Graphs without a 3-connected subgraph are 4-colorable

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Abstract

In 1972, Mader showed that every graph without a 3-connected subgraph is 4-degenerate and thus 5-colorable. We show that the number 5 of colors can be replaced by 4, which is best possible.

1 Introduction

Throughout the paper all graphs are finite and simple, and we only use standard notions and notations. We recall that a graph is *k-connected* if it has at least $k+1$ vertices and no separator with at most $k-1$ vertices. In 1972, Mader [1] proved the following theorem.

Theorem 1.1. *For every integer $k \geq 1$, every graph with average degree at least $4k$ contains a $(k+1)$-connected subgraph.*

Focusing on the case $k = 2$ of Theorem 1.1, call a graph *fragile* if it contains no 3-connected subgraph. From Theorem 1.1, every fragile graph contains a vertex of degree at most 7. By restricting the proof of Mader to the case $k = 2$, it is easy to show that all fragile graphs G on at least 4 vertices satisfy $|E(G)| \leq 2.5|V(G)| - 5$ (we supply the proof in Section 3 for the sake of completeness). So the average degree of G is smaller than 5. Thus every fragile graph contains a vertex of degree at most 4, and this is best possible as shown by the graph in Figure 1. Every fragile graph is therefore 5-colorable.

Despite recent progress on related questions, there is no available proof that the number 5 of colors can be improved. Scott and Seymour announced that they have a proof that for all $m \geq 4$, every graph with chromatic number $m+1$ contains a 3-connected subgraph with chromatic number m, see [2] (that also contains a thorough literature review) where this is referred to as a personal communication. The objective of this paper is to prove the following theorem that is a particular case of the result claimed by Scott and Seymour.

Theorem 1.2. *Every graph without 3-connected subgraph is 4-colorable.*

Theorem 1.2 is best possible as shown by the graph in Figure 1. The proof of Theorem 1.2 is given in Section 2. Several remarks and open questions are presented in Section 3.

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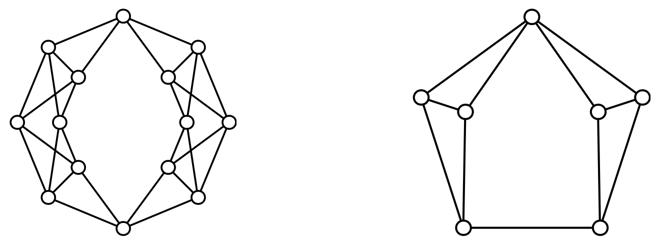

Figure 1: Graphs with no 3-connected subgraph.

2 Proof of Theorem 1.2

To prove Theorem 1.2, we shall establish the following stronger statement. By *k-coloring of a graph G*, we mean a function c that associates to each vertex of G an integer in $\{1, \dots, k\}$ and such that for all edges xy of G, $c(x) \neq c(y)$.

Theorem 2.1. *Every fragile graph G satisfies the following four conditions.*

(1) For all non-adjacent $x, y \in V(G)$, G admits a 4-coloring c such that $c(x) = c(y)$.

(2) For all distinct $x, y \in V(G)$, G admits a 4-coloring c such that $c(x) \neq c(y)$.

(3) For all distinct $x, y, z \in V(G)$, G admits a 4-coloring c such that $c(x) \notin \{c(y), c(z)\}$.

(4) For all distinct $x, y, z \in V(G)$ that are not pairwise adjacent, G admits a 4-coloring c such that $|\{c(x), c(y), c(z)\}| = 2$.

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| \leq 3$, then G obviously satisfies conditions (1)–(4).

For the induction step, suppose $|V(G)| \geq 4$ and that the statement holds for every graph with less vertices than G. Since G is not 3-connected, there exist two induced subgraphs G_1, G_2 of G such that $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$, $V(G_1) \setminus V(G_2) \neq \emptyset$, $V(G_2) \setminus V(G_1) \neq \emptyset$, and $S = V(G_1) \cap V(G_2)$ has size at most 2. Moreover, since G is fragile, G_1 and G_2 are also fragile and, as $|V(G_1)|, |V(G_2)| < |V(G)|$, we may apply the induction hypothesis to both G_1 and G_2.

