Max Independent Set Remains NP-hard when Excluding a Planar Induced Minor

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Abstract

We show that there is a fixed planar graph H, namely the 5×5 grid, such that Max Independent Set remains NP-hard in H-induced-minor-free graphs. This refutes the Dallard–Milanič–Štorgel conjecture and a weakening of it by Gartland and Lokshtanov, and by Korhonen.

1 Introduction

Graph classes excluding a planar minor have bounded treewidth [13]. Thus, in them, any problem definable in MSO_2 logic (or its optimization variants) can be solved in polynomial time [4]. This cannot hold in classes excluding a planar *induced* minor (where induced minors are obtained by removing vertices and contracting edges). Indeed, a problem like Dominating Set is NP-complete on graphs excluding a 5-vertex path as an induced minor [1] (or, equivalently, as an induced subgraph).

However, no similar direct argument dismisses that possibility for the Max Independent Set problem (MIS for short). This led Dallard, Milanič, and Štorgel to ask whether a polynomial-time algorithm exists for MIS on any planar-induced-minor-free class.

▶ **Conjecture 1** (Dallard–Milanič–Štorgel [5]). *For every planar graph H, Max Independent Set can be solved in polynomial time on graphs excluding H as an induced minor.*

This question, turned into a conjecture, has garnered a large amount of related work in the past five years. (Let K_t denote the t-vertex clique, $K_{s,t}$ the biclique with s vertices against t vertices, P_t the t-vertex path, C_t the t-vertex cycle, and W_t the wheel obtained from C_t by adding a universal vertex.) Conjecture 1 was shown when H is K_5 minus an edge, $K_{2,t}$, W_4 [5], P_6 [8], or any windmill graph (a matching plus a universal vertex) [3].

A natural relaxation of Conjecture 1 to quasipolynomial time has seen further development.

▶ **Conjecture 2** (Gartland–Lokshtanov, Korhonen [11], [3, Question 2]). *For every planar graph H, Max Independent Set can be solved in quasipolynomial time on graphs excluding H as an induced minor.*

While Conjecture 1 is already open when $H = P_7$, Conjecture 2 is known when H is any path, any cycle [7], a disjoint union of triangles [2], or even $tC_3 \uplus C_4$ [3].

Korhonen and Lokshtanov have obtained a subexponential algorithm, as a byproduct of balanced separators of size $\mathcal{O}(\sqrt{m})$, where m is the number of edges.

▶ **Theorem 3** ([12, Corollary 1.2]). *For every planar graph H, Max Independent Set can be solved in $2^{\tilde{O}_H(n^{2/3})}$ time on n-vertex graphs excluding H as an induced minor.*

Gartland and Lokshtanov have proposed strengthenings of Conjectures 1 and 2 for a family of problems. Fix a positive integer t and a CMSO_2 sentence φ, i.e., a closed formula with universal and existential quantifications over vertices, vertex subsets, and edge subsets, and counting predicates modulo some fixed integers. The (tw $\leqslant t, \varphi$)-Weighted Maximum Induced Subgraph ((tw $\leqslant t, \varphi$)-WMIS) takes as input a vertex-weighted graph (G, w) and asks one either to find a set $S \subseteq V(G)$ such that

- $G[S]$ satisfies φ,
- the treewidth of $G[S]$ is at most t, and
- the sum of weights of vertices in S is maximized subject to the above conditions,

or to conclude that no such set S exists.

Special cases of the (tw $\leqslant t, \varphi$)-WMIS problem include, for instance, MIS, Feedback Vertex Set, and Even Cycle Transversal. It was conjectured that Conjecture 1 more generally holds for (tw $\leqslant t, \varphi$)-WMIS.

▶ **Conjecture 4** ([6, Conjecture 1.4.2]). *For every planar graph H, positive integer t, and CMSO$_2$ sentence φ, $(tw \leqslant t, \varphi)$-Weighted Maximum Induced Subgraph can be solved in polynomial time on H-induced-minor-free graphs.*

Our contributions. We refute Conjectures 1, 2, and 4 (unless P = NP for Conjectures 1 and 4, and unless NP $\subseteq$ QP for Conjecture 2) by showing the following.

