset in \mathcal{S} . In particular, there is one of the four vertices of C , say, v , such that $N[v]$ intersects at least $s/4$ subsets of \mathcal{S} .

We branch on two options: either we put v in (an initially empty set) I , and remove its closed neighborhood from G , or we remove v from G (without adding it to I). With the former choice, the size of \mathcal{S} drops by at least $s/4$, whereas with the latter, it drops by at least 1.

Even if fully expanded while $s > 0$, this binary branching tree has at most

$$\sum_{0 \leq i \leq 4q \log_{4/3} n} \binom{n}{i} = n^{O(q \log n)} \text{ leaves,}$$

since including a vertex in I can be done at most $4q \log_{4/3} n$ times within the same branch; thus, leaves can be uniquely described as binary words of length n with at most $4q \log_{4/3} n$ occurrences of, say, 1.

We retrospectively set $c \geq 1$ such that the number of leaves is at most $n^{cq \log n}$, running the algorithm thus far (when $q \geq 1$) takes at most time $n^{c+cq \log n}$, and calling Corollary 3.1 takes less time than $n^{c \log n}$. At each leaf of the branching, $s = 0$ holds, which means that the current graph does not admit q independent 4-vertex cycles. By the induction hypothesis, we can solve each such instance in time $n^{cq^2 \log n}$. Thus the overall running time is

$$n^{c+cq \log n} + n^{cq \log n} \cdot n^{cq^2 \log n} \leq n^{c+cq \log n} \cdot (n^{cq^2 \log n} + 1) \leq n^{c(q+1)^2 \log n - cq \log n - c \log n + c + \frac{1}{\log n}}.$$

Note that $n^{cq^2 \log n} \geq 1$ thus we could upperbound $n^{cq^2 \log n} + 1$ by $2n^{cq^2 \log n} = n^{cq^2 \log n + \frac{1}{\log n}}$. Since $c, q \geq 1$ and $\log n \geq 1$, it holds that $-cq \log n - c \log n + c + \frac{1}{\log n} \leq -2c + c + 1 \leq 0$. Hence we get the claimed running time of $n^{c(q+1)^2 \log n}$. \square

One may wonder if some other problems beside MIS become (much) easier on O_k -free graphs than in general. As $2K_2$ -free graphs are O_2 -free, one cannot expect a quasi-polynomial time algorithm for MINIMUM DOMINATING SET [Ber84, CP84], HAMILTONIAN CYCLE [Gol04], MAXIMUM CLIQUE [Pol74], and MINIMUM COLORING [KKTW01] since these problems remain NP-complete on $2K_2$ -free graphs. Nevertheless we give a quasi-polynomial time algorithm for 3-COLORING.

Theorem 3.4. *For every positive integer k , 3-COLORING can be solved in quasi-polynomial time $n^{O(k^2 \log n)}$ in n -vertex O_k -free graphs.*

Proof. We solve the more general LIST 3-COLORING problem, where, in addition, every vertex v is given a list $L(v) \subseteq \{1, 2, 3\}$ from which one has to choose its color. Note that when $L(v) = \emptyset$ for some vertex v , one can report that the instance is negative, and when $|L(v)| = 1$, v has to

be colored with the unique color in its list, and this color has to be deleted from the lists of its neighbors (once this is done, v might as well be removed from the graph). These reduction rules are performed as long as they apply, so we always assume that the current instance has only lists of size 2 and 3.

We follow the previous proof, and simply adapt the branching rule, and the value of s . Now s is defined as the sum taken over all vertex sets X consisting of q independent 4-vertex cycles (the cycles themselves need not be induced), of the sum of the list sizes of the vertices of X . Hence $8 \leq s \leq 12 \cdot n^{4q}$. There is a vertex $v \in C$ and a color $c \in L(v)$ such that c appears in at least $\frac{1}{2} \cdot \frac{1}{12} \cdot \frac{s}{4} = \frac{s}{96}$ of the lists of its neighbors. This is because all the lists have size at least 2, and are subsets of $\{1, 2, 3\}$, thus pairwise intersect. (Note that this simple yet crucial fact already breaks for LIST 4-COLORING.)

We branch on two options: either we color v with c , hence we remove color c from the lists of its neighbors or we commit to not color v by c , and simply remove c from the list of v . With the former choice, the size of \mathcal{S} drops by at least $s/96$, whereas with the latter, it drops by at least 1. The rest of the proof is similar with a possibly larger constant c . \square

4 Preliminary results

An important property of t -biclique-free graphs is that they are not dense (in the sense that they have a subquadratic number of edges).

Theorem 4.1 (K3v3ri, S3s, and Tur3n [KST54]). *For every integer $t \geq 2$ there is a constant c_t such that any n -vertex t -biclique-free graph contains at most $c_t n^{2-1/t}$ edges.*

The following lemma shows that for O_k -free graphs, being t -biclique-free is equivalent to a much stronger ‘large girth’ condition, up to the removal of a bounded number of vertices.

Lemma 4.2. *There is a function f such that for any integer $\ell \geq 3$ and any t -biclique-free and O_k -free graph G , the maximum number of vertex-disjoint cycles of length at most ℓ in G is at most $f(\ell, t, k)$.*

Proof. Let $N := (2c_t k \ell^2)^t$, where c_t is the constant of Theorem 4.1.

Assume for the sake of contradiction that G contains N vertex-disjoint cycles of length at most ℓ , which we denote by C_1, \dots, C_N . Let H be the graph with vertex set v_1, \dots, v_N , with an edge between v_i and v_j in H if and only if there is an edge between C_i and C_j in G . Since G is O_k -free, H has no independent set of size k . By Tur3n’s theorem [Tur41], H contains at least $\frac{N^2}{2k-2} - \frac{N}{2} \geq \frac{N^2}{2k} - \frac{N}{2}$ edges.

Consider the subgraph G' of G induced by the vertex set $\bigcup_{i=1}^N C_i$. The graph G' has $n \leq \ell N$ vertices, and $m \geq 3N + \frac{N^2}{2k} - \frac{N}{2} > \frac{N^2}{2k}$ edges. Note that by the definition of N , we have

$$m > \frac{N^2}{2k} = \frac{1}{2k} \cdot N^{2-1/t} \cdot N^{1/t} \geq \frac{1}{2k \ell^{2-1/t}} \cdot n^{2-1/t} \cdot 2c_t k \ell^2 \geq c_t n^{2-1/t},$$

which contradicts Theorem 4.1, since G' (as an induced subgraph of G) is t -biclique-free. \square

The *girth* of a graph G is the minimum length of a cycle in G (if G is acyclic, its girth is set to be infinite). We obtain the following immediate corollary of Lemma 4.2.

Corollary 4.3. *There is a function g such that for any integer $\ell \geq 3$, any t -biclique-free and O_k -free graph G contains a set X of at most $g(\ell, t, k)$ vertices such that $G - X$ has girth at least ℓ .*

Proof. Let f be the function of Lemma 4.2, and let $g(\ell, t, k) := (\ell - 1) \cdot f(\ell - 1, t, k)$. Consider a maximum collection of disjoint cycles of length at most $\ell - 1$ in G . Let X be the union of the vertex sets of all these cycles. By Lemma 4.2, $|X| \leq (\ell - 1)f(\ell - 1, t, k) = g(\ell, t, k)$, and by definition of X , the graph $G - X$ does not contain any cycle of length at most $\ell - 1$, as desired. \square

We now state (without proof) a slight variant of Lemma 4.2, which will be particularly useful at the end of the proof of our main result. A *banana* in a graph G is a pair of vertices joined by at least 2 disjoint paths whose internal vertices all have degree 2 in G .

Lemma 4.4. *There is a function f' such that any t -biclique-free and O_k -free graph G contains a set X of at most $f'(t, k) = O_t(k^t)$ vertices such that all bananas of G intersect X .*

The proof is the same as that of Lemma 4.2. We can take $f'(t, k) = 2f(2, t, k) = O(k^t)$, where f is the function of Lemma 4.2 and the implicit multiplicative constant in the $O(\cdot)$ depends on t . In all the applications of this lemma, t will be a small constant (2 or 3).

The *average degree* of a graph $G = (V, E)$, denoted by $\text{ad}(G)$, is defined as $2|E|/|V|$. Let us now prove that t -biclique-free and O_k -free graphs have bounded average degree. This can also be deduced³ from the main result of [KO04], but we include a short proof for the sake of completeness. Moreover, the decomposition used in the proof will be used again in the proof of our main result.

Lemma 4.5. *Every O_k -free graph G of girth at least 11 has average degree at most $2k$.*

Proof. We proceed by induction on k . When $k = 1$, G is a forest, with average degree less than 2. Otherwise, let C be a cycle of minimal length in G . Let N be the neighborhood of C , let S the second neighborhood of C , and let $R = V(G) \setminus (C \cup N)$. Thus $V(G)$ is partitioned into C, N, R , and we have $S \subseteq R$. Observe that there are no edges between C and R in G , so it follows that $G[R]$ is O_{k-1} -free, and thus $\text{ad}(G[R]) \leq 2k - 2$ by induction. Observe also that since G has girth at least 11 and C is a minimum cycle, the two sets N and S are both independent sets. Moreover each vertex of N has a unique neighbor in C , and each vertex in S has a unique neighbor in N . Indeed, in any other case we obtain a path of length at most 5 between two vertices of C , contradicting the minimality of C . It follows that C is the only cycle in $G[C \cup N \cup S]$, hence this graph has average degree at most 2. As a consequence, G has a partition of its edges into two subgraphs of average degree at most $2k - 2$ and at most 2, respectively, and thus $\text{ad}(G) \leq 2k - 2 + 2 = 2k$, as desired. \square

It can easily deduced from this result that every t -biclique-free O_k -free graph has average degree at most $h(t, k)$, for some function h (and thus chromatic number at most $h(t, k) + 1$).

