Maximum Independent Set when excluding an induced minor: $K_1 + tK_2$ and $tC_3 \uplus C_4$

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Abstract

Dallard, Milanič, and Štorgel [arXiv '22] ask if for every class excluding a fixed planar graph H as an induced minor, Maximum Independent Set can be solved in polynomial time, and show that this is indeed the case when H is any planar complete bipartite graph, or the 5-vertex clique minus one edge, or minus two disjoint edges. A positive answer would constitute a far-reaching generalization of the state-of-the-art, when we currently do not know if a polynomial-time algorithm exists when H is the 7-vertex path. Relaxing tractability to the existence of a quasipolynomial-time algorithm, we know substantially more. Indeed, quasipolynomial-time algorithms were recently obtained for the t-vertex cycle, C_t [Gartland et al., STOC '21] and the disjoint union of t triangles, tC_3 [Bonamy et al., SODA '23].

We give, for every integer t, a polynomial-time algorithm running in $n^{O(t^5)}$ when H is the friendship graph $K_1 + tK_2$ (t disjoint edges plus a vertex fully adjacent to them), and a quasipolynomial-time algorithm running in $n^{O(t^2 \log n)+t^{O(1)}}$ when H is $tC_3 \uplus C_4$ (the disjoint union of t triangles and a 4-vertex cycle). The former extends a classical result on graphs excluding tK_2 as an induced subgraph [Alekseev, DAM '07], while the latter extends Bonamy et al.'s result.

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1 Introduction

The Max Independent Set (MIS for short) problem asks, in its optimization form, for a largest *independent set* of its input graph G, i.e., a subset of pairwise non-adjacent vertices in G. In its decision form, the input is a graph G and an integer k, and the question is whether G admits an independent set of size at least k.

Besides the ubiquitous usefulness that such a fundamental problem has within combinatorial optimization, and notably in the areas of packing, scheduling, and coloring, MIS (or equivalently Maximum Clique, the same problem in the complement graph) has as wide a range of applications as possible, as evidenced for instance in map labeling [47], coding theory [17], spatial scheduling [26], genetic analysis [1], information retrieval [10], macromolecular docking [28], and sociometry [27] (also see Butenko's thesis [16]). It is thus unfortunate that this problem is not only hard to solve but also very reluctant to being approximated. Indeed, the decision version of MIS is NP-complete [29] and W[1]-complete [25], while its optimization version cannot be approximated within ratio $n^{1-\varepsilon}$ on n-vertex graphs, for any $\varepsilon > 0$, unless P=NP [35, 49].

In spite of this, theorists and practitioners have put a lot of effort in designing efficient algorithms for MIS. In parallel to generically solving MIS via integer programming, highly performant exact and heuristic MIS solvers have emerged in the recent years, based on diverse methods such as kernelization and evolutionary approaches [40], deep reinforcement learning [4], graph neural networks [44], and dataless training (where backpropagation is applied to a loss function based instead on the input) [9]. On the theory side, exact exponential algorithms have been developed for decades culminating in a running time below $1.2^n n^{O(1)}$ [48].

Another approach is to try and exploit the structure that the input graphs may have. Indeed, in all the aforementioned applications, inputs are not uniformly sampled over all n-vertex graphs: They instead bare some structural properties, and in some cases, might avoid some specific patterns. Graph theory[1] offers two main notions of *patterns* or containment: the natural and straightforward *subgraphs* (obtained by removing vertices and edges), and the deeper *minors* (further allowing to contract edges). Both notions come with an induced variant, when edge removals are disallowed, bringing the number of containment types to four. It is then sensible to determine the patterns H whose absence makes MIS (more) tractable. It turns out that this question is completely settled for subgraphs and minors.

For the subgraph containment, the argument is the following. By the grid minor theorem [45], the class of graphs excluding H as a subgraph has bounded treewidth if (and only if) all the connected components of H are paths and subdivided *claws* (i.e., stars with three leaves); thus MIS can be solved in polynomial-time in this class, for instance by Courcelle's theorem [21]. If instead H has a connected component which is not a path nor a subdivided claw, MIS remains NP-complete since such a class either contains all subcubic graphs or the $2|V(H)|$-subdivision of every graph, two families of graphs on which MIS is known to be NP-complete [43, 5, 8].

For minors, the dichotomy relies on the planarity of H. Indeed, if H is planar, then the class of graphs excluding H as a minor has bounded treewidth (again, mainly by the grid minor theorem), and MIS can be solved efficiently. If H is non-planar, then the H-minor-free graphs include all planar graphs for which MIS is known to be NP-complete [30].

The question is more intriguing for the induced containments, and the *induced subgraph* case has received a lot of attention. While it is known for a long time that if H is *not* the disjoint union of paths and subdivided claws, MIS remains NP-complete on graphs without H as an induced subgraph [43, 5], it has been conjectured that MIS is otherwise polynomial-time solvable. This has been proven when H is the 6-vertex path [34], a claw with exactly one edge subdivided [6, 41], or any disjoint union of claws [15]. The latter result extends a polynomial-time algorithm due to Alekseev when H is any disjoint union of edges [7]. The author indeed proves that the total number of maximal independent sets is polynomially bounded, and can be enumerated in polynomial time.

While we currently do not know of a polynomial-time algorithm when H is the 7-vertex path, Gartland and Lokshtanov [31] have obtained a quasipolynomial-time algorithm when

[1] We refer the reader to Section 2 for the relevant background in graph theory.

H is P_t, the t-vertex path, for any positive integer t; also see [42]. Supporting the existence of at least a quasipolynomial-time algorithm when H is $S_{i,j,k}$, the claw whose three edges are subdivided $i-1$, $j-1$, and $k-1$ times, respectively, a quasipolynomial-time approximation scheme (QPTAS) [20] and a polynomial-time algorithm among bounded-degree graphs [2] have been proposed. The parameterized complexity of MIS when excluding a fixed induced subgraph has been studied [22, 12, 13], but the mere statement of which H make the problem fixed-parameter tractable (and which ones keep it W[1]-complete) is unclear [13].