▶ **Theorem 5.** *MIS is NP-hard in graphs excluding the 5×5 grid as an induced minor.*

The proof is a remarkably simple reduction from Max Cut on general graphs F.

Look at Figure 1: this intermediate graph G has the strong product of a path with an edge (i.e., a ladder with diagonals) for each *row* (with each path occupying one of two sides), and a biclique between the two sides in each column. It is easy to show that a cycle in G contains two vertices in the same column and on opposite sides (mentally remove all the column bicliques and consider the possible neighbors of a leftmost vertex of the cycle). The union of the neighborhoods of two such vertices contains the entire biclique and thus separates the left-hand side from the right-hand side. Therefore, every graph with three vertex-disjoint nonadjacent cycles and a path between every pair of cycles avoiding the neighborhood of the third cycle cannot be an induced minor of G (the 5×5 grid has this property; see Figure 4). Indeed, consider the cycle whose neighborhood contains the biclique in between the bicliques of the other two cycles.

We then pick an appropriate induced subgraph G_F of G, where every odd-index column represents a vertex of F, and every edge of F "occupies" two rows. From the rows of edge $e = uv \in E(F)$, we only keep the vertices in columns between that of u and that of v, with the exception of four vertices; see Figure 2. This defines the graph G_F. Any independent set I in G_F defines a bipartition of $V(F)$ where the part of $v \in V(F)$ depends on the side of the column of v that intersects I. This is well-defined due to the column bicliques. Finally, the encoding of $e \in E(F)$ is such that one more vertex can be inserted in the independent set exactly when its endpoints are in distinct parts of the bipartition of $V(F)$; see Figure 5.

If we reduce from Max Cut on (sub)cubic graphs F, the previous reduction produces graphs with $\mathcal{O}(|V(F)|^2)$ vertices. Classic reductions thus imply the following, under the Exponential-Time Hypothesis (ETH), i.e., the assumption that there is a $\lambda > 1$ such that n-variable 3-SAT cannot be solved in time $\mathcal{O}(\lambda^n)$ [9].

▶ **Corollary 6.** *Unless the ETH fails, Max Independent Set requires time $2^{\Omega(\sqrt{n})}$ in n-vertex graphs excluding the 5×5 grid as an induced minor.*

Future work. Now, Conjectures 1 and 2 can be turned into dichotomy questions.

▶ **Question 1.** *Which planar graphs H are such that* Max Independent Set *can be solved in (quasi)polynomial time on graphs excluding H as an induced minor?*

In fact, we give a negative answer to Conjectures 1 and 2 for a graph simpler than the 5×5 grid: the cycle C_9 (with vertices $1, \dots, 9$ in cyclic order) with three additional vertices respectively adjacent to 1 and 2, 4 and 5, and 7 and 8.

One can also turn to approximation schemes. The following conjecture is not refuted by the present work.

▶ **Conjecture 7.** *For every planar graph H,* Max Independent Set *admits a quasipolynomial-time approximation scheme (QPTAS) on graphs excluding H as an induced minor.*

Conjecture 7 is actually known to be implied by a structural conjecture of Gartland and Lokshtanov.

▶ **Conjecture 8** (Gartland–Lokshtanov [6])**.** *For every planar graph H, there is a constant $k := k(H)$ such that every n-vertex H-induced-minor-free graph G admits a set $X \subseteq V(G)$ of size at most k such that the closed neighborhood of X is a balanced separator of G.*

A more adventurous (but still open) conjecture is the strengthening of Conjecture 7 to a polynomial-time approximation scheme (PTAS).

2 Hardness of MIS in Planar-Induced-Minor-Free Classes

For any two integers i, j, we set $[i, j] := \{k \in \mathbb{Z} \mid i \leqslant k \leqslant j\}$, and $[i] := [1, i]$. We use the standard graph-theoretic notation. For any graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively, of G. If $S \subseteq V(G)$, then $G[S]$ denotes the subgraph of G induced by S.

We denote by $\alpha(G)$ the *independence number* of G, that is, the size of a largest subset of vertices of G that are pairwise nonadjacent. We write $\mathsf{MaxCut}(G)$ for the number of edges in a maximum-cardinality cut of G.

