Induced Minors and Region Intersection Graphs

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Abstract

We show that for any positive integers g and t, there is a $K_6^{(1)}$-induced-minor-free graph of girth at least g that is not a region intersection graph over the class of K_t-minor-free graphs. This answers in a strong form the recently raised question of whether for every graph H there is a graph H' such that H-induced-minor-free graphs are region intersection graphs over H'-minor-free graphs.

1 Introduction

Inspired by the success of Robertson and Seymour's graph minor theory [17], a recent line of work aims to extend this theory to the realm of induced-minor-free classes.[1] Currently, far less is understood on classes excluding an induced minor than on those excluding a minor. While H-minor-free n-vertex graphs are known since the 90's to have treewidth $O_H(\sqrt{n})$ [1], foreshadowed a decade earlier by the Lipton–Tarjan planar separator theorem [13], only recently were H-induced-minor-free m-edge graphs shown to have treewidth $\widetilde{O}_H(\sqrt{m})$ [11].

There are several open questions (for simplicity, we phrase all of them as conjectures) on induced-minor-free classes.

- For any planar graph H, the independence number of any H-induced-minor-free graph can be computed in polynomial time (see [5, Question 8.2]).[2]
- For any planar graph H, every H-induced-minor-free graph admits a balanced separator dominated by a subset of size $O_H(1)$ (Gartland–Lokshtanov's conjecture [9]).
- For any planar graph H, every H-induced-minor-free graph has treewidth at most linear in its maximum degree (see [3]).
- For any graph H, the independence number admits a polynomial-time approximation scheme in H-induced-minor-free graphs.
- For any planar graph H, weakly sparse H-induced-minor-free classes have bounded twin-width (a special case is mentioned in [2]).

[1]All the relevant notions are defined in Section 2.

[2]Merely obtaining a quasipolynomial-time algorithm is also a wide open question.

- For any planar graph H, every H-induced-minor-free graph has treewidth at most linear in its Hadwiger number (see [4]).
- For any graph H, every H-induced-minor-free graph is quasi-isometric to an H-minor-free graph (a more general conjecture is found in [10]).

All these questions are open within classes of large girth, a condition which may make them more approachable. One more question, posed independently by Lokshtanov [14] and McCarty [15], is whether region intersection graphs could provide a bridge between the structure of minors and induced minors. A graph G is a *region intersection graph* (*RIG*) over a graph H if there exists a collection $\mathcal{R} = (R_v \subseteq H : v \in V(G))$ of connected subgraphs of H such that $uv \in E(G)$ if and only if $V(R_u) \cap V(R_v) \neq \emptyset$. We call H the *host graph* of G.

Question 1. *Is every graph class excluding a fixed induced minor included in the region intersection graphs of a class excluding a fixed minor?*

If true, one could then work with the host graph and benefit from its decomposition given by the Graph Minor Structure Theorem [18]. Wiederrecht asked a related question of whether one can *determine* if a given induced-minor-free class is a region intersection graph over a minor-free class [20].

Region intersection graphs were introduced by Lee [12] as a generalization of the well-studied class of string graphs (intersection graph of curves on the plane). Indeed, a graph is a string graph if and only if it is a region intersection graph over some planar graph. The class of string graphs does not exclude any graph as a minor, but excludes any 1-subdivision of a non-planar graph as an induced minor [19]. More generally, Lee [12] proved the following relationship between region intersection graphs and minors.

Lemma 1 ([12])**.** *For every graph G, if a graph H is not a minor of G then any graph that contains $H^{(1)}$ as an induced minor is not a region intersection graph over G.*

Thus RIGs over an H-minor-free class are examples of classes excluding an induced minor. The theory on region intersection graphs, and mainly on string graphs, is more advanced than that of induced-minor-free graphs. For instance, RIGs over K_t-minor-free classes can be $O_t(1)$-vertex-colored (or $O_t(1)$-edge-colored) such that every monochromatic connected component has bounded weak diameter [12, 7]. Such a result is useful in various contexts, and it would resolve several conjectures for classes excluding a fixed induced minor (see for instance [11, 3]). One way to achieve that would be via a positive answer to Question 1.

Unfortunately, we answer Question 1 negatively, and perhaps more surprisingly, even within classes of arbitrarily large girth.