We eventually arrive at the *induced minor* containment, the topic of the current paper. As for minors, the class of all graphs excluding a non-planar graph H as an induced minor contains all planar graphs; hence MIS remains NP-complete in such a class. However we do not know of a *planar* graph H, for which MIS remains NP-complete on H-induced-minor-free graphs. This has led Dallard, Milanič, and Štorgel [23] to ask if such classes exist:

▶ **Question 1.** *Is it true that for every planar graph H, Max Independent Set can be solved in polynomial time in the class of graphs excluding H as an induced minor?*

A first observation is that avoiding H as an induced minor implies avoiding it as an induced subgraph. Thus Question 1 is settled for P_6, $S_{1,1,2}$, and $tS_{1,1,1}$ (where tG denotes the disjoint of t copies of G). The same authors [23] further obtain a polynomial-time algorithm when H is K_5^- (the 5-vertex clique minus an edge), $K_{2,t}$ (the bipartite complete graph with 2 vertices fully adjacent to t vertices), and $W_4 = K_1 + C_4$ (a 4-vertex cycle C_4 with a fifth vertex fully adjacent to the cycle). All three cases were shown by bounding the so-called *tree-independence number* (i.e., treewidth where *bag size* is replaced by *independence number of the subgraph induced by the bag*) [23], in which case a polynomial-time algorithm can be derived for MIS using the corresponding tree-decompositions [24]. They also show that this is as far as this sole technique can go: H-induced-minor-free graphs have bounded *tree-independence number* if and only if H is edgeless or an induced minor of K_5^-, $K_{2,t}$, or W_4 [23]. The framework of *potential maximal cliques* [14] and the *container method* has led to a polynomial-time algorithm when $H = C_5$ [19, 3].

Question 1 a beautiful question, and, if true, a very difficult one. Indeed, $H = P_7$, the 7-vertex path, is a very simple planar graph for which we currently do not know such a polynomial-time algorithm. A natural relaxation of Question 1 is to only request a quasipolynomial-time algorithm:

▶ **Question 2.** *Is it true that for every planar graph H, Max Independent Set can be solved in quasipolynomial time in the class of graphs excluding H as an induced minor?*

We know somewhat more about Question 2. There is a quasipolynomial-time algorithm for MIS in C_t-induced-minor-free graphs [32], building upon the $H = P_t$ case. Recently, Bonamy et al. [11] present a quasipolynomial-time algorithm when H is tC_3, i.e., the disjoint union of t triangles. See Table 1 for a summary of the introduction.

Expecting an affirmative solution to Question 1 or Question 2 may seem optimistic. However, as far as precise running time is concerned, we do know that for every planar H, MIS is probably not as difficult in H-induced-minor-free graphs as it is in general graphs. Indeed, Korhonen [39] describes a $2^{O(n/\log^c n)}$-time algorithm, for some constant $c > 0$, to solve MIS on n-vertex graphs excluding a fixed planar graph H as an induced minor. Assuming the Exponential-Time Hypothesis [36], such a running time is impossible in general graphs [37].

Our results. We make some progress regarding Questions 1 and 2. Our first contribution is, for every positive integer t, a polynomial-time algorithm when H is the *friendship graph*

H exluded as	subgraph	minor	induced subgraph	**induced minor**
in P	$tS_{t,t,t}$	planar	P_6, $S_{1,1,2}$, $tS_{1,1,1}$	K_5^-, $K_{2,t}$, W_4, C_5, $\boxed{K_1 + tK_2}$
only known in QP	–	–	P_t $(t \geqslant 7)$	C_t $(t \geqslant 6)$, $\boxed{tC_3 \uplus C_4}$
NP-c	$\neg tS_{t,t,t}$	non-planar	$\neg tS_{t,t,t}$	non-planar
open P vs NP-c	–	–	P_7, ...	P_7, C_6, ...
open QP vs NP-c	–	–	$S_{1,1,3}$, $S_{1,2,2}$ ...	$C_4 \uplus C_4$, ...

Table 1 The complexity of Max Independent Set when H is excluded as one of the four main types of patterns. "$\neg tS_{t,t,t}$" means that for no t does H is a subgraph of $tS_{t,t,t}$. Our results are framed.

$K_1 + tK_2$ (also called *Dutch windmill graph* or *fan*), i.e., t independent edges universally linked to a $2t + 1$-st vertex:

▶ **Theorem 1.** *Max Independent Set can be solved in polynomial-time $n^{O(t^5)}$ in n-vertex $K_1 + tK_2$-induced-minor-free graphs.*

This extends Alekseev's result [7] for graphs excluding tK_2 as an induced subgraph, or equivalently, as an induced minor. We indeed use this result to first derive a polynomial-time algorithm in subgraphs of $K_1 + tK_2$-induced-minor-free graphs G induced by vertices from a bounded number of breadth-first search (BFS) layers of G.

We then consider the connected components of our input graph G when deprived of a subset X of vertices inducing tK_2 and, subject to that property, maximizing the size of the largest connected component in $G - X$. We show that, due to this careful selection of X, every component C of $G - X$ admits an efficiently constructible path-decomposition $\mathcal{P}$ with *bounded adhesion* (i.e., any two distinct bags have a bounded intersection), each bag of which is contained in a bounded number of consecutive BFS layers of C. Hence MIS can be solved efficiently within a bag, by our opening step (see previous paragraph). This part is quite technical, but mostly to justify the existence of $\mathcal{P}$. The algorithm itself remains simple.

Section 3 is then obtained by exhaustively finding X and guessing its intersection X' with a maximum independent set of G, and performing dynamic programming on the connected components of $G - X$, deprived of $N(X')$. The dynamic-programming table is filled via the efficient algorithm when handling an induced subgraph contained in few BFS layers.

