Flip Distance to a Non-crossing Perfect Matching

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Abstract

A *perfect straight-line matching M* on a finite set P of points in the plane is a set of segments such that each point in P is an endpoint of exactly one segment. M is *non-crossing* if no two segments in M cross each other. Given a perfect straight-line matching M with at least one crossing, we can remove this crossing by a flip operation. The flip operation removes two crossing segments on a point set Q and adds two non-crossing segments to attain a new perfect matching M'. It is well known that after a finite number of flips, a non-crossing matching is attained and no further flip is possible. However, prior to this work, no non-trivial upper bound on the number of flips was known. If $g(n)$ (resp. $k(n)$) is the maximum length of the longest (resp. shortest) sequence of flips starting from any matching of size n, we show that $g(n) = O(n^3)$ and $g(n) = \Omega(n^2)$ (resp. $k(n) = O(n^2)$ and $k(n) = \Omega(n)$).

Van Leeuwen and Schoone showed with the same argument and the same definition of flip how to transform a Hamilton cycle to a non-crossing Hamilton cycle on a set of n points within $O(n^3)$ flips [17]. Therefore, we do not consider the main result (our upper bound on $g(n)$) as a new contribution, because the used technique is exactly the same.

We want to use these proceedings to draw attention again on this old problem and hope to stimulate research that will close the gap between the upper and lower bound.
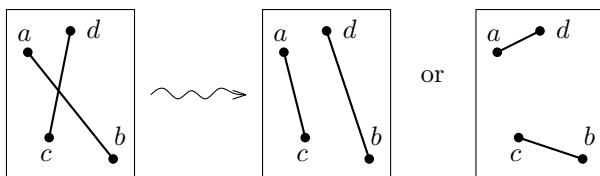
1 Introduction

Figure 1: Two crossing segments are replaced by two non-crossing segments. There are two ways to flip.

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Given $2n$ points in the plane in general position (no three points on a line), we define a *perfect straight-line non-crossing matching* as a set of n segments such that each point is incident to exactly one segment and no two segments intersect. Given $2n$ points in the plane, it is well-known that a perfect straight-line non-crossing matching always exists. One elegant argument to see this is to start with any perfect straight-line matching, potentially self-intersecting, and remove any crossing by a flip (see Figure 1). Although the total number of crossings might increase (see Figure 5), the sum of the length of all the segments decreases (see Figure 2). Thus, the process will eventually end with a perfect non-crossing matching.
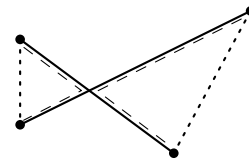

Figure 2: The two new edges (dotted) are shorter than the old edges (solid) since the dashed part to the left (resp. to the right) of the crossing is longer than the dotted segment on the left (resp. on the right).

A simpler argument is to take the first two points with lowest x-coordinate and connect them with a segment and continue with the remaining points by induction. Contrary to the first argument, this does not carry over to the bichromatic setting. In the bichromatic setting, we are given n red and n blue points and a *bichromatic matching* is a matching as above with the additional property that only segments linking points of different colors are allowed.

Motivated by this old folklore result, we investigate the question on the maximum and minimum number of flips that are necessary and sufficient to reach a straight-line non-crossing matching.

1.1 Preliminaries

From here on, P always denotes a set of $2n$ points in the plane and M a perfect straight-line matching on P. Given two points a and b we denote by $\mathrm{seg}(a, b)$ the segment with endpoints a and b. Given a perfect straight-line matching M with at least one crossing, we can remove this crossing by a flip operation. The flip operation removes any two crossing segments

$\mathrm{seg}(a, b)$ and $\mathrm{seg}(c, d)$ on a point set $Q = \{a, b, c, d\}$ and adds two non-crossing segments ($\mathrm{seg}(a, c)$ and $\mathrm{seg}(b, d)$) or ($\mathrm{seg}(a, d)$ and $\mathrm{seg}(b, c)$) to attain a new matching M'. Matching M is a *successor* of matching M' if we can construct M from M' by a single flip. We say that $\mathcal{M} = (M_0, \ldots, M_k)$ is a valid sequence of matchings, if each matching M_{i+1} is a successor of M_i and M_k is non-crossing. The number k denotes the length of $\mathcal{M}$. Given a set P of $2n$ points in the plane, we define:

$$f(M) = \max\{\, k : \exists \mathcal{M} \text{ of length } k \text{ with } M = M_0 \,\};$$

$$h(M) = \min\{\, k : \exists \mathcal{M} \text{ of length } k \text{ with } M = M_0 \,\}.$$