▶ **Theorem 5.** *MIS is NP-hard in graphs excluding the 5×5 grid as an induced minor.*

Proof. We reduce Max Cut in general graphs to Max Independent Set in graphs excluding a fixed planar induced minor. Let F be any n-vertex m-edge Max Cut instance. We first build an intermediate graph G with vertex set

$$\{(i, j, s) \mid i \in [2m],\ j \in [2n-1],\ s \in \{0, 1\}\},$$

where we see i as the row index, j as the column index, and s as the side, and edge set

$$\{(i, j, 0)(i', j, 1) \mid i, i' \in [2m],\ j \in [2n-1]\} \cup$$

$$\{(i, j, s)(i, j', s') \mid i \in [2m],\ j, j' \in [2n-1], |j - j'| = 1,\ s, s' \in \{0, 1\}\};$$

see Figure 1.

We arbitrarily order the vertices of F: $v_1, v_2, \dots, v_n$. We set $C_j := \{(i, j, s) \mid i \in [2m],\ s \in \{0, 1\}\}$ to be the *jth column*. We see C_{2j-1} as the column of v_j, for every $j \in [n]$. The columns of even index do *not* represent a vertex of F. Similarly, a *row* is any set $\{(i, j, s) \mid j \in [2n-1],\ s \in \{0, 1\}\}$ with $i \in [2m]$. We reserve two rows for each edge $e \in E(F)$, and denote them by $R'_{e,0}$ and $R'_{e,1}$. For convenience, we denote by $e^0, e^1 \in [2m]$ the row indices of $R'_{e,0}, R'_{e,1}$, respectively.

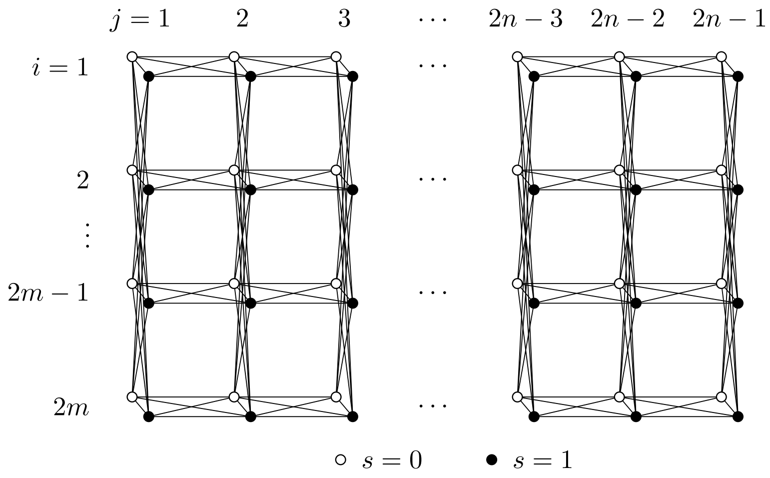

Figure 1 The intermediate graph G.

We finally define the following induced subgraph G_F of G. For every edge $e = v_jv_{j'} \in E(F)$ (with $j < j'$), we only keep from $R'_{e,0}$ the vertices $(e^0, 2j-1, 0)$, $(e^0, 2j'-1, 1)$, and all the vertices (e^0, j'', s) with $2j-1 < j'' < 2j'-1,\ s \in \{0,1\}$, and from $R'_{e,1}$ the vertices $(e^1, 2j-1, 1)$, $(e^1, 2j'-1, 0)$, and all the vertices (e^1, j'', s) with $2j-1 < j'' < 2j'-1,\ s \in \{0,1\}$. We denote by $R_{e,0} \subset R'_{e,0}$ and $R_{e,1} \subset R'_{e,1}$ the obtained subsets. This completes the construction; see Figure 2. We denote by S_0 the set of vertices of G_F whose last coordinate is 0 (on the side $s = 0$), and by S_1 the set of vertices of G_F on the side $s = 1$.