Corollary 4.6. *There is a function h such that every t -biclique-free O_k -free graph has average degree at most $h(t, k)$, and chromatic number at most $h(t, k) + 1$.*

Proof. Let G be a t -biclique-free O_k -free graph. By Corollary 4.3, G has a set X of at most $g(11, t, k)$ vertices such that $G - X$ has girth at least 11. Note that $\text{ad}(G) \leq \text{ad}(G - X) + |X| \leq \text{ad}(G - X) + g(11, t, k) \leq 2k + g(11, t, k)$, where the last inequality follows from Lemma 4.5.

Let $h(t, k) = 2k + g(11, t, k)$. As the class of t -biclique-free O_k -free graphs is closed under taking induced subgraphs, it follows that any graph in this class is $h(t, k)$ -degenerate, and thus $(h(t, k) + 1)$ -colorable. \square

³Using the more recent result of [Dvo18], it can be proved that the class of t -biclique-free and O_k -free graphs actually has *bounded expansion*, which is significantly stronger than having bounded average degree. This can also be deduced from our main result, as it implies that sparse O_k -free graphs have logarithmic separators, and thus polynomial expansion.

A *feedback vertex set* (FVS) X in a graph G is a set of vertices of G such that $G - X$ is acyclic. The minimum size of a feedback vertex set in G is denoted by $\text{fvs}(G)$. The classical Erdős-Pósa theorem [EP65] states that graphs with few vertex-disjoint cycles have small feedback vertex sets.

Theorem 4.7 (Erdős and Pósa [EP65]). *There is a constant $c > 0$ such that if a graph G contains less than k vertex-disjoint cycles, then $\text{fvs}(G) \leq ck \log k$.*

We use this result to deduce the following useful lemma.

Lemma 4.8. *There is a constant $c > 0$ such that the following holds. Let G consist of a cycle C , together with ℓ paths P_1, \dots, P_ℓ on at least 2 edges*

- *whose endpoints are in C , and*
- *whose internal vertices are disjoint from C , and*
- *such that the internal vertices of each pair of different paths P_i, P_j are pairwise distinct and non-adjacent.*

Suppose moreover that G is O_k -free (with $k \geq 2$) and has maximum degree at most $d + 2$. Then

$$\ell \leq c d k \log k.$$

Proof. Observe that each path P_i intersects or is adjacent to at most $2(d - 1) + 4d < 6d$ other paths P_j . It follows that there exist $s \geq \frac{\ell}{6d}$ of these paths, say P_1, \dots, P_s without loss of generality, that are pairwise non-intersecting and non adjacent. Consider the subgraph G' of G induced by the union of C and the vertex sets of the paths P_1, \dots, P_s . Since the paths P_i , $1 \leq i \leq s$, are non-intersecting and non-adjacent, and since G' does not contain k independent cycles, the graph G' does not contain k vertex-disjoint cycles. Let G'' be the graph obtained from G' by suppressing all vertices of degree 2 (i.e., replacing all maximal paths whose internal vertices have degree 2 by single edges). Observe that since G' does not contain k vertex-disjoint cycles, the graph G'' does not contain k vertex-disjoint cycles either. Observe also that G'' is cubic and contains $2s$ vertices. It was proved by Jaeger [Jae74] that any cubic graph H on n vertices satisfies $\text{fvs}(H) \geq \frac{n+2}{4}$. As a consequence, it follows from Theorem 4.7 that $\frac{2s+2}{4} \leq \text{fvs}(G'') \leq c'k \log k$ (for some constant c'), and thus $\ell \leq 12dc'k \log k = cdk \log k$ (for $c = 12c'$), as desired. \square

A *strict subdivision* of a graph is a subdivision where each edge is subdivided at least once.

Lemma 4.9. *There is a constant $c > 0$ such that for any integer $k \geq 2$, any strict subdivision of a graph of average degree at least $ck \log k$ contains an induced O_k .*

Proof. Note that if a graph G contains k vertex-disjoint cycles, then any strict subdivision of G contains an induced O_k . Hence, it suffices to prove that any graph with less than k vertex-disjoint cycles has average degree at most some function of k . By Theorem 4.7, there is a constant c' such that any graph G with less than k vertex-disjoint cycles contains a set X of at most $c'k \log k$ vertices such that $G - X$ is acyclic. In this case $G - X$ has average degree at most 2, and thus G has average degree at most $c'k \log k + 2 \leq ck \log k$ (for some constant c), as desired. \square

5 Logarithmic treewidth of sparse O_k -free graphs

Recall our main result.

Theorem 1.2. *Any t -biclique-free O_k -free graph on n vertices has a feedback vertex set of size $O_{t,k}(\log n)$.*

The proof of Theorem 1.2 relies on *the cycle rank*, which is defined as $r(G) = |E(G)| - |V(G)| + |C(G)|$ where $C(G)$ denotes the set of connected components of G . The cycle rank is exactly the number of edges of G which must be deleted to make G a forest, hence it is a trivial upper bound on the size of a minimum feedback vertex set. Remark the following simple properties.

Lemma 5.1. *The cycle rank is invariant under the following operations:*

1. *Deleting a vertex of degree 0 or 1.*
2. *Deleting a connected component which is a tree.*

We call *reduction* the operation of iteratively deleting vertices of degree 0 or 1, which preserves cycle rank by the above lemma. A graph is *reduced* if it has minimum degree 2, and the *core* of a graph G is the reduced graph obtained by applying reductions to G as long as possible. The inclusion-wise minimal FVS of G and of its core are exactly the same.

In a graph G , a vertex x is called ε -rich if $d(x) \geq \varepsilon \cdot r(G)$. Our strategy to prove Theorem 1.2 is to iteratively reduce the graph, find an ε -rich vertex, add it to the FVS and delete it from the graph. The following lemma shows that the cycle rank decreases by a constant factor each iteration, implying that the process terminates in logarithmically many steps.

Lemma 5.2. *In a reduced O_k -free graph, deleting a vertex of degree d decreases the cycle rank by at least $\frac{d-k+1}{2}$.*

Proof. In any graph G , deleting a vertex x of degree d decreases the cycle rank by $d-c$, where c is the number of connected components of $G-x$ which contain a neighbor of x . If G is O_k -free, then all but at most $k-1$ components of $G-x$ are trees. Furthermore, if T is a connected component of $G-x$ which is a tree, then T must be connected to x by at least two edges, as otherwise T must contain a vertex of degree 1 in G , which should have been deleted during reduction. Thus we have

$$2c - (k-1) \leq d. \tag{1}$$

Therefore the cycle rank decreases by at least $d - \frac{d+k-1}{2} = \frac{d-k+1}{2}$ as desired. \square

The existence of rich vertices is given by the following result.

Theorem 5.3. *For any k , there is some $\varepsilon_k > 0$ such that any O_k -free graph with girth at least 11 has an ε_k -rich vertex.*

Let us first prove Theorem 1.2 using Theorem 5.3.

Proof of Theorem 1.2. Fix k and t . Given a graph G which is O_k -free and t -biclique-free, we apply Lemma 4.2 to obtain a set X of size $f(11, t, k)$ such that $G' \stackrel{\text{def}}{=} G - X$ has girth at least 11. Thus, it suffices to prove the result for G' , and finally add X to the resulting FVS of G' . Since $\log r(G') \leq \log \binom{|V(G')|}{2} \leq 2 \log |V(G')|$, we have reduced the problem to the following.

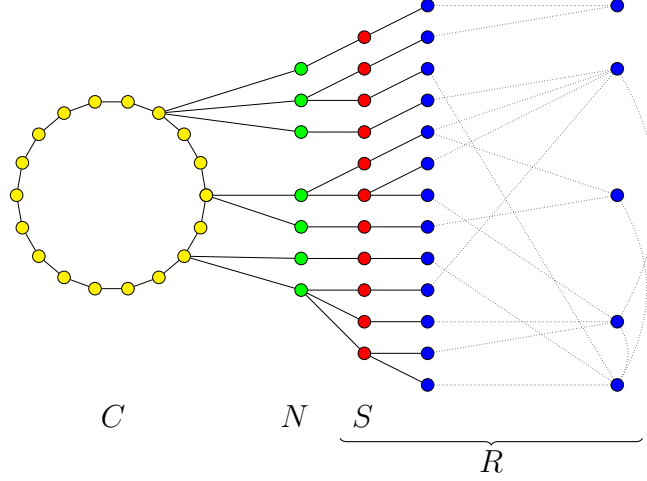


Figure 2: Subgraph of an O_4 -free graph G . $V(G)$ is partitioned into three sets C, N, R , where C is a shortest cycle, N is an independent set and first neighborhood of C , and R is O_3 -free. S is the second neighborhood of N . Gray lines correspond to induced paths where all internal vertices have degree 2.