Theorem 2. *For any positive integers t and g, there is a $K_6^{(1)}$-induced-minor-free graph of girth at least g that is not in RIG($\{H : H$ is K_t-minor-free$\}$).*

The bridge between induced-minor-freeness and minor-freeness (if it exists) is not given straightforwardly by region intersection graphs. Our construction for proving Theorem 2 is an extension of the so-called *Pohoata–Davies grids* [6, 16] (see Figure 1), a key family of graphs in the study of induced subgraphs and tree-decompositions.

Hopefully, our construction steers the search for a link between induced-minor-freeness and minor-freeness in a more fortunate direction.

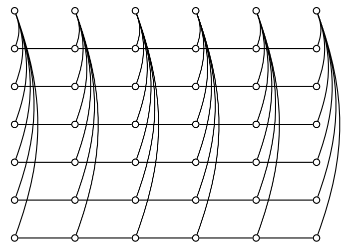

Figure 1: The Pohoata–Davies 6×6 grid.

2 Preliminaries

Given an integer i, we denote by $[i]$ the set of integers that are at least 1 and at most i.

2.1 Standard graph-theoretic notation

We denote by $V(G)$ and $E(G)$ the set of vertices and edges of a graph G, respectively. A graph H is a *subgraph* of a graph G, denoted by $H \subseteq G$, if H can be obtained from G by vertex and edge deletions. Graph H is an *induced subgraph* of G if H is obtained from G by vertex deletions only. For $S \subseteq V(G)$, the *subgraph of G induced by S*, denoted $G[S]$, is obtained by removing from G all the vertices that are not in S. Then $G - S$ is a short-hand for $G[V(G) \setminus S]$.

A set $X \subseteq V(G)$ is connected (in G) if $G[X]$ has a single connected component. The *girth* of a graph is the number of vertices of one of its shortest cycles, and ∞ if the graph is acyclic. A graph class is *weakly sparse* if it excludes $K_{t,t}$ as a subgraph for some finite integer t. A *balanced separator* of an n-vertex graph G is a set $X \subseteq V(G)$ such that $G - X$ has no connected component on more than $n/2$ vertices.

If G is a graph and ℓ is a positive integer, then $G^{(\ell)}$ denotes the *ℓ-subdivision* of G (replacing every edge of G by a path with $\ell + 1$ edges), and ℓG denotes the graph obtained from ℓ disjoint copies of G. We call the original vertices of $V(G)$ in $G^{(\ell)}$ *branching vertices*, and the added vertices (which have degree 2) *subdivision vertices*. We say that two disjoint sets $X, Y \subseteq V(G)$ are *anti-complete* if there is no edge in G with one end in X and the other in Y. The *diameter of G* is defined as $\max_{u,v \in V(G)} d_G(u, v)$, where $d_G(u, v)$ is the number of edges in a shortest path between u and v. The *weak diameter of S in G* for $S \subseteq V(G)$ is equal to $\max_{u,v \in S} d_G(u, v)$.

2.2 Tree-decomposition

A *tree-decomposition* of a graph G is a collection $\mathcal{T} = (W_x : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the vertices of a tree T, such that

- for every edge $uv \in E(G)$, some bag W_x contains both u and v, and
- for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in W_x\}$ induces a non-empty (connected) subtree of T.

The *width* of $\mathcal{T}$ is $\max\{|W_x|: x \in V(T)\} - 1$. The *treewidth* of G is the minimum width of a tree-decomposition of G. The *adhesion* of $\mathcal{T}$ is $\max\{|W_x \cap W_y|: xy \in E(T)\}$. The *torso* of a bag W_x (with respect to $\mathcal{T}$), denoted by $G\langle W_x \rangle$, is the graph obtained from the induced subgraph $G[W_x]$ by adding edges so that $W_x \cap W_y$ is a clique for each edge $xy \in E(T)$. A *path-decomposition* is a tree-decomposition in which the underlying tree is a path, simply denoted by the corresponding sequence of bags $(W_1, \dots, W_n)$.