Our second contribution is a quasipolynomial-time algorithm when H is, $tC_3 \uplus C_4$, the disjoint union of t triangles and a 4-vertex cycle:

▶ **Theorem 2.** *Max Independent Set can be solved in quasipolynomial-time $n^{O(t^2 \log n)+t^{O(1)}}$ in $tC_3 \uplus C_4$-induced-minor-free graphs.*

We first perform a quasipolynomial branching step to get rid of holes of size at most 6 (i.e., induced cycles of length 4, 5, or 6). We then assume that the input graph G is not $(t + 2)C_3$-induced-minor-free, for otherwise we conclude with Bonamy et al.'s algorithm [11]. Thus G, being $tC_3 \uplus C_4$-induced-minor-free, has to admit $(t + 2)C_3$ as an induced subgraph, i.e., a collection $T_1, \ldots, T_{t+2}$ of $t + 2$ pairwise vertex-disjoint and non-adjacent triangles. We define $S_{i,j}$, minimally separating T_i and T_j in the graph G deprived of the neighborhoods of the other triangles T_k (with $k \neq i, k \neq j$).

We show that each $S_{i,j}$ induces a clique. So does every intersection $N_{i,j}$ of the neighborhood of two distinct triangles T_i, T_j of the collection (this is where getting rid of the holes of length at most 6 comes into play). We can therefore exhaustively guess the intersection of a maximum independent set with the union of the sets $S_{i,j}$ and $N_{i,j}$ (for every $i < j \in [t+2]$). We finally observe that $G' = G - \bigcup_{i \neq j \in [t+2]} S_{i,j} \cup N_{i,j}$ is chordal, since the presence of a hole H in G' would imply the existence in G of t independent triangles in the non-neighborhood of H, a contradiction to the $tC_3 \uplus C_4$-induced-minor-freeness of G. We thus conclude by using a classic algorithm for MIS in chordal graphs [33, 46].

In Section 2 we introduce the relevant graph-theoretic background. In Section 3 we prove Theorem 1, and in Section 4 we prove Theorem 2.

2 Preliminaries

If $i \leqslant j$ are two integers, we denote by $[i, j]$ the set of integers $\{i, i+1, \ldots, j-1, j\}$, and by $[i]$, the set $[1, i]$. We denote by $V(G)$ and $E(G)$ the set of vertices and edges of a graph G, respectively. We denote by $G_1 \simeq G_2$ the fact that the two graphs G_1 and G_2 are *isomorphic*, i.e., equal up to renaming their vertex set. For $S \subseteq V(G)$, the *subgraph of G induced by S*, denoted $G[S]$, is obtained by removing from G all the vertices that are not in S (together with their incident edges). Then $G - S$ is a short-hand for $G[V(G) \setminus S]$. H is an *induced subgraph* of G if there is an $S \subseteq V(G)$ such that $G[S] \simeq H$.

For G a graph and $X \subseteq V(G)$, $E_G(X)$ (or simply $E(X)$) is a short-hand for $E(G[X])$. For G a graph and $X, Y \subseteq V(G)$ two disjoint sets, $E_G(X, Y)$ denotes the set of edges of $E(G)$ with one endpoint in X and the other endpoint in Y. We denote by $N_G(v)$ and $N_G[v]$, the open, respectively closed, neighborhood of v in G. For $S \subseteq V(G)$, we set $N_G(S) := \bigcup_{v \in S} N_G(v) \setminus S$ and $N_G[S] := N_G(S) \cup S$. We may omit the subscript if G is clear from the context. A *connected component* is a maximal connected induced subgraph.

Two cycles C_1, C_2 are said *independent* if they are vertex-disjoint and there is no edge between C_1 and C_2. A collection of cycles is *independent* if they are pairwise independent. Two vertex subsets $X, Y \subseteq V(G)$ *touch* if $X \cap Y \neq \emptyset$ or there is an edge $uv \in E(G)$ with $u \in X$ and $v \in Y$. Then two (or more) cycles are independent if and only if they do *not* touch. We say that $X, Y \subseteq V(G)$ *touch in Z* if $X \cap Y \cap Z \neq \emptyset$ or there is an edge $uv \in E(G)$ with $u \in X \cap Z$ and $v \in Y \cap Z$, or equivalently, if $X \cap Z$ and $Y \cap Z$ touch.

A graph H is a *induced minor* of a graph G if H can be obtained from G by a sequence of vertex deletions and edge contractions. A *minor* is the same but also allows edge deletions. Equivalently an induced minor H –with vertex set, say, $\{v_1, \ldots, v_{V(H)}\}$– of G can be defined as a vertex-partition $B_1, \ldots, B_{|V(H)|}$ of an induced subgraph of G, such that every $G[B_i]$ is connected and $v_i v_j \in E(H)$ if and only if $E_G(B_i, B_j) \neq \emptyset$ (i.e., when the disjoint sets B_i and B_j touch). Observe indeed that contracting each B_i into a single vertex (possible since each B_i induces a connected subgraph) results in H. A graph G (resp. a graph class) is said *H-induced-minor-free* if H is not an induced minor of G (resp. no graph of the class admits H as an induced minor).

We denote by C_ℓ the ℓ-vertex cycle, and by K_ℓ, the ℓ-vertex clique. A *hole* is an induced cycle of length at least four. A graph is *chordal* if it has no hole. For two disjoint sets $X, Y \subseteq V(G)$ in a graph G, an *(X, Y)-separator* is a (possibly empty) set $S \subseteq V(G) \setminus (X \cup Y)$ such that there is no path between X and Y in $G - S$. An (X, Y)-separator is *minimal* if no proper subset of it is itself an (X, Y)-separator.

The *disjoint union* $G_1 \uplus G_2$ of two graphs G_1, G_2 has vertex set $V(G_1) \uplus V(G_2)$ and edge set $E(G_1) \uplus E(G_2)$, where $V(G_1) \uplus V(G_2)$ presupposes that the vertices of G_1 and G_2 have disjoint labels. If $t \geqslant 2$ is an integer and G a graph, tG is the graph $G \uplus (t-1)G$, and $1G$ is simply G. The *join* $G_1 + G_2$ of two graphs G_1, G_2 has vertex set $V(G_1) \uplus V(G_2)$ and edge set $E(G_1) \uplus E(G_2) \uplus \{uv \ : \ u \in V(G_1), v \in V(G_2)\}$. In other words, the join of G_1 and G_2

is obtained from their disjoint union by adding all possible edges between G_1 and G_2.