If $S = \emptyset$, the induction step is obvious and we omit the details. So we may set $S = \{u, v\}$ (possibly $u = v$). We have to prove that for any of the precoloring conditions C among (1)–(4) on any given set $X \subseteq V(G)$ (namely, $X = \{x, y\}$ for conditions (1) and (2) and $X = \{x, y, z\}$ for conditions (3) and (4)) some appropriate 4-coloring exists. Suppose first that $X \subseteq V(G_1)$. Then, by the induction hypothesis, G_1 admits a coloring c_1 that satisfies C. By applying (1) or (2) to the vertices u and v of G_2 (or trivially if $u = v$), and up to a relabeling of the colors, we can force a coloring c_2 of G_2 such that $c_2(u) = c_1(u)$ and $c_2(v) = c_1(v)$. Note that the case when uv is an edge corresponds to the usual amalgamation of two colorings on a clique separator. Hence, $c_1 \cup c_2$ is a coloring of G that

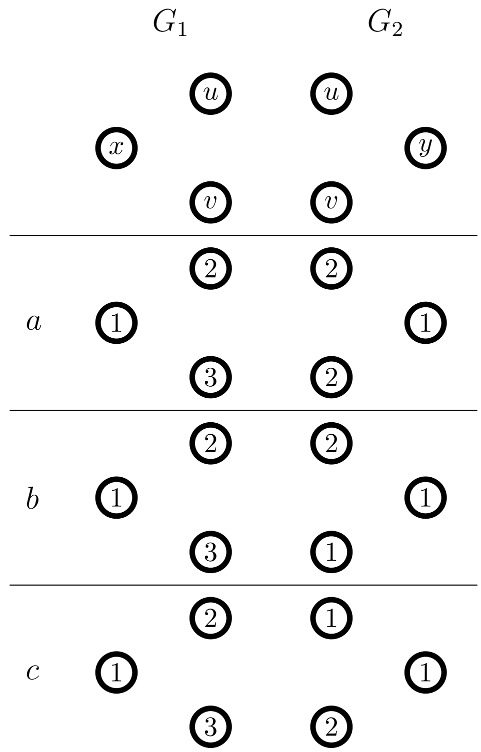

Figure 2: Colorings obtained in the proof of Claim 1.

satisfies C. The proof is similar when $X \subseteq V(G_2)$. Hence, from here on, we may assume that

$$X \text{ intersects both } V(G_1) \setminus V(G_2) \text{ and } V(G_2) \setminus V(G_1). \qquad (\star)$$

We now prove four claims, from which Theorem 2.1 trivially follows. Their proofs are easy when $u = v$, so we omit this case and assume $u \neq v$. Note that, unless specified otherwise, we shall make no assumption on whether u and v are adjacent.

Claim 1. *The graph G satisfies (1).*

Proof. By ($\star$), we may assume that $x \in V(G_1) \setminus V(G_2)$ and $y \in V(G_2) \setminus V(G_1)$. We build three colorings a_1, b_1 and c_1 of G_1 and three colorings a_2, b_2 and c_2 of G_2 that are represented in Figure 2 for the reader's convenience.

By (3) applied to x, u, v (in this order) in G_1, we obtain a coloring a_1 of G_1 such that $a_1(x) \notin \{a_2(u), a_2(v)\}$. Similarly, we obtain a coloring a_2 of G_2 such that $a_2(y) \notin \{a_2(u), a_2(v)\}$. Up to a relabeling, we may assume that $a_1(x) = a_2(y) = 1$, $a_1(u) = a_2(u) = 2$ and $a_1(v), a_2(v) \in \{2, 3\}$. If $a_1(v) = a_2(v)$, then $a_1 \cup a_2$ is a coloring of G that satisfies (1). Hence, up to symmetry, we may assume that $a_1(v) = 3$ and $a_2(v) = 2$.

By (3) applied to u, x, v in G_1, we obtain a coloring b_1 of G_1 such that $b_1(u) \notin \{b_1(x), b_1(v)\}$. Similarly, we obtain a coloring b_2 of G_2 such that $b_2(u) \notin \{b_2(y), b_2(v)\}$.