Consequently functions $g(n)$ and $k(n)$ are defined as:

$$g(n) = \max\{\, f(M) : M \text{ is a matching on } 2n \text{ points} \,\};$$

$$k(n) = \max\{\, h(M) : M \text{ is a matching on } 2n \text{ points} \,\}.$$

1.2 Results

We establish the following result:

Theorem 1 ([17]) *Let n be a large enough natural number. Then it holds:*

$$\frac{n^2 - n}{2} = \binom{n}{2} \le g(n) \le n^3.$$

This result immediately carries over to bichromatic matchings. We conjecture that $g(n) = \Theta(n^2)$.

Theorem 2 *Let n be a large enough natural number. Then it holds:*

$$n - 1 \le k(n) \le \frac{n^2}{2},$$

for some constants C.

Our proof of Theorem 2 does *not* carry over to the bichromatic case. However, we will see that the upper bound further holds if the crossing to flip is imposed at each step by an adversary and we may only choose which of the two flips (see Figure 1) we perform.

1.3 Related Work

The most relevant work is by van Leeuwen and Schoone, who showed the same upper bound on $g(n)$ for Hamilton cycles instead of matchings [17]. The study of Hamilton cycles is motivated as a post-processing step for algorithms that find a traveling salesman tour on a set of points. They show that they can improve the solution to be non-crossing in $O(n^3)$ steps.

For points in convex position, Oda and Watanabe established linear upper and lower bounds on $g(n)$, again on Hamilton cycles instead of matchings [15].

The combinatorial work on flip graphs of geometric structures is fairly large. See the survey by Bose and Hurtado [11] for an overview and some motivations.

Matchings, triangulations and spanning trees are commonly studied in recent work [1–9, 12, 13, 16]. Particularly interesting are triangulations of points that are in convex positions as they correspond to Catalan structures. Another interesting application comes from Lawson flips, which can be used to reach the Delaunay triangulation in $O(n^2)$ flips [14]. This can be used for the reverse search technique to enumerate triangulations [10].

2 Lower Bounds

We start with the lower bound for Theorem 1 and 2. Let ℓ and ℓ' be two parallel horizontal lines and let P be a set of $2n$ points, n of which are on ℓ and ℓ' respectively. In the following, we consider only matchings that connect points from ℓ to ℓ' (see Figure 3). Every
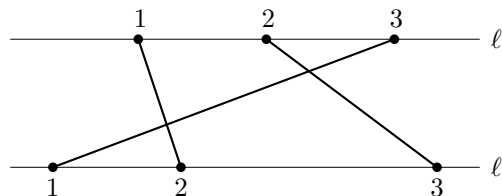

Figure 3: Matching corresponding to cycle (123).

such matching M can be interpreted as a permutation π_M and M is crossing free if and only if π_M is the identity. We can always do flips that correspond to an elementary step in bubble sort. Bubble sort on permutation π needs as many steps as the number of inversions of π. And, the number of inversions is at most $\binom{n}{2}$. A small perturbation of the point set ensures general position.

For the lower bound of Theorem 2, we define a *Schoone matching* on $2n$ points in convex position denoted by $p_1, p_2, \ldots, p_{2n}$ in counterclockwise order as follows. The points p_1 is linked to p_{n+1} and for each $i \in [2, n]$, p_i to p_{2n+2-i} (see Figure 4). Observe that
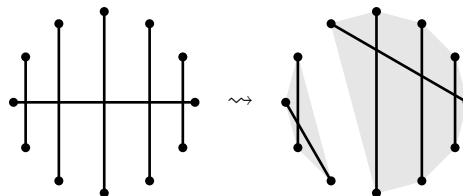

Figure 4: An initial configuration guaranteeing the lower bound of Theorem 2, and a possible flip.

a Schoone matching decomposes into two Schoone matchings after any flip. By this the total number of

crossings decreases by exactly one. This implies by induction that $h(M)$ equals the number of crossings minus one, for all Schoone matchings. Thus $n - 1 \leq k(n)$.