2.3 Minors, induced minors, and region intersection graphs

A graph H is a *minor* of a graph G if H is isomorphic to a graph that can be obtained from a subgraph of G by contracting edges. Equivalently, H is a minor of G if there exists a model $\mathcal{M} = (X_v \subseteq G: v \in V(H))$ of H in G which is a collection of disjoint connected subgraphs of G such that X_u and X_v are adjacent whenever $uv \in E(H)$. Each X_u is called a *branch set*. A graph H is an *induced minor* of a graph G if H is isomorphic to a graph that can be obtained from an induced subgraph of G by contracting edges. Equivalently, H is an induced minor of G if there is a model $\mathcal{M} = (X_v \subseteq G: v \in V(H))$ of H in G with the additional constraint that X_u and X_v are adjacent if and only if $uv \in E(H)$. A graph G is *H-minor-free* (resp. *H-induced-minor-free*) if H is not a minor (resp. an induced minor) of G.

Recall that a graph G is a *region intersection graph* over a graph H if there exists a collection $\mathcal{R} = (R_v \subseteq H: v \in V(G))$ of connected subgraphs of H such that $uv \in E(G)$ if and only if $V(R_u) \cap V(R_v) \neq \emptyset$. We denote by $\mathrm{RIG}(H)$ the class of graphs that are region intersection graphs over H. By extension, given a graph class $\mathcal{C}$, $\mathrm{RIG}(\mathcal{C})$ denotes the class of graphs that are region intersection graphs over some graph of $\mathcal{C}$.

2.4 Graph minor structure theorem

The Graph Minor Structure Theorem of Robertson and Seymour [18] states that every K_t-minor-free graph has a tree-decomposition with bounded-size adhesion such that each torso can be constructed using three ingredients: graphs on surfaces, vortices, and apices. To describe this formally, we need the following definitions.

Let G_0 be a graph embedded in a surface Σ. A closed disk D in Σ is *G_0-clean* if its only points of intersection with G_0 are vertices of G_0 that lie on the boundary of D. Let $x_1, \dots, x_b$ be the vertices of G_0 on the boundary of D in the order around D. A *D-vortex* (with respect to G_0) of a graph H is a path-decomposition $(W_1, \dots, W_b)$ of H such that $x_i \in W_i$ for each $i \in [b]$, and $V(G_0 \cap H) = \{x_1, \dots, x_b\}$.

For integers $g, p, a \geqslant 0$ and $k \geqslant 1$, a graph G is *(g, p, k, a)-almost-embeddable* if for some set $Z \subseteq V(G)$ with $|Z| \leqslant a$, there are graphs $G_0, G_1, \dots, G_p$ such that:

- $G - Z = G_0 \cup G_1 \cup \cdots \cup G_p$,
- $G_1, \dots, G_p$ are pairwise vertex-disjoint,
- G_0 is embedded in a surface Σ of Euler genus at most g,
- there are p pairwise disjoint G_0-clean closed disks $D_1, \dots, D_p$ in Σ, and
- for $i \in [p]$, there is a D_i-vortex $(W_1, \dots, W_{b_i})$ of G_i of width at most k.

The vertices in Z are called *apex* vertices—they can be adjacent to any vertex in G. A graph is *ℓ-almost-embeddable* if it is (g, p, k, a)-almost-embeddable for some $\ell \geqslant g, p, k, a$. A graph is *apex-free ℓ-almost-embeddable* if it is $(g, p, k, 0)$-almost-embeddable for some $\ell \geqslant g, p, k$.

Theorem 3 ([18]). *For every positive integer t, there exists an integer ℓ such that every K_t-minor-free graph has a tree-decomposition of adhesion at most ℓ such that each torso is ℓ-almost-embeddable.*

For every positive integer n, let A_n denote the *apex $n \times n$ grid*; that is, the graph obtained from the $n \times n$ grid by adding a universal vertex. The next theorem concerns the structure of apex-minor-free graphs. The statement is implied by a characterization of apex-minor-free graphs [8, Theorem 25, (6) $\Rightarrow$ (5)].

Theorem 4 ([8]). *For every positive integer ℓ, there exists some integer n such that every graph that has a tree-decomposition of adhesion at most ℓ where each torso is apex-free ℓ-almost-embeddable is A_n-minor-free.*

Finally, we will need the notion of *clique-sum*. Let k be a positive integer, $C_1 = \{v_1, \dots, v_k\}$, a clique in a graph G_1, $C_2 = \{w_1, \dots, w_k\}$, a clique in a graph G_2. A *k-clique-sum* of G_1 and G_2 is any graph G obtained from the disjoint union of G_1 and G_2 by identifying v_i and w_i for each $i \in [k]$ and then possibly deleting some edges in C_1 $(= C_2)$.