A *breadth-first search* (BFS) *layering* in G from a vertex $v \in V(G)$ (or from a connected set $S \subseteq V(G)$) is a partition of the remaining vertices into $L_1, L_2, \ldots$ such that every vertex of L_i is at distance exactly i of v (or S). Such an L_i is called a *BFS layer* of G (from v, or from S). Note that there cannot be an edge in G between L_i and L_j if $|i - j| > 1$.

A *path-decomposition* $\mathcal{P} = (B_1, \ldots, B_h)$ of a graph G is such that

- $\bigcup_{1 \leqslant i \leqslant h} B_i = V(G)$,
- for every $e \in E(G)$, there is some B_i that contains both endpoints of e, and
- whenever $v \in B_i \cap B_j$ with $i < j$, v is also in all B_k with $i < k < j$.

The sets B_i (for $i \in [h]$) are called the *bags* of $\mathcal{P}$, and the sets $B_i \cap B_{i+1}$ (for $i \in [h-1]$) the *adhesions* of $\mathcal{P}$. Path-decomposition $\mathcal{P}$ has maximum adhesion p if all of its adhesions have size at most p. Note that the adhesion $B_i \cap B_{i+1}$, if disjoint from $B_1 \cup B_h$, is a vertex cutset disconnecting B_1 from B_h.

3 Polynomial algorithm in $K_1 + tK_2$-induced-minor-free graphs

We first show how to solve Max Independent Set in $K_1 + tK_2$-induced-minor-free graphs of bounded diameter. More generally, we show the following.

▶ **Lemma 3.** *Let t, h be fixed non-negative integers. Let G be a $K_1 + tK_2$-induced-minor-free n-vertex graph, and $L_0 \subseteq V(G)$ such that $G[L_0]$ is connected. Let $L_1, L_2, \ldots, L_h$ be such that every vertex of L_i, for any $i \in [h]$, is at distance exactly i of L_0. Then Max Independent Set can be solved in polynomial time $n^{(2t-1)h+O(1)}$ on any induced subgraph of $G[\bigcup_{1 \leqslant i \leqslant h} L_i]$.*

Proof. For every $j \in [h]$, $G[L_j]$ has no tK_2 induced subgraph (or equivalently, induced minor). Indeed, $G[\bigcup_{0 \leqslant i \leqslant j-1} L_i]$ is a connected graph, hence $\bigcup_{0 \leqslant i \leqslant j-1} L_i$ can be contracted to a single vertex, and every vertex in L_j has at least one neighbor in L_{j-1}. Therefore a tK_2 induced subgraph in $G[L_j]$ would contradict the $K_1 + tK_2$-induced-minor-freeness of G.

Let H be any induced subgraph of $G[\bigcup_{1 \leqslant i \leqslant h} L_i]$. In particular $H[L_j]$ has also no tK_2 induced subgraph, for every $j \in [h]$. Hence, by a classical result of Alekseev [7], $H[L_j]$ has at most n^{2t-1} maximal independent sets, which can be listed in time $n^{2t+O(1)}$.

We thus exhaustively list every h-tuple $(I_1, \ldots, I_h)$ where, for every $j \in [h]$, I_j is a maximal independent set of $H[L_j]$, in time $n^{(2t-1)h+O(1)}$. Note that if there is an edge in H between L_i and L_j, then $|i - j| \leqslant 1$. As each I_j (for $j \in [h]$) is an independent set, $H' = H[\bigcup_{j \in [h]} I_j]$ is a bipartite graph as witnessed by the bipartition $(L_1 \cup L_3 \cup \ldots, L_2 \cup L_4 \cup \ldots)$. A maximum independent set I can thus be computed in polynomial time in H'. Indeed, by the Kőnig-Egerváry theorem [38], finding a maximum independent set in a bipartite graph boils down to finding a maximum matching, which can be done in polynomial time (and now, even in almost linear time [18]) by solving a maximum flow problem. We output the largest independent set I found among every run.

The correctness of the algorithm is based on the observation that a maximum independent set $I^\star$ of H intersects every L_i (for $i \in [h]$) at an independent set J_i, which by definition is contained in a maximal independent set I_i of $H[L_i]$. In the run when every maximal independent set I_i is guessed correctly, we obtain an independent set of the cardinality of $I^\star$. ◀

We say that G is *reduced* if it does not contain degree-1 vertices. If $e = uv$ is an edge of G, let $G \setminus e$ (resp. $S \setminus e$, $S \cup e$, for some $S \subseteq V(G)$) be the induced subgraph $G[V(G) - \{u, v\}] = G - \{u, v\}$ (resp. the sets $S \setminus \{u, v\}$, $S \cup \{u, v\}$). More generally, for a collection $e_1 = u_1v_1, \dots, e_k = u_kv_k$ of edges of G, we denote by $G \setminus \{e_1, \dots, e_k\}$ the induced subgraph $G[V(G) - \{u_1, v_1, \dots, u_k, v_k\}] = G - \{u_1, v_1, \dots, u_k, v_k\}$.

▶ **Lemma 4.** *Let G be a reduced connected $K_1 + tK_2$-induced-minor-free n-vertex graph. Let $X \subseteq V(G)$ maximize the size of a largest component of $G' := G - X$, among those sets X such that $G[X] \simeq tK_2$. Then for any $e \in E(G' - N_G(X))$ contained in a connected component C of G',*

1. *$C \setminus e$ is disconnected, and*
2. *each connected component of $C \setminus e$ contains a vertex in $N_G(X)$.*

Proof. As G is reduced, every vertex of X has degree at least two (in G). Thus G' cannot be connected, for otherwise, contracting in G the set $V(G')$ to a single vertex would form a $K_1 + tK_2$ induced minor. We thus know that G' has at least two connected components.