Up to a relabeling, we may assume that $b_1(x) = b_2(y) = 1$, $b_1(u) = b_2(u) = 2$ and $b_1(v), b_2(v) \in \{1, 3\}$. If $b_1(v) = b_2(v)$, then $b_1 \cup b_2$ is a coloring of G that satisfies (1). Hence, we may assume that $b_1(v) \neq b_2(v)$. If $b_2(v) = 3$, then $a_1 \cup b_2$ is a coloring of G that satisfies (1). Hence, we may assume that $b_1(v) = 3$ and $b_2(v) = 1$.

By (3) applied to v, x, u in G_1, we obtain a coloring c_1 of G_1 such that $c_1(v) \notin \{c_1(x), c_1(u)\}$. Similarly, we obtain a coloring c_2 of G_2 such that $c_2(v) \notin \{c_2(y), c_2(u)\}$. Up to a relabeling, we may assume that $c_1(x) = 1$ and either $c_1(u) = 1$ and $c_1(v) = 2$ or $c_1(u) = 2$ and $c_1(v) = 3$. Up to a relabeling, we may also assume that $c_2(y) = 1$ and either $c_2(u) = 1$ and $c_2(v) = 2$ or $c_2(u) = 2$ and $c_2(v) = 3$. If $c_1(u) = c_2(u)$, then $c_1 \cup c_2$ is a coloring of G that satisfies (1). Hence, we may assume that $c_1(u) \neq c_2(u)$. If $c_2(u) = 2$ (and so $c_2(v) = 3$), then $a_1 \cup c_2$ is a coloring of G that satisfies (1). Hence, we may assume that $c_1(u) = 2$, $c_1(v) = 3$, $c_2(u) = 1$ and $c_2(v) = 2$.

By (4) applied to x, u, v in G_1, we obtain a coloring d_1 of G_1 such that $|\{d_1(x), d_1(u), d_1(v)\}| = 2$ (note that x, u and v are not pairwise adjacent because $a_2(u) = a_2(v)$ implies $uv \notin E(G)$). Up to a relabeling, we may assume that $d_1(x) = 1$ and $\{d_1(x), d_1(u), d_1(v)\} = \{1, 2\}$. If $d_1(u) = 1$ and $d_1(v) = 2$, then $d_1 \cup c_2$ satisfies (1). And if $d_1(u) = 2$ and $d_1(v) = 1$, then $d_1 \cup b_2$ satisfies (1). Finally, if $d_1(u) = 2$ and $d_1(v) = 2$, then $d_1 \cup a_2$ satisfies (1). The claim is proved. □

Claim 2. *The graph G satisfies (3).*

Proof. If $x \in \{u, v\}$ (say $x = u$ up to symmetry), then by $(\star)$ we may assume that $y \in V(G_1) \setminus V(G_2)$ and $z \in V(G_2) \setminus V(G_1)$. By (3) applied separately to x, v and y in G_1 and to x, v and z in G_2, we obtain up to a relabeling a coloring a_1 of G_1 and a coloring a_2 of G_2 such that $a_1(x) = a_2(x) = 1$, $a_1(v) = a_2(v) = 2$, $a_1(y) \neq 1$ and $a_2(z) \neq 1$. Hence, $a_1 \cup a_2$ is a coloring of G that satisfies (3). We may therefore assume that $x \notin \{u, v\}$, and so up to symmetry that $x \in V(G_1) \setminus V(G_2)$.

Hence, by $(\star)$ and up to symmetry, we may restrict our attention to the following two cases.

Case 1: $x \in V(G_1) \setminus V(G_2)$ and $y, z \in V(G_2)$.

If $uv \in E(G)$, then by (3) applied to x, u and v and up to a relabeling, there exists a coloring a_1 of G_1 such that $a_1(x) = 1$, $a_1(u) = 2$, and $a_1(v) = 3$. The graph induced by u, v, y and z is not a complete graph on 4 vertices, because such a graph is 3-connected and would imply that G is not fragile. Hence, either $|\{u, v, y, z\}| \leq 3$ or there are non-adjacent vertices among u, v, y and z. In either case, there exists a coloring a_2 of G_2 that requires at most three colors for u, v, y, z (trivially if $|\{u, v, y, z\}| \leq 3$ or by applying (1) to a non-edge otherwise). Up to a relabeling, we may assume that $a_2(u) = 2$, $a_2(v) = 3$ and $\{a_2(y), a_2(z)\} \subseteq \{2, 3, 4\}$. Hence, $a_1 \cup a_2$ is a coloring of G satisfying (3). We may therefore assume from here on that $uv \notin E(G)$.