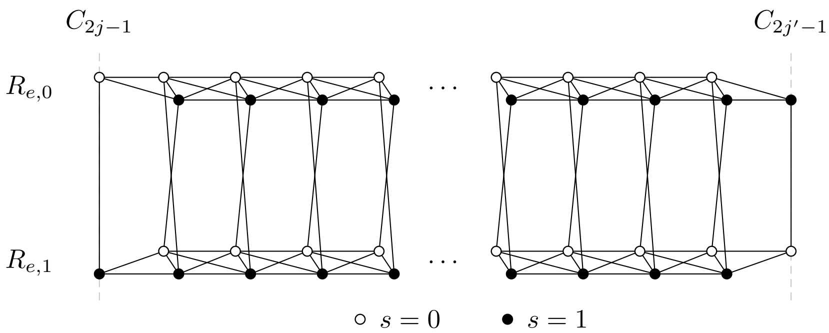

Figure 2 The restriction of G_F to the two rows reserved for $e = v_jv_{j'}$.

▷ Claim 9. Graph G excludes the 5×5 grid as an induced minor.

Proof. We first claim that every cycle of G contains two (adjacent) vertices in the same column but on different sides, i.e., of the form $(i, j, 0)$ and $(i', j, 1)$. Indeed, if one removes from G all the edges with endpoints in the same column but on different sides, the resulting graph G' is the disjoint union of $2m$ graphs Q_{2n-1} obtained from two $(2n-1)$-vertex paths P_{2n-1} by making the jth vertex of the first path adjacent to the $(j-1)$st and $(j+1)$st of the other path (when these vertices exist). In particular, consider a *leftmost* vertex $(\widehat{i}, \widehat{j}, s)$ (i.e., minimizing $\widehat{j}$) of a cycle C that does not contain a pair $(i, j, 0), (i', j, 1)$. Its two neighbors on C must be $(\widehat{i}, \widehat{j}+1, 0)$ and $(\widehat{i}, \widehat{j}+1, 1)$; these two vertices form such a pair. See Figure 3 for an illustration.

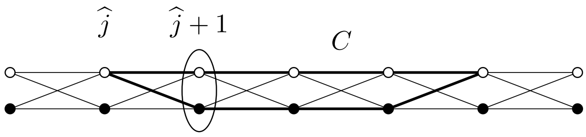

Figure 3 The leftmost-vertex argument in (a connected component of) G'.

Let H be any graph with three vertex-disjoint and nonadjacent cycles $\Gamma_1, \Gamma_2, \Gamma_3$ such that there is a path between any two cycles in the non-neighborhood of the third cycle. Note that the 5×5 grid is such a graph H; see Figure 4.

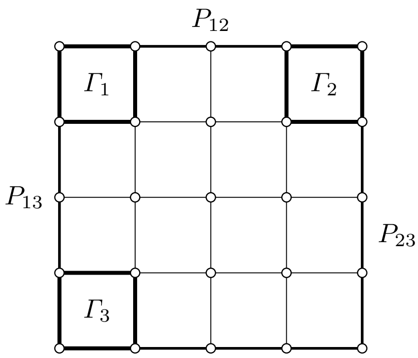

Figure 4 Three appropriate cycles in the 5×5 grid.

We claim that no such graph H is an induced minor of G. The subgraph of G induced by the union U_a of the branch sets representing $V(\Gamma_a)$ (for any $a \in [3]$) has to contain a cycle. By the first paragraph, there are three column indices $j_1, j_2, j_3 \in [2n-1]$ such that U_a contains a pair of the form $(\cdot, j_a, 0), (\cdot, j_a, 1)$ for each $a \in [3]$. Without loss of generality, let us assume that $j_1 \leqslant j_2 \leqslant j_3$. Since the cycles $\Gamma_1, \Gamma_2, \Gamma_3$ are nonadjacent, we further have $j_1 < j_2 < j_3$. Finally, note that, for any $i, i' \in [2m]$, the union of the closed neighborhoods of $(i, j_2, 0)$ and $(i', j_2, 1)$ in G contains C_{j_2}, and that the removal of C_{j_2} disconnects C_{j_1} from C_{j_3}. This contradicts the assumption that there is a path between Γ_1 and Γ_3 in the non-neighborhood of Γ_2. $\lhd$

In particular, the induced subgraph G_F of G excludes the 5×5 grid as an induced minor. We finish the proof by showing that computing the independence number of G_F would solve Max Cut for the arbitrary graph F.

$\rhd$ Claim 10. $\alpha(G_F) = \mathsf{MaxCut}(F) + 2 \sum_{v_j v_{j'} \in E(F),\ j<j'} (j'-j)$.