Let C_1 be a largest connected component of G'. Since G is connected, there exists a shortest path P in G from $V(C_1)$ to $V(G') \setminus V(C_1)$. Say, that P ends in the connected component $C_2 \neq C_1$ of G'. Path P has to have some internal vertices in X, but since $G[X] \simeq tK_2$, it follows that there is an edge e^* in $E(X)$ (but not necessarily in P) incident to both $N_G(V(C_1))$ and $N_G(V(C_2))$.

For every edge $e \in E(G' - N_G(X))$ in component C (possibly C_1), $C \setminus e$ is disconnected. Indeed, for the sake of contradiction, suppose that $C \setminus e$ is connected, and consider $X' := (X \setminus e^*) \cup e$. By assumption, $G[X'] \simeq tK_2$. Furthermore, the connected component $G - X'$ containing e^* is strictly larger than C_1, as it contains $(V(C_1) \setminus e) \cup e^*$ and $V(C_2) \setminus e$, disjoint sets of size at least $|V(C_1)|$ and at least one, respectively. This contradicts the maximality of X, and establishes the first item.

We now prove the second item, also by contradiction. Suppose that there is a connected component C' of $C \setminus e$ that does not contain a vertex in $N_G(X)$. We will reach a contradiction by showing that C' contains an edge e' not incident to $N_G(X)$, and such that $C \setminus e'$ is connected (and conclude in light of the previous paragraph).

Let $L_i \subseteq V(C')$ be the *i-th neighborhood of e* in C', i.e., the vertices at distance i of one endpoint of e, and at least i of the other endpoint. We consider the *last layer* L_k, i.e., such that $L_k \neq \emptyset$ and $L_{k+1} = \emptyset$. If L_k contains an edge e' then removing the endpoints of this edge does not disconnect C' (and hence C) since each vertex in L_k has a neighbor in L_{k-1} (with the convention that L_0 consists of the endpoints of e) and $G[L_0 \cup L_1 \cup \cdots \cup L_{k-1}]$ is connected.

If L_k does not contain an edge, then each vertex in L_k has two neighbors in L_{k-1}. This is because G is reduced, and by assumption that no vertex of C' has a neighbor in X. Hence, removing the endpoints of any edge e' incident to a vertex in L_k does not disconnect C' (nor C), since each vertex in L_k has at least one neighbor in L_{k-1} which is not an endpoint of e'. In either case, e' is an edge of $C - N_G(X)$ that does not disconnect C, which we showed is not possible. ◀

We now prove the main technical result of the section.

▶ **Proposition 5.** *Let G be a reduced connected $K_1 + tK_2$-induced-minor-free n-vertex graph. Let $X \subseteq V(G)$ maximize the size of a largest component of $G' := G - X$, among those sets X such that $G[X] \simeq tK_2$. Then for every connected component C of G', a path-decomposition $\mathcal{P}$ of C such that*

- *every bag of $\mathcal{P}$ is contained in $O(t^4)$ consecutive BFS layers, and*
- *every adhesion of $\mathcal{P}$ is of size at most $2t^2$,*

can be computed in time $n^{O(1)}$.

Proof. Let v be a vertex in $N_G(X) \cap V(C)$ and, for any positive integer s, let L_s be the vertices at distance s from v in C. Let q be the largest distance between v and a vertex of C. We set $f(t) := t^2(6t^2+2) = O(t^4)$. We will show that, for every $s \in [q - f(t)]$, there is a vertex cutset of size at most $2t^2$ separating L_s from $L_{s+f(t)}$.

We show that any long-enough induced path (such as a shortest path from L_s to $L_{s+f(t)}$) has some edges with both endpoints in $V(C) \setminus N_G(X)$.

▷ Claim 6. Any induced path P in C contains less than $3t^2$ vertices in $N_G(X)$.

Proof. If there are $3t^2$ vertices on P with a neighbor in X, then there are at least $3t$ vertices $w_1, \dots, w_{3t}$ on P that are neighbors of a fixed edge $e \in E(X)$. For each $i \in [t]$, contract every edge of P between w_{3i-2} and w_{3i-1} but one, say e_i. Contract e, and call z the resulting vertex. The vertex z and the t edges e_i contradict the fact that G is $K_1 + tK_2$-induced-minor-free. ◁

For an edge $e \in E(C)$ we denote by $\text{dist}(e, v)$ the length of a shortest path from an endpoint of e to v. We build a collection of paths P_i and edges $e_i \in E(P_i)$, for $i = 1, 2, \dots$, while they are well-defined, in the following way.

Let P_1 be a shortest path from L_s to $L_{s+f(t)}$. Let $e_1 \in E(P_1)$ minimize $e \mapsto \text{dist}(e, v)$, among those edges of P_1 with both endpoints in $V(C) \setminus N_G(X)$. By Claim 6 there are less than $6t^2$ edges on P_1 with an endpoint in $N_G(X)$, hence $\text{dist}(e_1, v) \leqslant s + 6t^2$. We denote by Q_1 the maximal subpath of P_1 starting in L_s and not containing an endpoint of e_1 (that is, stopping just before reaching an endpoint of e_1). For the next iteration, we work in $C \setminus e_1$ (recall that this stands for C deprived of the two endpoints of e_1).

We now describe in general the i-th iteration. Let P_i be a shortest path from L_s to $L_{s+f(t)}$ in $C \setminus \{e_1, e_2, \dots, e_{i-1}\}$. Let e_i be the first edge of P_i (when starting from L_s) such that $\text{dist}(e_i, v) \geqslant \text{dist}(e_{i-1}, v) + 2$ and e_i has no endpoint in $N_G(X)$. Note that by Claim 6, $\text{dist}(e_i, v) \leqslant \text{dist}(e_{i-1}, v) + 2 + 6t^2$. Let Q_i be the maximal subpath of P_i starting in L_s and not containing an endpoint of e_i. See Figure 1.