Claim 5.4. *For any k , there is a constant c_k such that if G is an O_k -free graph with girth at least 11, then $\text{fvs}(G) \leq c_k \cdot \log r(G)$.*

Let us now assume that G is as in the claim, and consider its core H , for which $r(H) = r(G)$ and $\text{fvs}(H) = \text{fvs}(G)$. Consider an ε_k -rich vertex x in H with ε_k as in Theorem 5.3. If $r(G) \geq 2k \cdot \varepsilon_k^{-1}$, then $d(x) \geq 2k$, hence by Lemma 5.2, deleting x decreases the cycle rank of G by at least

$$\frac{d(x) - k + 1}{2} \geq \frac{d(x)}{4} \geq \frac{\varepsilon_k}{4} r(G). \quad (2)$$

Thus, as long as the cycle rank is more than $2k \cdot \varepsilon_k^{-1}$, we can find a vertex whose deletion decreases the cycle rank by a constant multiplicative factor. After logarithmically many steps, we have $\text{fvs}(G) \leq r(G) \leq 2k \cdot \varepsilon_k^{-1}$. In the end, the feedback vertex set consists of at most $f(11, t, k)$ vertices in X , logarithmically many rich vertices deleted in the induction, and at most $2k \cdot \varepsilon_k^{-1}$ vertices for the final graph. \square

We now focus on proving Theorem 5.3. Let G be an O_k -free graph with girth at least 11. Consider C the shortest cycle of G , N the neighborhood of C , and $R \stackrel{\text{def}}{=} G - (C \cup N)$ the rest of the graph (see Figure 2). Remark that there is no edge between C and R , hence $G[R]$ is an O_{k-1} -free graph. As a special case, if $k = 2$, then $G[R]$ is a forest. We will show that in general, it remains possible to reduce the problem to the case where $G[R]$ is a forest, which is our main technical theorem.

Theorem 5.5. *For any k , there is some $\delta_k > 0$ such that if G is a connected O_k -free graph with girth at least 11, and furthermore $G[R]$ is a forest where R is as in the decomposition described above, then G has a δ_k -rich vertex.*

Theorem 5.5 will be proved in Section 8. In the remainder of this section, we assume Theorem 5.5 and explain how Theorem 5.3 can be deduced from it.

Proof of Theorem 5.3. The proof is by induction on k . Let $\delta_k > 0$ be as in Theorem 5.5, and let $\varepsilon_{k-1} > 0$ be as in Theorem 5.3, obtained by induction hypothesis. We fix

$$\varepsilon_k \stackrel{\text{def}}{=} \min \left\{ \frac{\varepsilon_{k-1}}{20}, \frac{\delta_k}{20}, \frac{\delta_k}{5(k+1)}, \frac{1}{30(k-2)} \right\}. \quad (3)$$

Let G be any O_k -free graph with girth at least 11. Reductions preserve all the hypotheses of the claim, and the value of $r(G)$, hence we can assume G to be reduced. Consider the decomposition C, N, R as previously described. We construct a subset $F \subset R$ inducing a rooted forest in G such that the only edges from F to $R \setminus F$ are incident to roots of F , and each root of F is incident to at most one such edge.

Claim 5.6. *If $F \subset R$ has the former property and $F' \subset R \setminus F$ induces a forest in G , then $F \cup F'$ induces a forest in G .*

Proof. Each connected component of $G[F]$ has a single root, which is the only vertex which can be connected to F' . \lrcorner

We construct F inductively, starting with an empty forest, and applying the following rule as long as it applies: if $x \in R \setminus F$ is adjacent to at most one vertex in $R \setminus F$, we add x to F , and make it the new root of its connected component in F . The condition on F obviously holds for $F = \emptyset$. When adding x , by Claim 5.6, $F \cup \{x\}$ is still a forest. Furthermore, if $y \in F \cup \{x\}$ is adjacent to $R \setminus (F \cup \{x\})$, then either $y = x$ or y was a root before the addition of x , and is not adjacent to x , and therefore x and y are in distinct connected components of $F \cup \{x\}$. In either case, y is a root of $F \cup \{x\}$ as required.

We now denote by F the forest obtained when the previous rule no longer applies, and let $R' = R \setminus F$.

Claim 5.7. *The graph $G[R']$ has minimum degree at least 2.*

Proof. A vertex of degree less than 2 in $G[R']$ should have been added to F . \lrcorner

Claim 5.8. *The graph $G[C \cup N \cup F]$ is connected.*

Proof. It suffices to show that each connected component T of $G[F]$ is connected to N . Each such component T is a tree. If T consists of a single vertex v , then v is the root of T and has at most one neighbor in R' by definition. Since G is reduced, v has degree at least 2 in G , hence it must be connected to N .

If T contains at least two vertices, then it contains at least 2 leaves, and in particular at least one leaf v which is not the root of T . The vertex v has a single neighbor in R (its parent in T), and thus similarly as above it must have a neighbor in N . \lrcorner

Define B as the set of vertices of R' adjacent to $N \cup F$, and let A be the set of edges between $N \cup F$ and B .

Claim 5.9. *If $|A| \leq \frac{9}{10}r(G)$, then G has an ε_k -rich vertex.*

Proof. Deleting A from G decreases the cycle rank by at most $|A|$, hence $r(G - A) \geq r(G)/10$. Since $G[C \cup N \cup F]$ and $G[R']$ are unions of connected components of $G - A$, we have

$$r(G - A) = r(G[C \cup N \cup F]) + r(G[R']).$$

Thus either $G[C \cup N \cup F]$ or $G[R']$ has cycle rank at least $r(G)/20$. If it is $G[C \cup N \cup F]$, then we can apply Theorem 5.5 to find a $(\delta_k/20)$ -rich vertex, and if it is $G[R']$, then we can apply the induction hypothesis to find a $(\varepsilon_{k-1}/20)$ -rich vertex. In either case, this gives an ε_k -rich vertex. \lrcorner

Thus we can now assume that $|A| \geq \frac{9}{10}r(G)$.

Let B_1 , resp. B_2 , be the set of vertices of B incident to exactly one, resp. at least two edges of A , and let $A_1, A_2 \subseteq A$ be the set of edges of A incident to B_1, B_2 respectively. Remark that A_1, A_2 and B_1, B_2 partition A and B respectively, and $|A_1| = |B_1|$.

Claim 5.10. *If $|A_2| \geq \frac{4}{9}|A|$, then G has an ε_k -rich vertex.*

Proof. Assume that $|A_2| \geq \frac{4}{9}|A|$, and thus $|A_2| \geq \frac{2}{5}r(G)$.

By Lemma 4.5, G is $2k$ -degenerate, hence it can be vertex-partitioned into $k+1$ forests. Consider this partition restricted to B_2 , and choose $B_3 \subseteq B_2$ which induces a forest and maximizes the set A_3 of edges incident to B_3 . Thus $|A_3| \geq |A_2|/(k+1) \geq \frac{2}{5(k+1)}r(G)$. By Claim 5.6, $F \cup B_3$ is a forest, hence Theorem 5.5 applies to $G[C \cup N \cup F \cup B_3]$.

By Claim 5.8, $G[C \cup N \cup F]$ is connected, thus adding the vertices B_3 and the edges A_3 increases the cycle rank by $|A_3| - |B_3|$. This quantity is at least $|A_3|/2$ since any vertex of B_3 is incident to at least two edges of A_3 , and each edge of A_3 is incident to exactly one vertex of B_3 . Thus Theorem 5.5 yields a vertex of degree at least

$$\delta_k \cdot r(G[C \cup N \cup F \cup B_3]) \geq \frac{|A_3|}{2} \delta_k \geq \frac{1}{5(k+1)} \delta_k \cdot r(G) \geq \varepsilon_k \cdot r(G) \quad (4)$$

as desired. \lrcorner

Thus we can now assume that $|A_1| \geq \frac{5}{9}|A|$, and thus $|B| \geq \frac{5}{9}|A| \geq \frac{1}{2}r(G)$.

Let X , resp. Y , be the set of vertices of B with degree at least 3, resp. exactly 2, in $G[R']$. By Claim 5.7, this is a partition of B .

Claim 5.11. *If $|X| \geq |B|/5$, then G has an ε_k -rich vertex.*

Proof. Assume that $|X| \geq |B|/5$, and thus $|X| \geq \frac{1}{10}r(G)$.

The cycle rank is lowerbounded by the following sum:

$$r(G[R']) \geq |E(G[R'])| - |R'| = \frac{1}{2} \sum_{x \in R'} (d_{G[R']}(x) - 2). \quad (5)$$

By Claim 5.7, every term in the sum is non-negative, and each $x \in X$ contributes by at least $1/2$ to the sum. Thus $r(G[R']) \geq |X|/2 \geq \frac{1}{20}r(G)$, and the induction hypothesis applied to $G[R']$ (which is O_{k-1} -free) yields an $(\varepsilon_{k-1}/20)$ -rich vertex, which is also ε_k -rich. \lrcorner

Thus we can now assume that $|Y| \geq \frac{4}{5}|B| \geq \frac{2}{5}r(G)$.