Proof. We first show that

$$\alpha(G_F) \geqslant \mathsf{MaxCut}(F) + 2 \sum_{v_j v_{j'} \in E(F),\ j<j'} (j'-j).$$

Let (A_0, A_1) be a bipartition of $V(F)$ realizing the maximum cut. We identify A_0 (resp. A_1) with the side $s=0$ (resp. $s=1$). We build the independent set I as follows.

For every $v_j \in A_0$ and every incident edge $e = v_j v_{j'} \in E(F)$, add to I the unique vertex of $(R_{e,0} \cup R_{e,1}) \cap C_{2j-1} \cap S_0$. Note that this vertex is in $R_{e,0}$ if $j < j'$ and in $R_{e,1}$ if $j' < j$.

Symmetrically, for every $v_j \in A_1$ and every incident edge $e = v_j v_{j'} \in E(F)$, add to I the unique vertex of $(R_{e,0} \cup R_{e,1}) \cap C_{2j-1} \cap S_1$.

At this stage, for every $e = v_j v_{j'} \in E(F)$ with, say, $j < j'$, I intersects $R_{e,0} \cup R_{e,1}$ at exactly two vertices: one in C_{2j-1} and one in $C_{2j'-1}$. Say that the former (the one in C_{2j-1}) is in $R_{e,s}$, and that the latter is in $R_{e,s'}$, with $s, s' \in \{0,1\}$. We add the following vertices of $R_{e,0} \cup R_{e,1}$ to the independent set I:

- for every $v_{j''} \in A_0$ with $j'' \in [j+1, j'-1]$, we add the vertex $(e^s, 2j''-1, 0)$;
- for every $v_{j''} \in A_1$ with $j'' \in [j+1, j'-1]$, we add the vertex $(e^s, 2j''-1, 1)$;
- for every $j'' \in [j, j'-2]$, we add $(e^{1-s}, 2j'', 0)$ to I;
- in addition, if $s = s'$, we add $(e^{1-s}, 2j'-2, 0)$ to I.

(Note that for the last two items, we could also have consistently picked the corresponding vertices in S_1. In fact, we could have kept only a single vertex at the intersection of a row with a column of even index, but we decided against that to simplify the description of G.) This finishes the construction of I. It is indeed an independent set as we observed the following rules:

- in every column, we only picked vertices from one side, and
- we never picked two vertices in consecutive columns of the same row.

We now determine the size of I. For every $e = v_j v_{j'} \in E(F)$ with $j < j'$, $I \cap (R_{e,0} \cup R_{e,1})$ contains exactly one vertex in every column intersected by $R_{e,0} \cup R_{e,1}$ except possibly $C_{2j'-2}$, in which it contains at most one vertex. Thus

$$|I \cap (R_{e,0} \cup R_{e,1})| \in \{2(j'-j), 2(j'-j)+1\}.$$

Furthermore, if e is in the cut (A_0, A_1), then $s = s'$ and $|I \cap (R_{e,0} \cup R_{e,1})| = 2(j'-j)+1$. See Figure 5 for an illustration. Therefore,

$$|I| = \mathsf{MaxCut}(F) + 2 \sum_{v_j v_{j'} \in E(F),\ j<j'} (j'-j), \quad \text{as desired.}$$

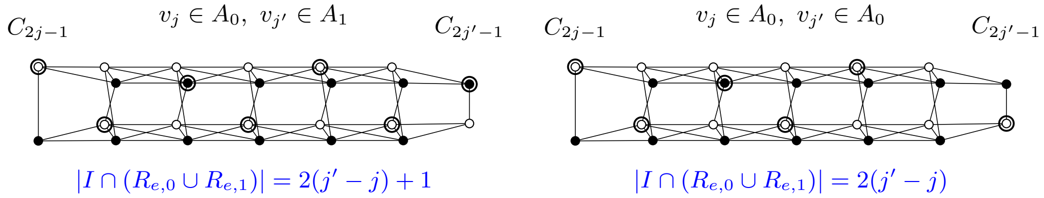

Figure 5 Example of $I \cap (R_{e,0} \cup R_{e,1})$ (circled vertices) when v_j and $v_{j'}$ are on opposite sides of the bipartition (A_0, A_1) (left) and when they are on the same side (right).