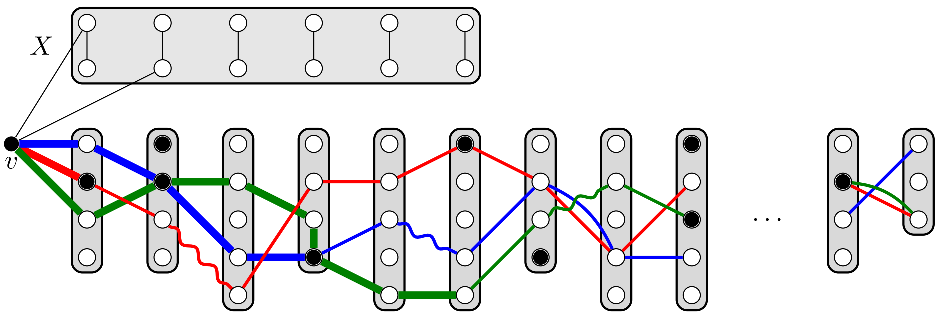

Figure 1 Illustration of X (for $t = 6$) and the first $f(t)$ layers (from left to right) of the connected component C of $G - X$ rooted at v. Vertices of C with a neighbor in X are filled. We represented the first three iterations: P_1, e_1 (in red), P_2, e_2 (in blue), P_3, e_3 (in dark green). Not to clutter the figure, we do not represent all the edges, and we code the different labels: edges e_i are squiggly, and subpaths Q_i are thicker.

Let $e_1, \dots, e_k$ be the eventual collection of edges. In principle, the while loop stops when:

(*i*) L_s is disconnected from $L_{s+f(t)}$ in $C \setminus \{e_1, \dots, e_k\}$, or

(*ii*) there is no edge $e_{k+1} \in E(P_{k+1})$ such that $\text{dist}(e_{k+1}, v) \geqslant \text{dist}(e_k, v) + 2$ and e_{k+1} has no endpoint in $N_G(X)$.

▷ Claim 7. It holds that $k \leqslant t^2$. In particular, as $f(t) = t^2(6t^2+2)$, (*ii*) cannot hold.

Proof. Assume for the sake of contradiction that $k \geqslant t^2+1$. By Lemma 4 (using both items), for each i there exists a vertex $v_i \in V(C) \cap N_G(X)$ disconnected from v in $C \setminus e_i$. Since $k \geqslant t^2+1$, there are $t+1$ vertices $v_{a_1}, \ldots, v_{a_{t+1}}$ all adjacent to an endpoint of the same fixed edge $e \in E(X)$. Let C' be the component containing v in $C \setminus \{e_{a_1}, \ldots, e_{a_t}\}$, and let us contract (in G) set $V(C')$ to a single vertex, say w. Since $e_{a_{t+1}}$ is, by construction, an edge of C', vertex w is adjacent to an endpoint of e.

Observe also that, for every $i \in [t]$, each subpath Q_{a_i} is contained in C'. Vertex w is adjacent to an endpoint u_{a_i} of e_{a_i}, for every $i \in [t]$. Let C_{a_i} be the vertex set of the connected component of $C \setminus e_{a_i}$ containing v_{a_i}. Let R_{a_i} be a path from u_{a_i} to v_{a_i} in the subgraph of G induced by C_{a_i} plus the endpoints of e_{a_i}. For each $i \in [t]$, if R_{a_i} has at least two edges, contract the edge wu_{a_i}, and all the edges of R_{a_i} but the last two; the last one being $f_{a_i} = y_{a_i}v_{a_i}$. Finally contract e, and the edge between e and w. We call the resulting vertex z.

We claim that z and the edges f_{a_i} make a $K_1 + tK_2$ induced minor in G. One can see that z is adjacent to v_{a_i} via $e \in E(X)$, and to y_{a_i} via w and the path R_{a_i}. We shall justify that the edges f_{a_i} form an induced matching in G. Indeed, suppose there is an edge between some f_{a_i} and f_{a_j} with $i \neq j \in [t]$. Then in $C \setminus e_{a_i}$, there is a path from v to v_{a_i} via Q_j, a contradiction. ◁

Claim 7 implies that the $2k \leqslant 2t^2$ endpoints of $e_1, \ldots, e_k$ form a vertex cutset disconnecting L_s from $L_{s+f(t)}$. We can now build the path-decomposition $\mathcal{P}$. Recall that the BFS search from v gives rise to q layers, $L_1, \ldots, L_q$ (outside v).

For $j \in [\lfloor q/f(t) \rfloor]$, let S_j be the vertex cutset of size at most $2t^2$ (and obtained as detailed above) disconnecting $L_{(j-1)f(t)+1}$ from $L_{jf(t)}$. We denote by $L_{s \to s'}$ the set $\bigcup_{s \leqslant h \leqslant s'} L_h$.

- Let $B_1 \subseteq V(C)$ consist of v, S_1, plus all the vertices of connected components of $C[L_{1 \to f(t)}] - S_1$ that do not intersect $L_{f(t)}$.
- For j going from 2 to $\lfloor q/f(t) \rfloor - 1$, let $B_j \subseteq V(C)$ consist of $S_j \cup S_{j+1}$ plus the vertices not already present in one of $B_1, \ldots, B_{j-1}$ of all the connected components of
$$C[L_{(j-1)f(t)+1 \to (j+1)f(t)}] - (S_j \cup S_{j+1})$$
that do not intersect $L_{(j+1)f(t)}$.
- Let finally $B_{\lfloor q/f(t) \rfloor} \subseteq V(C)$ consist of $S_{\lfloor q/f(t) \rfloor}$ plus the vertices of all the connected components of $C[L_{(\lfloor q/f(t) \rfloor -1)f(t)+1 \to q}] - S_{\lfloor q/f(t) \rfloor}$ that intersect L_q.