Suppose that there exists a coloring a_1 of G_1 such that $a_1(x) \neq a_1(u) = a_1(v)$. So, up to a relabeling, we may assume $a_1(x) = 1$ and $a_1(u) = a_1(v) = 2$. Then by (1) applied to u and v in G_2, there exists a coloring a_2 of G_2 such that $a_2(u) = a_2(v)$. Hence, $|\{a_2(u), a_2(v), a_2(y), a_2(z)\}| \leq 3$. So, up to a relabeling, we may assume that $a_2(u) =$

$a_2(v) = 2$ and $\{a_2(y), a_2(z)\} \subseteq \{2, 3, 4\}$. So $a_1 \cup a_2$ is a coloring of G that satisfies (3). We may therefore assume that no coloring as a_1 exists.

Hence, when applying (3) to x, u and v, up to a relabeling, we obtain a coloring b_1 of G_1 such that $b_1(x) = 1$, $b_1(u) = 2$ and $b_1(v) = 3$. And when applying (4) to x, u and v (which is allowed since $uv \notin E(G)$), up to a relabeling and to the symmetry between u and v, we obtain a coloring c_1 of G_1 such that $c_1(x) = 1$, $c_1(u) = 1$ and $c_1(v) = 2$.

By (2) applied to u and v, there exists a coloring d_2 of G_2 such that $d_2(u) \neq d_2(v)$. If $|\{d_2(u), d_2(v), d_2(y), d_2(z)\}| \leq 3$, then up to a relabeling, we may assume that $d_2(u) = 2$, $d_2(v) = 3$ and $\{d_2(y), d_2(z)\} \subseteq \{2, 3, 4\}$, So $b_1 \cup d_2$ is a coloring of G that satisfies (3). And if $|\{d_2(u), d_2(v), d_2(y), d_2(z)\}| = 4$, then we may assume up to a relabeling that $d_2(u) = 1$, $d_2(v) = 2$, $d_2(y) = 3$ and $d_2(z) = 4$, so $c_1 \cup d_2$ is a coloring that satisfies (3).

Case 2: $x, y \in V(G_1) \setminus V(G_2)$ and $z \in V(G_2) \setminus V(G_1)$.

By (3) applied to x, y and u, up to a relabeling, we obtain a coloring a_1 of G_1 such that $a_1(x) = 1$, $a_1(y) = 2$ and $a_1(u) \in \{2, 3\}$. If $a_1(v) \neq 1$, then color 1 is not used on u or v under a_1. By (1) or (2) applied to u and v, we obtain up to a relabeling a coloring a_2 of G_2 such that $a_2(u) = a_1(u)$ and $a_2(v) = a_1(v)$. Thus, color 1 is not used on u or v under a_2 either and so, up to a relabeling, we may assume that $a_2(z) \neq 1$. Hence $a_1 \cup a_2$ is a coloring of G that satisfies (3). We may therefore assume that $a_1(v) = 1$.

By (3) applied to v, u and z, up to a relabeling, we obtain a coloring b_2 of G_2 such that $b_2(v) = 1$, $b_2(u) = a_1(u)$ and $b_2(z) \neq 1$. Hence $a_1 \cup b_2$ is a coloring of G that satisfies (3). □

Claim 3. *The graph G satisfies (2).*

Proof. By Claim 2, we may apply (3) to x, y and any vertex of G. We obtain a coloring of G that satisfies (2). □

Claim 4. *The graph G satisfies (4).*

Proof. By ($\star$), we may assume that $x \in V(G_1) \setminus V(G_2)$ and $y \in V(G_2) \setminus \{u\}$ and $z \in V(G_2) \setminus V(G_1)$.

Suppose that $uv \in E(G)$. Then by (3) applied to x, u and v and up to a relabeling, there exists a coloring a_1 of G_1 such that $a_1(x) = 1$, $a_1(u) = 2$ and $a_1(v) = 3$. By (3) applied to u, y and z (that are distinct since $y \neq u$ and $z \in V(G_2) \setminus V(G_1)$) and up to a relabeling, we obtain a coloring a_2 of G_2 such that $a_1(u) = 2$, $a_1(v) = 3$ and $\{a_2(y), a_2(z)\}$ is either $\{3, 1\}$, $\{3\}$ or $\{4\}$. In either case, $a_1 \cup a_2$ is a coloring of G satisfying (4). We may therefore assume from here on that $uv \notin E(G)$.