Let Z be the set of vertices of R' that either are in Y or have degree at least 3 in $G[R']$. Remark that Z is exactly the set of vertices of R' with degree at least 3 in G . In $G[R']$, a *direct path* is a path whose endpoints are in Z , and whose internal vertices are not in Z . In particular, internal vertices of a direct path have degree 2. A direct path need not be induced, as its endpoints may be adjacent. As a degenerate case, we consider a cycle that contains a single vertex of Z to be a direct path whose two endpoints are equal. One can naturally construct a multigraph G_Z with vertex set Z and whose edges correspond to direct paths in $G[R']$. Remark that vertices of Z have the same degree in G_Z and in $G[R']$.

Any $y \in Y$ has two neighbors x_1, x_2 in G_Z . In degenerate cases, it may be that $x_1 = x_2 \neq y$ (multi-edge in G_Z), in which case $G[R']$ contains a banana between y and x_1 , or that $x_1 = x_2 = y$ (loop in G_Z), in which case there is a cycle C_y which is a connected component of $G[R']$, and such that y is the only vertex of Z in C_y . We partition Y into Y_i, Y_e as follows: for y, x_1, x_2 as above, if $x_1, x_2 \in Y$, then we place y in Y_i , and otherwise (x_1 or x_2 is in $Z \setminus Y$) we place y in Y_e .

Claim 5.12. *If $|Y_e| \geq \frac{3}{4}|Y|$, then G has an ε_k -rich vertex.*

Proof. Assume $|Y_e| \geq \frac{3}{4}|Y|$, and thus $|Y_e| \geq \frac{3}{10}r(G)$.

By definition, any vertex of Y_e is adjacent in G_Z to some vertex of $Z \setminus Y$. Thus, using that $d_{G_Z}(z) = d_{G[R']}(z)$ for any $z \in Z$, we obtain

$$\sum_{z \in Z \setminus Y} d_{G[R']}(z) \geq |Y_e|. \quad (6)$$

Recall inequality (5) on cycle rank:

$$r(G[R']) \geq \frac{1}{2} \sum_{x \in R'} (d_{G[R']}(x) - 2). \quad (7)$$

By Claim 5.7, the terms of this sum are non-negative. Thus, restricting it to $Z \setminus Y$, we have

$$r(G[R']) \geq \frac{1}{2} \sum_{z \in Z \setminus Y} (d_{G[R']}(z) - 2). \quad (8)$$

By definition of Z , vertices of $Z \setminus Y$ have degree at least 3 in $G[R']$. Thus, each term of the previous sum satisfies $d_{G[R']}(z) - 2 \geq d_{G[R']}(z)/3$. It follows using (6) that

$$r(G[R']) \geq \frac{1}{2} \sum_{z \in Z \setminus Y} \frac{d_{G[R']}(z)}{3} \geq \frac{|Y_e|}{6} \geq \frac{1}{20}r(G). \quad (9)$$

Thus the induction hypothesis applied to $G[R']$ (which is O_{k-1} -free) yields an $(\varepsilon_{k-1}/20)$ -rich vertex, which is also ε_k -rich. \lrcorner

Thus we can now assume that $|Y_i| \geq \frac{1}{4}|Y| \geq \frac{1}{10}r(G)$.

We now consider the induced subgraph H of $G[R']$ consisting of Y , and direct paths joining vertices of Y . Thus H has maximum degree 2, and since $G[R']$ is O_{k-1} -free, at most $k-2$ components of H are cycles, the rest being paths. Remark that the endpoints of paths in H correspond exactly to Y_e . Also, each connected component of H must contain at least one vertex of Y .

We perform the following cleaning operations, in this order:

- In each cycle of H , pick an arbitrary vertex and delete it, so that all connected components are paths.
- Iteratively delete a vertex of degree 0 or 1 which is not in Y , so that the endpoints of paths are all in Y .
- Delete any isolated vertex.

Let H' be the subgraph of H obtained after these steps.

Claim 5.13. *All but $3(k-2)$ vertices of Y_i are internal vertices of paths of H' .*

Proof. If $y \in Y_i$ belongs to a path of H , then it must be an internal vertex of this path, and the path is unaffected by the cleaning operations. Thus it suffices to prove that in each cycle of H , at most 3 vertices of Y_i are deleted or become endpoints of paths during the clean up.

Let C' be a cycle of H . If C' contains no more than 2 vertices of Y_i , there is nothing to prove. Remark in this case that C' is entirely deleted by the clean up. Otherwise, let x be the vertex deleted from H (which may be in Y_i), and let y_1, y_2 be the first vertices of Y_i strictly before and after x in the cyclic order of C' . Since C' has at least 3 vertices of Y_i , x, y_1, y_2 are all distinct. Then, it is clear that the cleaning operations transform C' into a path with endpoints y_1, y_2 , such that any $y \in Y_i \cap C'$ distinct from x, y_1, y_2 is an internal vertex of this path. \lrcorner

We now add H' to F , which yields a forest by Claim 5.6. Recall that vertices of Y are adjacent to $N \cup F$, and all endpoints of paths of H' are in Y . Thus, in $G[C \cup N \cup F \cup H']$, every vertex of H' has degree at least 2, and vertices of Y_i in the interior of paths of H' have degree at least 3. Since $G[C \cup N \cup F]$ is connected by Claim 5.8, the addition of H' does not change the number of connected components. Using Claim 5.13, this implies that

$$r(G[C \cup N \cup F \cup H']) \geq |Y_i| - 3(k - 2). \quad (10)$$

We finally apply Theorem 5.5 to $G[C \cup N \cup F \cup H']$ to obtain a vertex with degree at least

$$\delta_k \cdot (|Y_i| - 3(k - 2)).$$

Since G contains vertices of degree at least 2, we can always assume that $\varepsilon_k \cdot r(G) \geq 2$, and thus

$$|Y_i| \geq \frac{1}{10} \cdot 2\varepsilon_k^{-1} \geq \frac{1}{5} \cdot 30(k - 2) = 6(k - 2). \quad (11)$$

It follows that $|Y_i| - 3(k - 2) \geq |Y_i|/2$, and the previous argument yields a vertex of degree at least $\frac{\delta_k}{2} |Y_i| \geq \frac{\delta_k}{20} r(G)$, which is an ε_k -rich vertex. \square

6 Cutting trees into stars and paths

The setting in all this section is the following. We consider a graph H which is the disjoint union of a forest F and an independent set N , such that the neighborhood of N in F is a subset $S \subseteq V(F)$ containing all the leaves of F , and each vertex of S has a unique neighbor in N . Moreover H is O_k -free (and it can be deduced from the definition that H is 3-biclique-free).

A path is a special case of subdivided star. The *center* of a subdivided star is the vertex of degree at least 3, if any. If none, the subdivided star is a path, and its *center* is a vertex of degree 2 that belongs to S , if any, and an arbitrary vertex otherwise. We say a forest $F' \subseteq F$ is *S' -clean*, for some $S' \subseteq S$, if $V(F') \cap S' = L(F')$, where $L(F')$ denotes the set of leaves of F' . We define being *quasi- S' -clean* for a subdivided star as intersecting S' at exactly its set of leaves, plus possibly its center. Formally, a subdivided star T is *quasi- S' -clean* if $L(T) \subseteq V(T) \cap S' \subseteq L(T) \cup \{c\}$ where c is the center of T . The *degree* of a subdivided star is the degree of its center. A forest $F' \subseteq F$ of subdivided stars is *quasi- S' -clean*, for some $S' \subseteq S$, if all its connected components are quasi- S' -clean (subdivided stars).

Our approach in this section is summarized in Figure 3. We start with an arbitrary forest F and a subset S of vertices including all the leaves of F (the vertices of S are depicted in white, while the vertices of $F - S$ are depicted in black). We first extract quasi- S -clean subdivided stars (Lemma 6.1). We then extract quasi- S -clean subdivided stars of large degree, or S -clean paths (Lemma 6.3). Finally we extract S -clean subdivided stars of large degree or paths (Corollary 6.7). At each step the number of vertices of S involved in the induced subgraph of F we consider is linear in $|S|$.

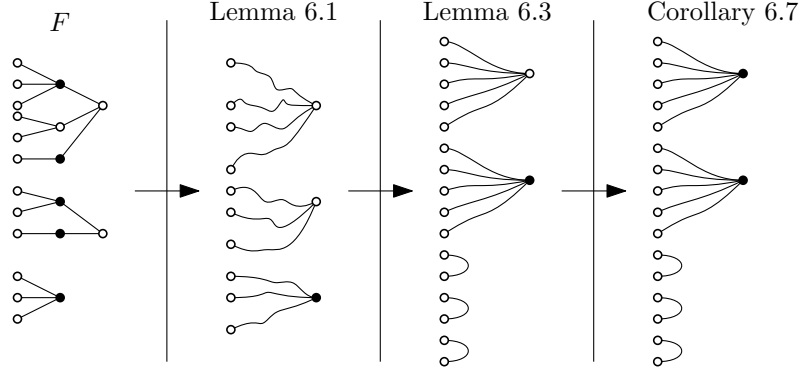


Figure 3: A visual summary of Section 6.

Lemma 6.1. *There is a subset $F^* \subseteq F$ containing at least $\frac{1}{2}|S|$ vertices of S such that each connected component of $H[F^*]$ is a quasi- S -clean subdivided star.*

Proof. We first use the following claim.