Let $\mathcal{P}$ be the path-decomposition $(B_1, B_2, \ldots, B_{\lfloor q/f(t) \rfloor})$. $\mathcal{P}$ is indeed a path-decomposition, since our process entirely covers $V(C)$, and by virtue of S_j separating $L_{(j-1)f(t)+1}$ from $L_{jf(t)}$. By construction,

- every bag intersects at most $2f(t) = O(t^4)$ layers of BFS from vertex v, and
- for every $j \in [\lfloor q/f(t) \rfloor - 1]$, $B_j \cap B_{j+1} = S_j$, so every adhesion has size at most $2t^2$.

Finally note that once X is found, one can find the path-decomposition $\mathcal{P}$ of C in time $n^{O(1)}$, since this only involves computing (at most n) shortest paths. ◀

We can now wrap up, using Proposition 5 and Lemma 3.

▶ **Theorem 1.** *Max Independent Set can be solved in polynomial-time $n^{O(t^5)}$ in n-vertex $K_1 + tK_2$-induced-minor-free graphs.*

Proof. Let G be our $K_1 + tK_2$-induced-minor-free n-vertex input graph. As including vertices of degree 1 in the independent set is a safe reduction rule, we can assume that G is reduced. By dealing with the possibly several connected components of G separately, we can further assume that G is connected. In time $n^{O(t)}$ we find $X \subseteq V(G)$ that maximizes the size of a largest component of $G' := G - X$, among those sets X such that $G[X] \simeq tK_2$.

We exhaustively guess the intersection X' of a fixed maximum independent set of G with the set X, with an extra multiplicative factor of 2^{2t}. We are now left with solving MIS separately in $C' := C - N_G(X')$ for each connected component C of G'. By Proposition 5, we obtain in time $n^{O(1)}$ a path-decomposition $\mathcal{P} = (B_1, \ldots, B_p)$ of C', such that

- every B_i (for $i \in [p]$) is contained in $O(t^4)$ consecutive BFS layers *of* C, and
- every adhesion $A_i := B_i \cap B_{i+1}$ (for $i \in [p-1]$) is of size at most $2t^2$.

Indeed, removing $N_G(X')$ from C (and its path-decomposition) preserves those properties.

Let us define A_0, A_p to be empty. We proceed to the following dynamic programming. For $i \in [0, p]$, and for any $S \subseteq A_i$, $T[i, S]$ is meant to eventually contain an independent set I of $C'[\bigcup_{1 \leqslant j \leqslant i} B_j]$ of maximum cardinality among those such that $I \cap B_i = S$. We set $T[0, \emptyset] = \emptyset$, and observe that it is the only entry of the form $T[0, \cdot]$.

We fill this table by increasing value of $i = 1, 2 \ldots, p$. Assume that all entries of the form $T[i', \cdot]$ are properly filled for $i' < i$. For every $S \subseteq A_i$, $T[i, S]$ is filled in the following way. For every $S' \subseteq A_{i-1}$, if $S \cup S'$ is an independent set, we compute, by Lemma 3 (with $L_1, \ldots, L_h$ being the $O(t^4)$ consecutive BFS layers *of* C containing B_i, and L_0 being the connected set, *in* C, formed by the union of all the previous layers), a maximum independent set I_i in $C'[B_i] - N[S \cup S']$ in time $n^{O(ht)} = n^{O(t^5)}$. We finally set $T[i, S] = T[i-1, S'] \cup I_i \cup S$ for a run that maximizes the cardinality of $T[i-1, S'] \cup I_i$.

It takes time $p \cdot 2^{O(t^2)} \cdot n^{O(t^5)} = n^{O(t^5)}$ to completely fill T. Eventually $T[p, \emptyset]$ contains a maximum independent set of C'. We return the union of X' and of the maximum independent sets of C' found for each connected component C of G'. The overall running time is $n^{O(t^5)}$. ◀

4 Quasipolynomial algorithm in $tC_3 \uplus C_4$-induced-minor-free graphs

We first show that the existence of a short hole allows for a quasipolynomial-time branching scheme in $tC_3 \uplus C_4$-induced-minor-free graphs.

▶ **Lemma 8.** *Let G be an n-vertex $tC_3 \uplus C_4$-induced-minor-free graph, and some fixed constant ℓ. While there is a hole H of length at most ℓ and a tC_3 induced subgraph in G, Max Independent Set admits a quasipolynomial branching step, running in $n^{O_\ell(t \log n)}$.*

Proof. Let $C(G) := \{X \in \binom{V(G)}{3t} \;:\; G[X] \simeq tC_3\}$ be the collection of vertex subsets inducing t disjoint triangles, and assume $\mu(G) := |C(G)| > 0$, and H is a hole of G of length at most ℓ. As G is $tC_3 \uplus C_4$-induced-minor-free, $N[V(H)]$ intersects every $X \in C(G)$. In particular, there is a vertex $v \in V(H)$ such that $N[v]$ intersects at least a $1/\ell$ fraction of the $X \in C(G)$.

We branch on two options: either we take v in an (initially empty) solution, and remove its closed neighborhood from G, or we remove v from G (without adding it to the solution). With the former choice, measure μ drops by at least a $1/\ell$ fraction (and the number of vertices of G decreases by at least 1), and with the latter choice, the number of vertices drops by 1. This branching is exhaustive. We simply need to argue about its running time.

Note that each option can be done at most n times, while the first option cannot be done more than $\log_\ell(n^{3t}) = O_\ell(t \log n)$ times. Hence the branching tree has at most $\binom{n}{\log_\ell(n^{3t})} = n^{O_\ell(t \log n)}$ leaves. ◀

The previous lemma permits us to get rid of short holes, which turns out useful in some corner case.

▶ **Theorem 2.** *Max Independent Set can be solved in quasipolynomial-time* $n^{O(t^2 \log n)+t^{O(1)}}$ *in* $tC_3 \uplus C_4$*-induced-minor-free graphs.*

Proof. We apply the quasipolynomial branching step of Lemma 8 with $\ell = 6$, until the input n-vertex graph G has no longer holes of length at most 6, or tC_3 induced subgraph.