3 Upper Bounds

Before we prove the upper bound, observe in Figure 5 that the number of crossings might increase after a flip. It is also possible that a segment that has dis-
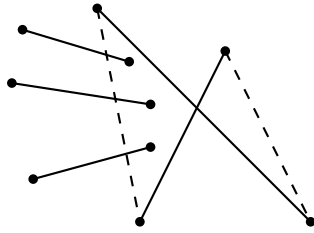

Figure 5: After the depicted flip, the number of crossings goes from 1 to 3.

appeared after a flip reappear after some more flips (see Figure 6). These two observations suggest that
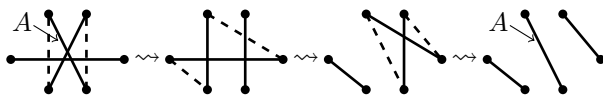

Figure 6: Segment A disappears and reappears.

there is no straightforward way of getting a good upper bound.

For the upper bound of Theorem 1, we define a potential function $\Phi_{\mathcal{L}}(M)$ that depends on a well-chosen set of lines $\mathcal{L}$. We show that $\Phi_{\mathcal{L}}(M) \leq 4n^3$ and that $\Phi_{\mathcal{L}}$ decreases by at least four after any flip. The potential function $\Phi_{\mathcal{L}}(M)$ is defined as the number of intersections between a line of $\mathcal{L}$ and a segment of M. We define $\mathcal{L}$ as follows. Given two points $p, q \in P$ let ℓ be the supporting line of p and q. We add to $\mathcal{L}$ the two lines slightly above and below ℓ (see Figure 7).
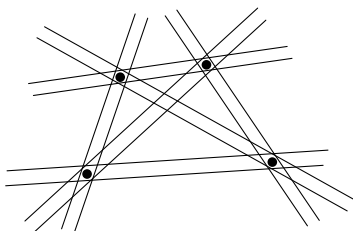

Figure 7: Construction of $\mathcal{L}$.

It holds that $|\mathcal{L}| = 2\binom{2n}{2} \leq 4n^2$. As any line and segment can cross at most once it follows $\Phi_{\mathcal{L}}(M) \leq |\mathcal{L}| \cdot |M| = 4n^3$. It remains to show that the number of segment-line intersections decreases by at least four in any flip. Consider two crossing segments A and B
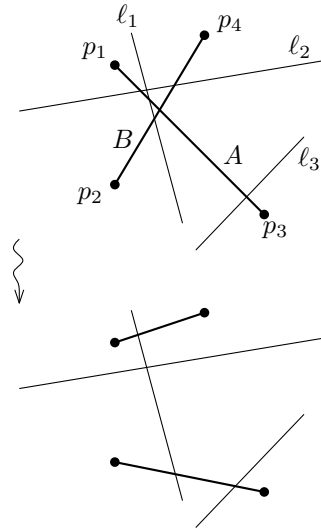

Figure 8: Flipping A and B yields fewer segment-line intersections.

on points $Q = \{p_1, \ldots, p_4\}$ as in Figure 8. Note that there are only three combinatorial types of lines intersecting the convex hull of Q. Either a line separates p_1 and p_2 from p_3 and p_4 as ℓ_1; a line separates p_1 and p_4 from p_2 and p_3 as ℓ_2; or a line separates one point from the other three as ℓ_3. For every type of lines the number of intersections does not increase after flipping A and B. It is also easy to see that the number of intersections decreases by two for lines of type ℓ_1 or ℓ_2 after flipping A and B. By definition of $\mathcal{L}$, there exists for every crossing of two segments at least two lines of type ℓ_1 and at least two lines of type ℓ_2. Thus $\Phi_{\mathcal{L}}(M)$ decreases by at least four as claimed.

For the upper bound of Theorem 2, we define a different set of lines $\mathcal{K}$, which contains one vertical line between any two consecutive points ordered in x-direction, see Figure 9. It follows that $\Phi_{\mathcal{K}}(M) \leq$
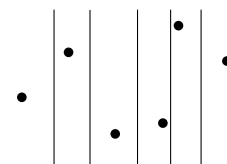

Figure 9: The set $\mathcal{K}$.

n^2 since $|\mathcal{K}| = n - 1$. We have to show that $\Phi_{\mathcal{K}}$ decreases by at least two after each flip. Let A and B be two crossing segments on the points p_1, p_2, p_3, p_4 ordered by x-coordinate. Then we replace A and B by $\text{seg}(p_1, p_2)$ and $\text{seg}(p_3, p_4)$, see Figure 10. It is clear that at least one line ℓ between p_2 and p_3 is not crossed after the flip and was crossed twice before the flip.
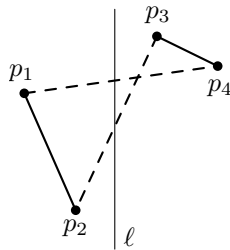
Figure 10: The number of crossings between ℓ and the segments of the matchings decreases by 2.

Acknowledgments

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