Claim 6.2. *There is a set of edges $X \subseteq E(H)$ such that every connected component of $H[F] \setminus X$ is either a quasi- S -clean subdivided star or a single vertex that does not belong to S .*

Proof. We proceed greedily, starting with $X = \emptyset$. While $H[F] \setminus X$ contains a component T and an edge $e \in T$ such that the two components of $T - e$ each contains either no vertex of S or at least two vertices of S , we add e to X .

Observe that in $H[F]$, every connected component with at least one edge contains at least 2 vertices of S . Throughout the process of defining X , every connected component of $H[F] \setminus X$ with at least one edge contains either 0 or at least 2 vertices of S .

At the end of the process, for any connected component T of $H[F] \setminus X$ with at least one edge, all the leaves of T belong to S . Otherwise, the edge incident to the leaf of T that is not in S can be added to X .

Thus, $H[F] \setminus X$ does not contain any component with more than one vertex of degree at least 3, since otherwise any edge on the path between these two vertices would have been added to X , yielding two components containing at least 2 leaves, and thus at least 2 vertices of S .

Observe also that if $H[F] \setminus X$ contains a component T with a vertex $v \in S$ that has degree 2 in T , then T is a path containing exactly 3 vertices of S , and thus T is a subdivided star whose center and leaves are in S , and whose other internal vertices are not in S . \square

To conclude, we need to select connected components of $H[F] \setminus X$ with more than one vertex and that are pairwise non-adjacent in H . Consider the minor G_F of F obtained by contracting each connected component of $H[F] \setminus X$ into a single vertex and deleting those that are a single vertex not in S . Since F is a forest, the graph G_F is a forest. We weigh each vertex of G_F by the number of elements of S that the corresponding connected component of $H[F] \setminus X$ contains. Since G_F is a forest, there is an independent set $\{u_1, u_2, \dots, u_p\}$ that contains at least half the total weight. The connected components corresponding to u_1, u_2, \dots, u_p together form a forest F^* with the required properties. \square

We observe that subdivided stars of small degree can be transformed into paths for a low price, as follows. A *subdivided star forest* is a forest whose components are subdivided stars (possibly paths).

Lemma 6.3. *For every $S' \subseteq S$, quasi- S' -clean subdivided star forest $F' \subseteq F$, and integer $D \geq 2$, there is a subdivided star forest $F'' \subseteq F'$ such that every connected component of $H[F'']$ is either an S' -clean path or a quasi- S' -clean subdivided star of degree at least D . Additionally, F'' contains at least $\frac{2|S' \cap F'|}{D}$ vertices of S' .*

Proof. We define F'' from F' as follows. Consider a connected component T of $H[F']$. If the center of T has degree at least D , we add T to F'' . Consider now the case where T is a quasi- S' -clean subdivided star whose center c has degree less than D . If $c \in S'$, we select a non-edgeless path $P \subseteq T$ between c and S' , and add P to F'' . If $c \notin S'$, we select two internally-disjoint paths $P_1, P_2 \subseteq T$ between c and S' , and add $P_1 \cup P_2$ to F'' . Note that $P_1 \cup P_2$ yields an S' -clean path.

To see that F'' contains at least $\frac{2|S' \cap F'|}{D}$ vertices of S' , we simply observe that out of a maximum of $(D - 1) + 1$ vertices of S' in a component T , we keep at least 2 in F'' . This adds up to $\frac{2|S'|}{D}$ vertices of S' since connected components of $H[F']$ are disjoint by definition. \square

Lemma 6.4. *If each vertex of N has degree less than $\frac{1}{8k}|S|$, one of the following holds.*

- *there is a subset S' of S and a subset F_2 of F such that F_2 contains $\frac{1}{32}|S|$ vertices of S' , and each connected component of $H[F_2]$ is an S' -clean subdivided star.*
- *there is a subset F_3 of F such that every connected component of F_3 is a quasi- S -clean subdivided star of degree at most 4 and F_3 contains at least $\frac{1}{8}|S|$ vertices of S .*

Proof. Let F^* be the forest obtained from Lemma 6.1, which contains at least $\frac{1}{2}|S|$ vertices of S , and such that each component of $H[F^*]$ is a quasi- S -clean subdivided star or an S -clean path. We define the *label* of a vertex of S to be its only neighbor in N .

Claim 6.5. *There is a subset F_1 of F^* containing at least $\frac{1}{4}|S|$ vertices of S , such that no subdivided star of F_1 has its center and one of its endpoints sharing the same label.*

Proof. Let ℓ be the maximum integer such that there exist ℓ subdivided stars S_1, S_2, \dots, S_ℓ in $H[F^*]$ and ℓ different labels $v_1, \dots, v_\ell \in N$, such that for any $1 \leq i \leq \ell$, S_i has its center and at least one of its endpoint labeled v_i . Note that in this case G contains ℓ independent cycles, and thus $\ell < k$ by assumption.

For any $1 \leq i \leq \ell$, remove all the leaves u of F^* that are labeled v_i , and also remove the maximal path of $H[F^*]$ ending in u . By assumption, there are at most $\frac{1}{8k}|S|$ such vertices u for each $1 \leq i \leq \ell$, and thus we delete at most $k \cdot \frac{1}{8k}|S| \leq \frac{1}{8}|S|$ vertices of S from F^* . We also delete the centers that have no leaves left (there are at most $k \cdot \frac{1}{8k}|S| \leq \frac{1}{8}|S|$ such deleted centers). Let F_1 be the resulting subset of F^* . Note that F_1 contains at least $|F^* \cap S| - 2 \cdot \frac{1}{8}|S| \geq (\frac{1}{2} - \frac{1}{4})|S| = \frac{1}{4}|S|$ vertices of S . \lrcorner

We can assume that a subset Y of at least $\frac{1}{8}|S|$ vertices of S in the forest F_1 obtained from Claim 6.5 are involved in a quasi- S -clean subdivided star of degree at least 5. Indeed, otherwise at least $\frac{1}{8}|S|$ vertices of S in the forest F_1 obtained from Claim 6.5 are involved in a quasi- S -clean subdivided star of degree at most 4 (note that an S -clean path is an S -clean subdivided star), and in this case the second outcome of Lemma 6.4 holds.

For each label $v \in N$, we choose uniformly at random with probability $\frac{1}{2}$ whether v is a *center label* or a *leaf label*. We then delete all the subdivided stars of F_1 whose center is labeled with a leaf label, and all the leaves whose label is a center label. Moreover, we delete from N all the vertices that are a center label, and let S' be the set of vertices of S whose neighbor in N is not deleted.

Take a vertex u of Y . If u is a center of a subdivided star, then the probability that u is not deleted is at least $\frac{1}{2}$. If u is a leaf, u is kept only if u and the center of the subdivided star it belongs to (which has by construction a different label) are correctly labeled, so u is kept with probability at least $\frac{1}{4}$. Overall, each vertex u of Y has probability at least $\frac{1}{4}$ to be kept. Thus the expectation of the fraction of vertices of Y not deleted is at least $\frac{1}{4}$, thus we can find an assignment of the labels to leaf labels or center labels, such that a subset $Z \subseteq Y$ with $|Z| \geq \frac{1}{4}|Y|$ survives.

We then iteratively delete vertices of degree 1 that do not belong to S' and all vertices of degree 0. Let F_2 be the resulting forest. Note that S' contains only the endpoints of stars with a leaf label, thus the forest F_2 is S' -clean. It remains to argue that F_2 contains a significant fraction of vertices of S . Note that a connected component of F_1 is deleted if and only if it contains at most one element of Z . Every such component has at least 4 elements in $Y \setminus Z$, hence there are at most $\frac{1}{4} \cdot \frac{3}{4}|Y| = \frac{3}{16}|Y|$ such components. It follows that F_2 contains at least $|Z| - \frac{3}{16}|Y| \geq \frac{1}{4}|Y| - \frac{3}{16}|Y| \geq \frac{1}{16}|S|$ elements of $Z \subseteq S$. \square

We now have all the ingredients to obtain the following two corollaries.

Corollary 6.6. *For any $D \geq 2$, there is a subset $F^* \subseteq F$ containing at least $\frac{1}{2D}|S|$ vertices of S such that each*

1. F^* induces a quasi- S -clean subdivided star forest whose components all have degree at least D , or
2. F^* induces an S -clean path forest.

Corollary 6.6 follows from Lemma 6.1 by applying Lemma 6.3 and observing that one of the two outcomes contains half the corresponding vertices in S .

Corollary 6.7. *Let $D \geq 2$. If each vertex of N has degree less than $\frac{1}{8k}|S|$, then there are $F'' \subseteq F$, $S' \subseteq S$ such that F'' contains at least $\frac{1}{32D}|S|$ vertices of S' and one of the following two cases apply.*

1. F'' induces an S' -clean subdivided star forest whose components all have degree at least D , or
2. F'' induces an S' -clean path forest.

Similarly, Corollary 6.7 follows from Lemma 6.4 by applying Lemma 6.3 and observing that one of the two outcomes contains half the corresponding vertices in S .

7 Trees, stars, and paths

In the proof of Theorem 5.5, we will apply Corollaries 6.6 and 6.7 several times, and divide our graph into two parts: a union of subdivided stars on one side, and a union of subdivided stars or paths on the other side. We now explain how to find a rich vertex in this context.

We start with the case where subdivided stars appear on both sides.