Suppose that there exists a coloring a_1 of G_1 such that $a_1(x) \neq a_1(u) = a_1(v)$. Then up to a relabeling we may assume that $a_1(x) = 1$ and $a_1(u) = a_1(v) = 2$. By (1) applied to u and v in G_2, there exists up to a relabeling a coloring a_2 of G_2 such that $a_2(u) = a_2(v) = 2$. If $a_2(y) = a_2(z)$, then up to relabeling, we may assume that $a_2(y) = a_2(z) \neq 1$, so (4) is satisfied by $a_1 \cup a_2$. And if $a_2(y) \neq a_2(z)$, then up to a relabeling, we may assume $a_2(y) = 1$ or $a_2(z) = 1$, and (4) is again satisfied by $a_1 \cup a_2$. We may therefore assume that no coloring as a_1 exists.

Hence, when applying (3) to x, u and v, up to a relabeling, we obtain a coloring b_1 of G_1 such that $b_1(x) = 1$, $b_1(u) = 2$ and $b_1(v) = 3$. And when applying (4) to x, u and v (which is allowed since $uv \notin E(G)$), up to a relabeling and to the symmetry between u and v, we obtain a coloring c_1 of G_1 such that $c_1(x) = 1$, $c_1(u) = 1$ and $c_1(v) = 2$.

On the other hand, by (2) applied to u and v, there exists a coloring d_2 of G_2 such that $d_2(u) \neq d_2(v)$. If $d_2(y) = d_2(z)$, then up to a relabeling, we may assume that $d_2(u) = 2$, $d_2(v) = 3$ and $d_2(y) \neq 1$. Thus, $b_1 \cup d_2$ is a coloring that satisfies (4). Hence, from here on, we may assume that $d_2(y) \neq d_2(z)$.

If $|\{d_2(u), d_2(v), d_2(y), d_2(z)\}| \geq 3$, then we may assume up to a relabeling that $d_2(u) = 2$, $d_2(v) = 3$ and $1 \in \{d_2(y), d_2(z)\}$, so $b_1 \cup d_2$ is a coloring that satisfies (4). If $|\{d_2(u), d_2(v), d_2(y), d_2(z)\}| = 2$, then up to a relabeling, we may assume that $d_2(u) = 1$, $d_2(v) = 2$, so that $\{d_2(y), d_2(z)\} = \{1, 2\}$. So $c_1 \cup d_2$ is a coloring of G that satisfies (4). $\square$

Theorem 2.1 immediately follows from Claims 1 to 4. $\square$

3 Conclusion and open questions

We collect here several remarks and open questions.

3.1 Fragile graphs have average degree less than 5

As announced in the introduction, we recall the proof that every fragile graph G on at least 4 vertices satisfies $|E(G)| \leq 2.5|V(G)| - 5$. When G has 4 vertices, the inequality holds since the graph on 4 vertices and 6 edges is a complete graph and is 3-connected. For the induction step, we decompose G into G_1 and G_2 as in the previous section. If $|V(G_1)| \leq 3$, then G contains a vertex x of degree at most 2. Hence,

$$|E(G)| \leq |E(G \setminus x)| + 2 \leq 2.5|V(G \setminus x)| - 5 + 2 = 2.5(|V(G)| - 1) - 3 \leq 2.5|V(G)| - 5.$$

We may therefore assume that $|V(G_1)| \geq 4$ and symmetrically $|V(G_2)| \geq 4$. Hence the induction hypothesis can be applied to both G_1 and G_2 so that the result follows from these inequalities:

$$\begin{aligned}
|E(G)| &\leq |E(G_1)| + |E(G_2)| \\
&\leq 2.5|V(G_1)| - 5 + 2.5|V(G_2)| - 5 \\
&= 2.5(|V(G_1)| + |V(G_2)|) - 10 \\
&\leq 2.5(|V(G)| + 2) - 10 \\
&= 2.5|V(G)| - 5.
\end{aligned}$$

3.2 Girth conditions

It is easy to prove by induction that every fragile graph of girth at least 4 on at least 3 vertices satisfies $|E(G)| \leq 2|V(G)| - 4$ (the proof is as in Section 3.1). This implies that every fragile graph with girth at least 4 contains a vertex of degree at most 3, so is 4-colorable. We tried to improve this bound, but we instead found a fragile graph with girth 4 and chromatic number 4, as we now present.