In time $n^{O(t)}$, we exhaustively look for a collection of pairwise vertex-disjoint and non-adjacent triangles $T_1, T_2, \dots, T_{t+2}$ in G. If such a collection does not exist, G is $(t+2)C_3$-induced-minor-free. Indeed, the absence of $(t+2)C_3$ as an induced subgraph implies that at least one of the $(t+2)$ independent cycles realizing a $(t+2)C_3$ induced minor is of length at least four. This is ruled out by the assumption that G is $tC_3 \uplus C_4$-induced-minor-free. (Note here that only $(t+1)$ independent cycles would suffice.)

We can thus assume that a collection $T_1, T_2, \dots, T_{t+2}$ exists, for otherwise, we can conclude with the quasipolynomial-time algorithm, running in $n^{O(t^2 \log n)+t^{O(1)}}$, of Bonamy et al. [11] for Max Independent Set in graphs with a bounded number of independent cycles (here, $(t+2)C_3$-induced-minor-free). In turn, as G contains a tC_3 (even $(t+2)C_3$) induced subgraph, we can, in light of the first paragraph, further assume that all the cycles of G have length either 3 or at least 7. We refer the reader to Figure 2 for a visual summary of the next two paragraphs.

For every pair T_i, T_j (with $i < j \in [t+2]$), let $S_{i,j}$ be a minimal (T_i, T_j)-separator in $G - \bigcup_{k \in [t+2] \setminus \{i,j\}} N(T_k)$. We first observe that $S_{i,j}$ is a clique of G. Suppose, for the sake of contradiction, that $u, v \in S_{i,j}$ are distinct and non-adjacent. Consider $P_{u,i}, P_{v,i}, P_{u,j}, P_{v,j}$ four chordless paths between u and T_i, v and T_i, u and T_j, v and T_j, respectively, all of which intersect $S_{i,j}$ exactly once. These paths exist since $S_{i,j}$ is a minimal (T_i, T_j)-separator. Since $P_{u,i}, P_{v,i}$ touch in T_i, and $P_{u,j}, P_{v,j}$ touch in T_j, we can build a cycle C out of those four paths. There is by assumption no edge between u and v, nor between $(P_{u,i} \cup P_{v,i}) \setminus \{u, v\}$ and $(P_{u,j} \cup P_{v,j}) \setminus \{u, v\}$. Therefore, a subset of $V(C)$ induces a hole H. As C, hence H, does not intersect the neighborhood of $\bigcup_{k \in [t+2] \setminus \{i,j\}} T_k$, the set $\{V(H)\} \cup \{T_k \ : \ k \in [t+2] \setminus \{i,j\}\}$ contradicts the $tC_3 \uplus C_4$-induced-minor-freeness of G.

Let $N_{i,j}$ be the set $N(T_i) \cap N(T_j)$ for each pair $i < j \in [t+2]$. Observe that the sets $N_{i,j}$ need not be disjoint, and that when T_i and T_j are at distance at least 3 apart, $N_{i,j}$ is empty. We notice that $N_{i,j}$ is a clique, for otherwise we can exhibit an induced cycle of length 4, 5, or 6 in G (hence a hole of length at most 6).

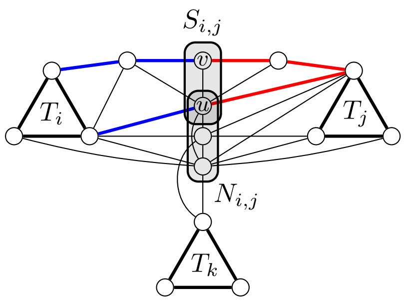

Figure 2 Illustration of the sets $S_{i,j}$ and $N_{i,j}$, and the paths $P_{u,i}$ (bottom) and $P_{v,i}$ (top) in blue, and the paths $P_{u,j}$ (bottom) and $P_{v,j}$ (top) in red. A non-edge uv in $S_{i,j}$ would imply, from these paths, that a hole exists in the non-neighborhood of the other triangles T_k, contradicting $tC_3 \uplus C_4$-minor-freeness. The absence of hole of length at most 6 implies that $N_{i,j}$ is also a clique.

We claim that $G' := G - (\bigcup_{i<j \in [t+2]} S_{i,j} \cup N_{i,j})$ is chordal. Indeed assume there is a hole H' in G'. The $tC_3 \uplus C_4$-induced-minor-freeness implies that H' intersects at least two sets

$N[T_i]$ and $N[T_j]$. Thus, let P be a subpath of H' whose endpoints are in two distinct $N(T_i)$ and $N(T_j)$, and no internal vertex of P lies in some $N[T_k]$ with $k \notin \{i, j\}$. As G' does not include any vertex of $\bigcup_{i'<j' \in [t+2]} N_{i',j'}$, it further holds that no vertex of P (*at all*) lies in some $N[T_k]$ with $k \notin \{i, j\}$. Therefore, path P contradicts that $S_{i,j}$ separates T_i and T_j in $G - \bigcup_{k \in [t+2] \setminus \{i,j\}} N(T_k)$.

We can now describe the rest of the algorithm after the collection $T_1, T_2, \ldots, T_{t+2}$ is found. We greedily compute the minimal separators $S_{i,j}$. We exhaustively try every subset $S \subseteq \bigcup_{i<j \in [t+2]} S_{i,j} \cup N_{i,j}$ that is an independent set. Such a set S contains at most one vertex in each $S_{i,j}$ and each $N_{i,j}$, as we have established that each $S_{i,j}$ and each $N_{i,j}$ form a clique. Hence there are $n^{O(t^2)}$ such sets S. For each S, we compute a maximum independent set I in the *chordal* graph $G - (N[S] \cup \bigcup_{i<j \in [t+2]} S_{i,j} \cup N_{i,j})$ in linear time (see [33, 46]). We finally output the set $S \cup I$ maximizing $|S \cup I|$. Note that the overall running time is $n^{O(t^2 \log n) + t^{O(1)}}$. ◀

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