Lemma 7.1 (Star-star lemma). *Let $c > 0$ be the constant of Lemma 4.9. Let H be an O_k -free graph whose vertex set is the union of two sets L, R , such that*

- $S = L \cap R$ is an independent set,

- there are no edges between $L \setminus S$ and $R \setminus S$, and
- L (resp. R) induces in H a disjoint union of subdivided stars, whose centers have average degree at least $3ck \log k$, and whose set of leaves is precisely S .

Then H contains a vertex of degree at least $\frac{1}{2^{f'(3,k)}}|S| = \Omega(\frac{1}{k^3}|S|)$, where f' is the function of Lemma 4.4.

Proof. Note that H is 3-biclique-free (but might contain some $K_{2,2}$ subgraph) and O_k -free. By Lemma 4.4, there is a set X of at most $f'(3, k)$ vertices of H such that all bananas of H intersect X . Since the centers of the subdivided stars are the only vertices of degree larger than 2 in H , we can assume that X is a subset of the centers of the subdivided stars.

Assume first that less than $\frac{1}{2}|S|$ vertices of S are leaves of subdivided stars centered in an element of X . Let $S' \subseteq S$ be the leaves of the subdivided stars whose center is not in X (note that $|S'| \geq \frac{1}{2}|S|$), and remove from the subdivided stars of $H[L]$ and $H[R]$ all branches whose endpoint is not in S' to get new sets of vertices L', R' . The centers of the resulting S' -clean subdivided stars now have average degree at least $\frac{1}{2} \cdot 3ck \log k > ck \log k$. We denote the resulting S' -clean subdivided stars of $H[L']$ by S_1, S_2 , etc. and their centers by s_1, s_2 , etc. Similarly, we denote the resulting S' -clean subdivided stars of $H[R']$ by S'_1, S'_2 , etc. and their centers by s'_1, s'_2 , etc. Observe that by the definition of X , for any two centers s_i, s'_j , there is at most one vertex $u \in S'$ which is a common leaf of S_i and S'_j .

Let B be the bipartite graph with partite set s_1, s_2, \dots and s'_1, s'_2, \dots , with an edge between s_i and s'_j if and only if some vertex of S' is a common leaf of S_i and S'_j . Note that B has average degree more than $ck \log k$, and some induced subgraph of H (which is O_k -free) contains a strict subdivision of B . This contradicts Lemma 4.9.

So we can assume that at least $\frac{1}{2}|S|$ vertices of S are leaves of subdivided stars centered in an element of X . Then some vertex of X has degree at least $\frac{1}{2^{f'(3,k)}}|S|$, as desired. \square

We now consider the case where subdivided stars appear on one side, and paths on the other.

Lemma 7.2 (Star-path lemma). *Let $c > 0$ be the constant of Lemma 4.9. Let H be an O_k -free graph whose vertex set is the union of two sets L, R , such that*

- $S = L \cap R$ is an independent set,
- there are no edges between $L \setminus S$ and $R \setminus S$,
- L induces in H a disjoint union of paths, whose set of endpoints is precisely S , and
- R induces in H a disjoint union of subdivided stars, whose centers have average degree at least $4ck \log k$, and whose set of leaves is precisely S .

Then H contains a vertex of degree at least $\frac{1}{3^{f'(2,k)}}|S| = \Omega(\frac{1}{k^2}|S|)$, where f' is the function of Lemma 4.4.

Proof. Note that H is 2-biclique-free and O_k -free. By Lemma 4.4, there is a set X of at most $f'(2, k)$ vertices of H such that all bananas of H intersect X . Since the centers of the subdivided stars are the only vertices of degree more than 2 in H , we can assume that X is a subset of the centers of the subdivided stars.

Assume first that less than $\frac{1}{3}|S|$ vertices of S are leaves of subdivided stars centered in an element of X . Then there are at least $\frac{1}{6}|S|$ paths in $H[L]$ whose endpoints are not leaves of

stars centered in X . Let $S' \subseteq S$ be the endpoints of these paths (note that $|S'| \geq \frac{1}{3}|S|$), and remove from the subdivided stars of $H[R]$ all branches whose endpoint is not in S' to get R' . The centers of the resulting S' -clean subdivided stars in $H[R']$ now have average degree at least $\frac{1}{3} \cdot 4ck \log k > ck \log k$. We denote these subdivided stars by S_1, \dots, S_t , and their centers by s_1, \dots, s_t .

Given two centers s_i, s_j , we say that a pair $u_i, u_j \in S'$ is an $\{i, j\}$ -route if u_i is a leaf of S_i , u_j is a leaf of S_j , and there is a path with endpoints u_i, u_j in $H[L]$. Observe that by the definition of X , for every pair s_i, s_j , there is at most one $\{i, j\}$ -route.

Let G be the graph with vertex set s_1, \dots, s_t , with an edge between s_i and s_j if and only if there is an $\{i, j\}$ -route. Note that G has average degree more than $ck \log k$, and some induced subgraph of H (which is O_k -free) contains a strict subdivision of G . This contradicts Lemma 4.9.

So we can assume that at least $\frac{1}{3}|S|$ vertices of S are leaves of subdivided stars centered in an element of X . Then some vertex of X has degree at least $\frac{1}{3f'(2,k)}|S|$, as desired. \square

From the two previous lemmas and Lemma 6.1 we deduce the following.

Lemma 7.3 (Star-tree lemma). *There is a constant $c > 0$ such that the following holds. Let H be an O_k -free and t -biclique-free graph whose vertex set is the union of two sets L, R , such that*

- $S = L \cap R$ is an independent set partitioned into S_P, S_T ,
- there are no edges between $L \setminus S$ and $R \setminus S$,
- L induces in H a disjoint union of subdivided stars, whose centers have average degree at least $(8ck \log k)^2$, and whose set of leaves is equal to S , and
- R induces in H the disjoint union of
 - paths on a vertex set R_P , whose set of endpoints is equal to S_P , and
 - a tree T on a vertex set R_T such that S_T is a subset of leaves of T .

Then H contains a vertex of degree at least $\Omega(\frac{1}{k^4 \log k}|S|)$.

Proof. Let $c > 0$ be the constant of Lemma 4.9. Assume first that $|S_T| \leq 1$. Then since the subdivided stars of L have average degree at least $(8ck \log k)^2$, we have $|S_P| = |S| - |S_T| \geq (8ck \log k)^2 - 1 \geq 1$ and thus $|S_P| \geq \frac{1}{2}|S|$. By removing the branch of a subdivided star of L that has an endpoint in S_T (if any), we obtain a set of S_P -clean subdivided stars of average degree at least $\frac{1}{2} \cdot (8ck \log k)^2 \geq 4ck \log k$. By Lemma 7.2, we get a vertex of degree at least $\Omega(\frac{1}{k^2}|S_P|) = \Omega(\frac{1}{k^2}(|S|))$, as desired. So in the remainder we can assume that $|S_T| \geq 2$.

Let T' be the subtree of T obtained by repeatedly removing leaves that are not in S_T . Since $|S_T| \geq 2$, $L(T') = S_T$. Observe that $F' = T' \cup R_P$ is an S -clean forest (with $L(F') = S$), thus any S -quasi-clean subforest of F' is S -clean. It follows from Corollary 6.6 (applied to S, F' , and $D = 4ck \log k$) that F' contains a subset F^* containing at least $\frac{1}{2 \cdot 4ck \log k}|S|$ vertices of S , such that $H[F^*]$ induces either (1) an S -clean forest of path, or (2) an S -clean forest of subdivided stars of degree at least $4ck \log k$.

We denote this intersection of S and F^* by S^* , and we remove in the subdivided stars of $H[L]$ all branches whose endpoint is not in S^* to get a new set of vertices $L^* \subset L$. By assumption, the average degree of the subdivided stars in L^* is at least $\frac{(8ck \log k)^2}{8ck \log k} = 8ck \log k \geq 4ck \log k$.

In case (1) above we can now apply Lemma 7.2, and in case (2) we can apply Lemma 7.1. In both cases we obtain a vertex of degree at least $\Omega(\frac{1}{k^3}|S^*|) = \Omega(\frac{1}{k^4 \log k}|S|)$, as desired. \square

8 Proof of Theorem 5.5

We start by recalling the setting of Theorem 5.5. The graph G is a connected O_k -free graph of girth at least 11, and C is a shortest cycle in G . The neighborhood of C is denoted by N , and the vertex set $V(G) \setminus (C \cup N)$ is denoted by R . The subset of R consisting of the vertices adjacent to N is denoted by S . Since C is a shortest cycle, of size at least 11, each vertex of S has a unique neighbor in N , and a unique vertex at distance 2 in C . Moreover N and S are independent sets. In the setting of Theorem 5.5, R is a forest.

Our goal is to prove that there is a vertex whose degree is linear in the cycle rank $r(G)$. To this end, we assume that G has maximum degree at most $\delta \cdot r(G)$, for some $\delta > 0$, and prove that this yields a contradiction if δ is a small enough function of k .

By Lemma 5.1, we can assume that G is reduced, i.e., contains no vertex of degree 0 or 1. If G consists only of the cycle C , then $r(G) = 1$ and the theorem is immediate. Thus we can assume that N is non-empty, which in turn implies that S is non-empty since G is reduced.

Using that G is connected, remark that

$$r(G) = |E(G)| - |V(G)| + 1 = 1 + \frac{1}{2} \sum_{v \in V(G)} (d(v) - 2) \quad (12)$$

We start by proving that the cardinality of S is at least the cycle rank $r(G)$.