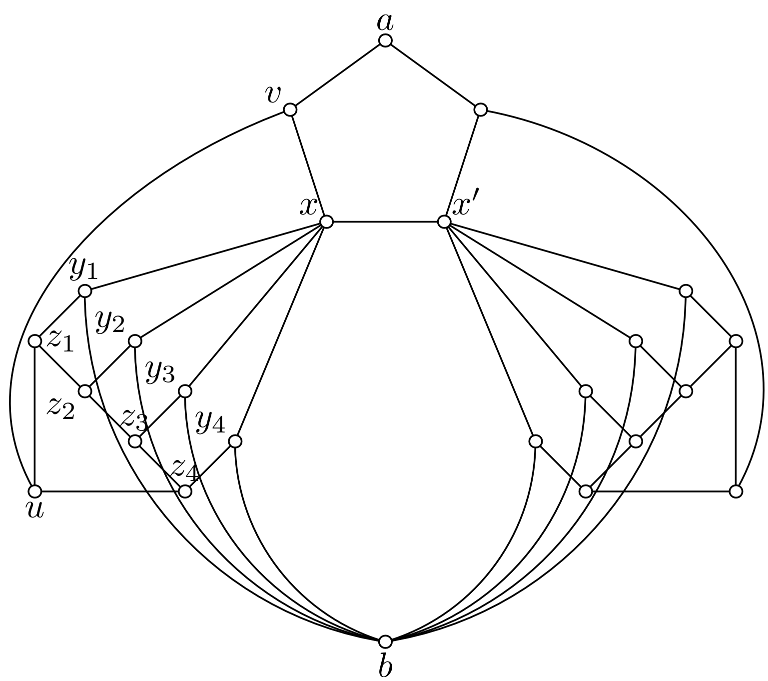

Figure 3: The graph G_1.

Let G_1 be the graph represented in Figure 3. It has girth 4 and is 2-degenerate; so in particular it is fragile and has chromatic number at most 3. For all 3-colorings of G_1, vertices a and b receive different colors. Indeed, suppose for a contradiction that for some 3-coloring of G_1, a and b receive the same color, say color 1. Then, one of x and x', say x up to symmetry, must receive a color different from 1, say color 2. So, the vertices $y_1, \dots, y_4$ must all receive the same color, say color 3. It follows that the vertices $z_1, \dots, z_4$ are colored with color 1 and 2 alternately. Hence, u receives color 3. Now, v has three neighbors, namely a, x and u that are colored with colors 1, 2 and 3 respectively, a contradiction.

It follows that the triangle-free graph G_2 represented in Figure 4 is not 3-colorable, but it is fragile since $\{a', b'\}$ is a separator, and G_1 is 2-degenerate even if two vertices adjacent to a and b are added. This raises the following question: Is there a finite girth that makes fragile graphs 2-degenerate? The same question can be asked with *2-degenerate* replaced by *3-colorable*. In Figure 5, a fragile graph with girth 6 and minimum degree 3 is presented.

Trivially, a graph G is fragile if and only if every subgraph H of G is either on at most 3