We now show that

$$\alpha(G_F) \leqslant \mathsf{MaxCut}(F) + 2 \sum_{v_j v_{j'} \in E(F),\ j<j'} (j'-j).$$

We fix some independent set I of G_F. In every column C_{2j-1} of v_j, I can only contain vertices from one side (because of the biclique between $S_0 \cap C_{2j-1}$ and $S_1 \cap C_{2j-1}$). The set I thus defines a partition (A_0, A_1) of $V(F)$ where $v_j \in V(F)$ is in A_0 if $I \cap C_{2j-1} \subseteq S_0$, and $v_j \in A_1$ if instead $I \cap C_{2j-1} \subseteq S_1$ (vertices whose columns do not intersect I are assigned arbitrarily to either part). For every $e = v_j v_{j'} \in E(F)$ with $j < j'$, we partition $R_{e,0} \cup R_{e,1}$ into $D_j, D_{j+1}, \ldots, D_{j'-1}, D_{j'}$, where

- $D_j := C_{2j-1} \cap (R_{e,0} \cup R_{e,1})$, and
- for every $j'' \in [j+1, j']$, $D_{j''} := (C_{2j''-2} \cup C_{2j''-1}) \cap (R_{e,0} \cup R_{e,1})$.

Observe that I can contain at most one vertex in D_j, and at most two vertices in each $D_{j''}$ with $j'' \in [j+1, j']$. This implies that $|I \cap (R_{e,0} \cup R_{e,1})| \leqslant 1 + 2(j' - j)$; see Figure 6.

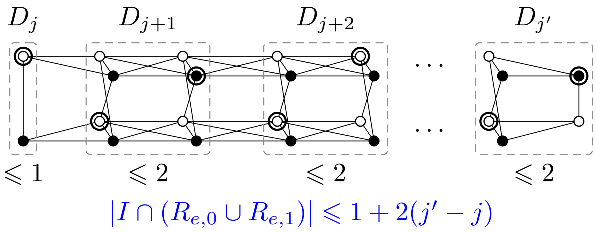

Figure 6 The partition $D_j, \dots, D_{j'}$ and an upper bound on $|I \cap (R_{e,0} \cup R_{e,1})|$.

Furthermore, if v_j and $v_{j'}$ are on the same side of (A_0, A_1), we claim that

$$|I \cap (R_{e,0} \cup R_{e,1})| \leqslant 2(j' - j).$$

Indeed, in each of the two rows, at least one of its two endpoint vertices cannot belong to I, because the endpoints of a row are on opposite sides. After deleting that forbidden endpoint, the row occupies $2(j'-j)$ consecutive columns. Pairing these columns consecutively partitions the remaining row into $j' - j$ cliques. Hence I contains at most $j' - j$ vertices in each row.

We conclude that $|I| \leqslant \mathsf{MaxCut}(F) + 2 \sum_{v_j v_{j'} \in E(F),\ j < j'} (j' - j)$. ◁

One can easily compute $2 \sum_{v_j v_{j'} \in E(F),\ j<j'} (j' - j)$ in polynomial time. So we conclude that solving MIS on G_F is as hard as solving Max Cut on F. ◀

We obtain the following corollary by reducing from Max Cut on cubic graphs.

▶ **Corollary 6.** *Unless the ETH fails, Max Independent Set requires $2^{\Omega(\sqrt{n})}$ time in n-vertex graphs excluding the 5×5 grid as an induced minor.*

Proof. By the Sparsification Lemma [10], under the ETH, N-variable 3-SAT-B for some constant B (3-SAT where each variable appears at most B times) has no $2^{o(N)}$-time algorithm. The standard linear-size occurrence-splitting construction underlying [14, Lemma 5], followed by the reduction of [14, Theorem 13], gives a linear-size reduction to Max Cut on cubic graphs. We finally observe that applying the reduction in the proof of Theorem 5 to cubic Max Cut instances F produces MIS instances with $n \leqslant 12|V(F)|^2$ vertices, so $n = \mathcal{O}(N^2)$. ◀

AI disclosure.

The reduction of Theorem 5 was suggested by GPT-6 Pro after, embarrassingly, a mere 23-minute think. We simplified and improved the original complicated argument that the constructed instances exclude the 20×20 grid as an induced minor. The write-up is entirely due to the authors.

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