Claim 8.1. $|S| \geq r(G)$, and thus G has maximum degree at most $\delta|S|$.

Proof. Observe that $\frac{1}{2} \sum_{v \in C \cup N} (d(v) - 2) = \frac{1}{2}|S|$. Furthermore $\frac{1}{2} \sum_{v \in R} (d(v) - 2)$ is equal to $\frac{1}{2}|S|$ minus the number of connected components of $G[R]$, as R induces a forest and each vertex of S has a unique neighbor outside of R . Since R is non-empty, it follows from (12) that $r(G) \leq |S|$. We assumed that G has maximum degree at most $\delta \cdot r(G)$ which is at most $\delta|S|$, as desired. \square

In the remainder of the proof, we let $c > 0$ be a sufficiently large constant such that Lemmas 7.3 and 4.8 both hold for this constant.

We consider $\delta < \frac{1}{8k}$, and use Claim 8.1 to apply Corollary 6.7 to the subgraph H of G induced by N and $F = R$ (which is O_k -free), with $D = 2 \cdot (8ck \log k)^2$. We obtain subsets $N' \subseteq N$, $R'' \subseteq R$ such that if we define S' as the subset of $S \cap R''$ with a neighbor in N' , we have $|S'| \geq \frac{1}{32D}|S|$ and at least one of the following two cases apply.

1. Each connected component of $H[R'']$ is an S' -clean subdivided star of degree at least D ,
or
2. Each connected component of $H[R'']$ is an S' -clean path.

We first argue that the second scenario holds.

Claim 8.2. *Each connected component of $H[R'']$ is an S' -clean path.*

Proof. Assume for a contradiction that Case 2 does not apply, hence Case 1 applies.

Let G_1 be the subgraph of G induced by $C \cup N' \cup R''$ (see Figure 4, left). Since $|C| \geq 11$ and vertices of C have disjoint second neighborhoods in S' , there exists a vertex $v^* \in C$ that sees at most $\frac{1}{11}|S'|$ vertices of S' in its second neighborhood. If we remove from G_1 the vertex v^* , its neighborhood $N(v^*) \subseteq N'$, its second neighborhood $N^2(v^*) \subseteq S'$, and the corresponding

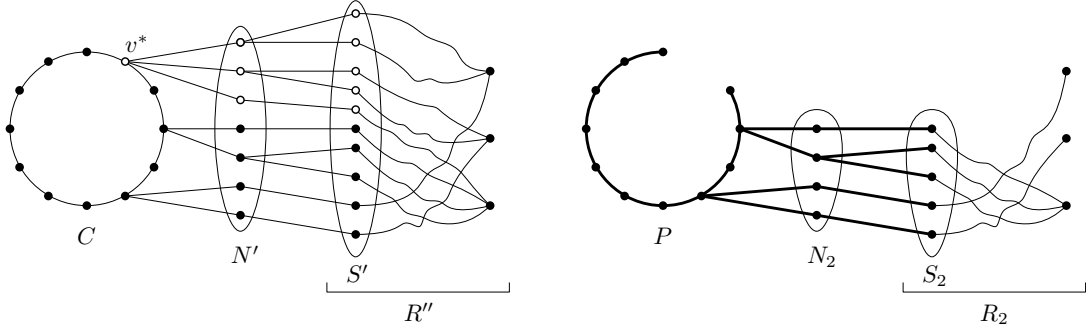


Figure 4: The graphs G_1 (left) and G_2 (right) in the proof of Claim 8.2.

branches of the subdivided stars of R'' , we obtain a graph G_2 whose vertex set is partitioned into a path $P = C - v^*$, its neighborhood $N_2 = N' - N(v^*)$, and the rest of the vertices R_2 (which includes the set $S_2 = S' - N^2(v^*)$), with the property that each component of $G_2[R_2]$ is an S_2 -clean subdivided star (see Figure 4, right). More importantly,

$$|S_2| \geq \frac{10}{11}|S'| \geq \frac{10}{11} \cdot \frac{1}{32D}|S| \geq \frac{1}{36D}|S|,$$

and the average degree of the centers of the subdivided stars is at least $\frac{10}{11}D \geq (8ck \log k)^2$.

Observe that $P \cup N_2 \cup S_2$ induces a tree in G_2 , such that all leaves of $G_2[P \cup N_2 \cup S_2]$ except at most two (the two neighbors of v^* on C) lie in S_2 , and non leaves of the tree are not in S_2 . We can now apply Lemma 7.3 with $R = P \cup N_2 \cup S_2$ and $L = R_2$. It follows that G_2 contains a vertex of degree at least $\Omega(\frac{1}{k^4 \log k}|S_2|) = \Omega(\frac{1}{k^6 \log^3 k}|S|) > \delta|S|$. Since G_2 is an induced subgraph of G , this contradicts Claim 8.1. \square

We denote the connected components of $H[R'']$ by P_1, \dots, P_ℓ , with $\ell \geq \frac{1}{64D}|S|$.

Claim 8.3. *There is a vertex u^* in C which has at least $\frac{1}{16(8ck \log k)^3}|S|$ endpoints of the paths P_1, \dots, P_ℓ in its second neighborhood, where $c > 0$ is the constant of Lemma 4.8.*

Proof. Assume for the sake of contradiction that each vertex of C has less than $\frac{1}{16(8ck \log k)^3}|S|$ endpoints of the paths P_1, \dots, P_ℓ in its second neighborhood.

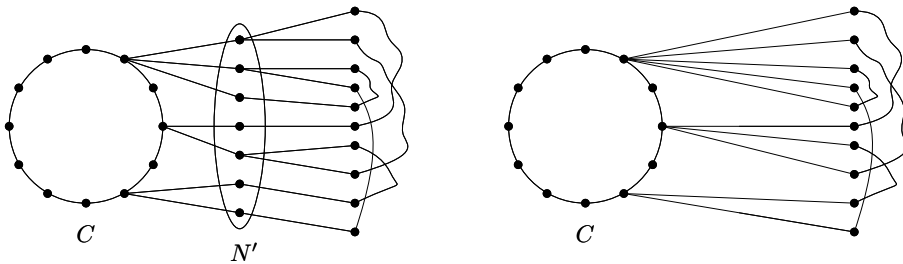


Figure 5: The graphs G_3 (left) and G_4 (right) in the proof of Claim 8.3.

Let G_3 be subgraph of G induced by $C \cup N'$ and $\bigcup_{i=1}^{\ell} V(P_i)$ (see Figure 5, left), and let G_4 be the graph obtained from G_3 by contracting each vertex of N' with its unique neighbor in C (i.e., G_4 is obtained from G_3 by contracting disjoint stars into single vertices), see Figure 5, right. Note that since G is O_k -free, G_3 and G_4 are also O_k -free (from the structural properties of C , N , and S , each cycle in G_4 can be canonically associated to a cycle in G_3 , and for any

set of independent cycles in G_4 , the corresponding cycles in G_3 are also independent). By our assumption, each vertex of C in G_4 has degree at most $\frac{1}{16(8ck \log k)^3} |S| + 2$, and G_4 consists of the cycle C together with $\ell \geq \frac{1}{64D} |S|$ paths whose endpoints are in C and whose internal vertices are pairwise disjoint and non-adjacent. By Lemma 4.8, it follows that

$$\frac{1}{64D} |S| < \ell \leq c \cdot \frac{1}{16(8ck \log k)^3} |S| \cdot k \log k,$$

and thus $D > 2(8ck \log k)^2$, which contradicts the definition of $D = 2(8ck \log k)^2$. \square

Claim 8.4. *If the vertices in $N[u^*]$ have average degree at least $(8ck \log k)^2$ in S' , then G contains a vertex of degree at least $\delta |S|$.*

Proof. The key idea of the proof of the claim is to consider the neighbors of u^* as the centers of stars (L) in Claim 7.3. In order to do that, we consider the subgraph G_5 of G induced by

- the path $C - u^*$,
- $N(u^*)$ and the paths P_i ($1 \leq i \leq \ell$) with at least one endpoint in the second neighborhood $N^2(u^*)$ of u^* (call these paths P'_1, \dots, P'_t), and
- the neighbors of the endpoints of the paths P'_1, \dots, P'_t in N .

All the components of $G_5 - N(u^*)$ are either paths P'_i with both endpoints in $N^2(u^*)$, or a tree whose leaves are all in $N^2(u^*)$ (except at most two leaves, which are the two neighbors of u^* in C). See Figure 6, right, for an illustration, where the vertices of $N^2(u^*)$ are depicted with squares and the components of $G_5 - N(u^*)$ are depicted with bold edges.

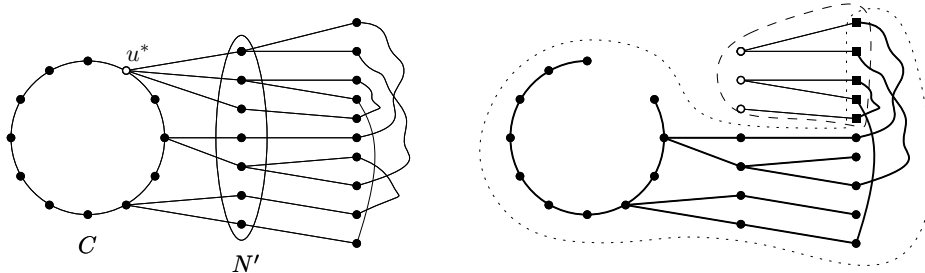


Figure 6: The graphs G_3 with the vertex u^* (left) and the graph G_5 (right) in the proof of Claim 8.4.