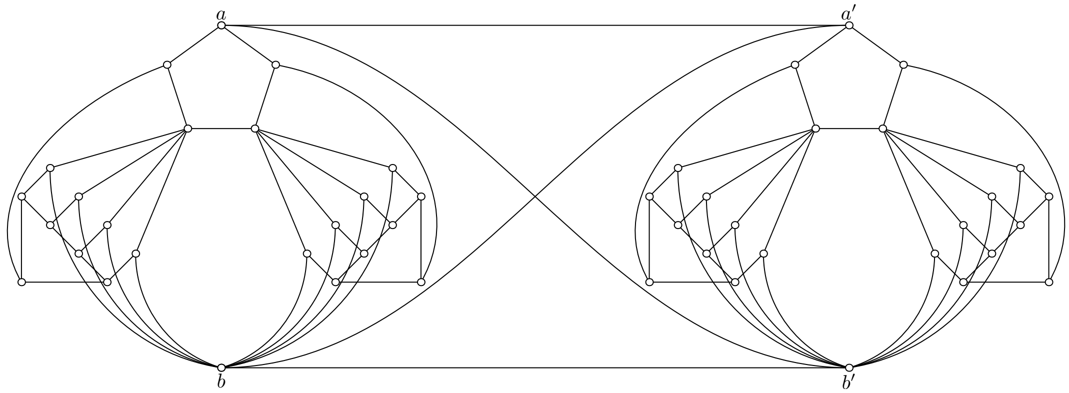

Figure 4: The graph G_2.

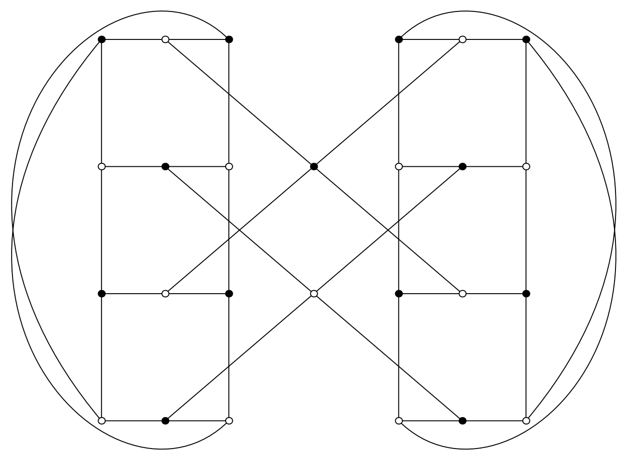

Figure 5: A bipartite fragile graph with girth 6 and minimum degree 3.

vertices or admits a separator of size at most 2. In fragile graphs of girth at least 4, one can further impose the separator to be an independent set.

Lemma 3.1. *A graph G with girth at least 4 is fragile if and only if every subgraph H of G is either on at most 2 vertices or admits an independent separator of size at most 2.*

Proof. We prove the statement by induction on $|V(G)|$. The equivalence can be checked to hold on graphs of up to 3 vertices. If $|V(G)| \geq 4$, then since G is not 3-connected, it admits a separator S of size at most 2. Suppose that S is not independent, so $S = \{u, v\}$ and $uv \in E(G)$. Let C be a connected component of $G \setminus S$. Since G has girth at least 4, no vertex of C is adjacent to both u and v. Hence, if $|C| = 1$, G admits a separator of size 1 (and therefore independent). So we may assume that $|C| \geq 2$. So, by the induction hypothesis, $G[S \cup C]$ admits an independent separator S'. It is easy to check that S' is also a separator of G. $\square$

3.3 Algorithms

By subdividing twice every edge of any graph G, a fragile graph G' is obtained. Poljak [3] proved that $\alpha(G') = \alpha(G) + |E(G)|$. It follows that a polynomial-time algorithm that computes a maximum independent set for any fragile graph would yield a similar algorithm for all graphs. This proves that computing a maximum independent set in a fragile graph is NP-hard.

We also observe that, in G', every edge uv becomes a path $ux_{uv}y_{uv}v$. Consider the graph G'' obtained from G' by adding, for every vertex x_{uv}, a new vertex x'_{uv} adjacent to u, x_{uv} and y_{uv}. It is easy to check that G'' is fragile and for all 3-colorings of G'' and all edges uv of G, u and v have different colors (in G''). It follows that if G'' is 3-colorable, then so is G. Conversely it is easy to check that if G is 3-colorable, so is G''. This proves that deciding whether a fragile graph is 3-colorable is NP-complete. By the same kind of argument, we can prove that deciding whether a graph is 3-colorable stays NP-complete even when we restrict ourselves to triangle-free graphs. To see this, pick any graph G, remove all edges uv, and replace them by a copy of the graph G_1 with a identified to u and b identified to v. This yields a triangle-free fragile graph that is 3-colorable if and only if G is 3-colorable.

Our proof that every fragile graph is 4-colorable yields an algorithm that actually computes a 4-coloring. But as far as we can see, a crude implementation of this algorithm would run in exponential time and we do not know if a polynomial-time algorithm exists.

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