By considering the vertices of $N(u^*)$ and their neighbors in S' as stars (whose centers, depicted in white in Figure 6, right, have average degree at least $(8ck \log k)^2$) we can apply Lemma 7.3, and obtain a vertex of degree at least $\Omega(\frac{1}{k^4 \log k} |S'|) \geq \Omega(\frac{1}{k^6 \log^3 k} |S|) \geq \delta |S|$ in G_5 (and thus in G), which contradicts Claim 8.1. \square

Observe that if the vertices of $N(u^*)$ have average degree at most $(8ck \log k)^2$ in S' , then u^* has degree at least $\frac{1}{16(8ck \log k)^3} |S| \geq \delta |S|$. If not, by Claim 8.4, G also contains a vertex of degree at least $\delta |S|$. Both cases contradict Claim 8.1, and this concludes the proof of Theorem 5.5. \square

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References

- [AAC⁺22] Tara Abrishami, Bogdan Alecu, Maria Chudnovsky, Sepehr Hajebi, Sophie Spirkl, and Kristina Vušković. Induced subgraphs and tree decompositions V. One neighbor in a hole, 2022.
- [ACHS22] Tara Abrishami, Maria Chudnovsky, Sepehr Hajebi, and Sophie Spirkl. Induced subgraphs and tree-decompositions III. Three-path-configurations and logarithmic tree-width. 2022.
- [Ale82] Vladimir E Alekseev. The effect of local constraints on the complexity of determination of the graph independence number. *Combinatorial-algebraic methods in applied mathematics*, pages 3–13, 1982.
- [AWZ17] Stephan Artmann, Robert Weismantel, and Rico Zenklusen. A strongly polynomial algorithm for bimodular integer linear programming. In Hamed Hatami, Pierre McKenzie, and Valerie King, editors, *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017*, pages 1206–1219. ACM, 2017.
- [BBB⁺18] Marthe Bonamy, Edouard Bonnet, Nicolas Bousquet, Pierre Charbit, and Stéphan Thomassé. EPTAS for max clique on disks and unit balls. In Mikkel Thorup, editor, *59th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2018, Paris, France, October 7-9, 2018*, pages 568–579. IEEE Computer Society, 2018.
- [BBB⁺21] Marthe Bonamy, Édouard Bonnet, Nicolas Bousquet, Pierre Charbit, Panos Giannopoulos, Eun Jung Kim, Paweł Rzażewski, Florian Sikora, and Stéphan Thomassé. EPTAS and subexponential algorithm for maximum clique on disk and unit ball graphs. *J. ACM*, 68(2), jan 2021.
- [Ber84] Alan A Bertossi. Dominating sets for split and bipartite graphs. *Information processing letters*, 19(1):37–40, 1984.
- [BFMR14] Adrian Bock, Yuri Faenza, Carsten Moldenhauer, and Andres J. Ruiz-Vargas. Solving the stable set problem in terms of the odd cycle packing number. In Venkatesh Raman and S. P. Suresh, editors, *34th International Conference on Foundation of Software Technology and Theoretical Computer Science, FSTTCS 2014, December 15-17, 2014, New Delhi, India*, volume 29 of *LIPICs*, pages 187–198. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2014.
- [BGK⁺18] Édouard Bonnet, Panos Giannopoulos, Eun Jung Kim, Paweł Rzażewski, and Florian Sikora. QPTAS and subexponential algorithm for maximum clique on disk graphs. In Bettina Speckmann and Csaba D. Tóth, editors, *34th International Symposium on Computational Geometry, SoCG 2018, June 11-14, 2018, Budapest, Hungary*, volume 99 of *LIPICs*, pages 12:1–12:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.
- [BKTW22] Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width I: tractable FO model checking. *J. ACM*, 69(1):3:1–3:46, 2022.
- [CFH⁺20] Michele Conforti, Samuel Fiorini, Tony Huynh, Gwenaél Joret, and Stefan Weltge. The stable set problem in graphs with bounded genus and bounded odd cycle

- packing number. In Shuchi Chawla, editor, *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020*, pages 2896–2915. SIAM, 2020.
- [CFK⁺15] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015.
- [Cou90] Bruno Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Information and Computation*, 85(1):12–75, 1990.
- [CP84] Derek G. Corneil and Yehoshua Perl. Clustering and domination in perfect graphs. *Discrete Applied Mathematics*, 9(1):27–39, 1984.
- [Dav22] James Davies. Oberwolfach report 1/2022. doi:10.4171/OWR/2022/1., 2022.
- [DMŠ21] Clément Dallard, Martin Milanič, and Kenny Štorgel. Tree decompositions with bounded independence number and their algorithmic applications. *CoRR*, abs/2111.04543, 2021.
- [DP20] Zdeněk Dvořák and Jakub Pekárek. Induced odd cycle packing number, independent sets, and chromatic number, 2020.
- [Dvo18] Zdeněk Dvořák. Induced subdivisions and bounded expansion. *European Journal of Combinatorics*, 69:143–148, 2018.
- [EP65] Paul Erdős and Lajos Pósa. On independent circuits contained in a graph. *Canadian Journal of Mathematics*, 17:347–352, 1965.
- [FJWY21] Samuel Fiorini, Gwenaël Joret, Stefan Weltge, and Yelena Yuditsky. Integer programs with bounded subdeterminants and two nonzeros per row. In *62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7-10, 2022*, pages 13–24. IEEE, 2021.
- [GKPT12] Petr A. Golovach, Marcin Kaminski, Daniël Paulusma, and Dimitrios M. Thilikos. Induced packing of odd cycles in planar graphs. *Theor. Comput. Sci.*, 420:28–35, 2012.
- [GLP⁺21] Peter Gartland, Daniel Lokshtanov, Marcin Pilipczuk, Michal Pilipczuk, and Pawel Rzazewski. Finding large induced sparse subgraphs in $c_{>t}$ -free graphs in quasipolynomial time. In Samir Khuller and Virginia Vassilevska Williams, editors, *STOC '21: 53rd Annual ACM SIGACT Symposium on Theory of Computing, Virtual Event, Italy, June 21-25, 2021*, pages 330–341. ACM, 2021.
- [Gol04] Martin Charles Golumbic. *Algorithmic graph theory and perfect graphs*. Elsevier, 2004.
- [IPZ01] Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *J. Comput. Syst. Sci.*, 63(4):512–530, 2001.
- [Jae74] François Jaeger. On vertex-induced forests in cubic graphs. In *Proceedings of the 5th Southeastern Conference on Combinatorics, Graph Theory and Computing*, pages 501–512, 1974.

- [KKTW01] Daniel Král, Jan Kratochvíl, Zsolt Tuza, and Gerhard J Woeginger. Complexity of coloring graphs without forbidden induced subgraphs. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 254–262. Springer, 2001.
- [KO04] Daniela Kühn and Deryk Osthus. Induced subdivisions in $K_{s,s}$ -free graphs of large average degree. *Combinatorica*, 24(2):287–304, 2004.
- [Kor22] Tuukka Korhonen. Grid induced minor theorem for graphs of small degree. *CoRR*, abs/2203.13233, 2022.
- [KST54] Tamás Kövári, Vera T. Sós, and Paul Turán. On a problem of K. Zarankiewicz. *Colloquium Mathematicum*, 3:50–57, 1954.
- [Le17] Ngoc-Khang Le. personal communication. 2017.
- [NSS22] Tung Nguyen, Alex Scott, and Paul D. Seymour. Induced paths in sparse cycle-touching graphs. 2022.
- [Pil11] Michał Pilipczuk. Problems parameterized by treewidth tractable in single exponential time: A logical approach. In Filip Murlak and Piotr Sankowski, editors, *Mathematical Foundations of Computer Science 2011 - 36th International Symposium, MFCS 2011, Warsaw, Poland, August 22-26, 2011. Proceedings*, volume 6907 of *Lecture Notes in Computer Science*, pages 520–531. Springer, 2011.
- [Pol74] Svatopluk Poljak. A note on stable sets and colorings of graphs. *Commentationes Mathematicae Universitatis Carolinae*, 15(2):307–309, 1974.
- [Ree99] Bruce A. Reed. Mangoes and blueberries. *Combinatorica*, 19(2):267–296, 1999.
- [RS86] Neil Robertson and Paul D. Seymour. Graph minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(1):92–114, 1986.
- [RS95] Neil Robertson and Paul D. Seymour. Graph minors. XIII. The disjoint paths problem. *J. Comb. Theory, Ser. B*, 63(1):65–110, 1995.
- [ST21] Ni Luh Dewi Sintiar and Nicolas Trotignon. (Theta, triangle)-free and (even hole, K_4)-free graphs - Part 1: Layered wheels. *J. Graph Theory*, 97(4):475–509, 2021.
- [Tro22] Nicolas Trotignon. Oberwolfach report 1/2022. doi:10.4171/OWR/2022/1., 2022.
- [Tur41] Paul Turán. On an external problem in graph theory. *Matematikai és Fizikai Lapok*, 48:436–452, 1941.