3 Proof of Theorem 2

In this section, we prove Theorem 2 first for graphs of girth 5. We then explain how the construction can be generalized so that the result holds for arbitrarily large girth.

Theorem 5. *For every positive integer t, there is a $K_6^{(1)}$-induced-minor-free graph G of girth 5 such that G is not in RIG($\{H : H$ is K_t-minor-free$\}$).*

We fix any positive integer t. By Theorem 3, there exists some integer $\ell := \ell(t)$ such that every K_t-minor-free graph has a tree-decomposition of adhesion at most ℓ where each torso is ℓ-almost-embeddable. By Theorem 4, there exists some integer $n := n(\ell)$ such that every graph that has a tree-decomposition of adhesion at most ℓ where each torso is apex-free ℓ-almost-embeddable is A_n-minor-free. We may assume that $n \geqslant \ell + 1$. We now construct our graph G.

Construction of G. Since the $n \times n$ grid is K_5-minor-free, the apex $n \times n$ grid A_n is K_6-minor-free. Let B_n be $nA_n^{(1)}$, that is, the disjoint union of n copies of the 1-subdivision of A_n, also equal to the 1-subdivision of the disjoint union of n copies of A_n. We now set a total order $\prec$ of $V(B_n)$, and a traceable (i.e., admitting a Hamiltonian path) spanning supergraph B'_n of B_n, whose Hamiltonian path defines the successor relation of $\prec$.

The vertices of each copy of $A_n^{(1)}$ appear consecutively along $\prec$. The graph B'_n is obtained by adding to each copy of $A_n^{(1)}$ the red edges of Figure 2. Note that this includes an edge between the top-left vertex of the grid and the apex of the previous copy of $A_n^{(1)}$ (leftmost

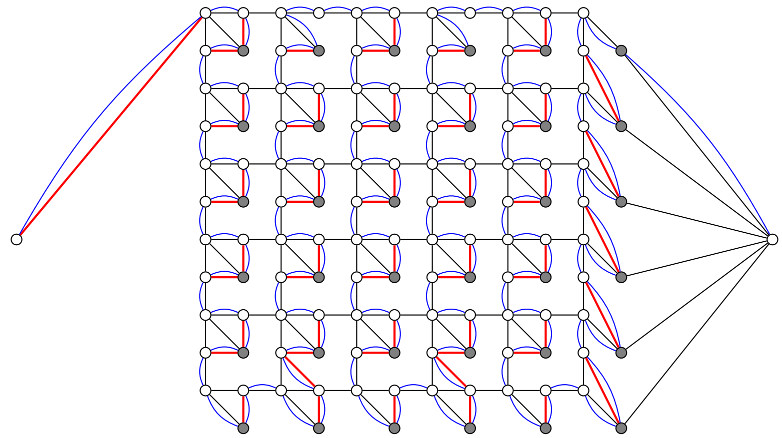

Figure 2: The graphs B_n, B'_n and the order $\prec$. We only represented one entire copy of $A_n^{(1)}$. Black edges represent B_n. Together with the red edges, they form B'_n. Every vertex filled in gray is adjacent to the apex vertex to the right (we only drew some of these edges for legibility). The Hamiltonian path of B'_n in blue defines the successor relation of $\prec$.

vertex in the figure). The order $\prec$ within each $A_n^{(1)}$ is given by the Hamiltonian path in blue, starting at the top-left vertex of the grid to the apex. Like B_n, the graph B'_n is also K_6-minor-free. The graph B'_n is not part of the construction and we will only use it in the proof of Lemma 7.

To finish the construction, we add to B_n the disjoint union of n paths $P_1, \dots, P_n$ of length $2|V(B_n)| - 1$, and make for every $i \in [|V(B_n)|]$ and $j \in [n]$, the $(2i-1)$-st vertex of P_j, denoted by $p_{j,i}$, adjacent to the i-th vertex of B_n along $\prec$, denoted by b_i. Call G the resulting graph. As a side note, if we replaced each copy of $A_n^{(1)}$ in G by K_1, then the graph obtained is a Pohoata–Davies Grid (see Figure 1).

The following three lemmas prove Theorem 5.

Lemma 6. *G has girth at least 5.*

Proof. B_n is the 1-subdivision of a simple graph, hence has girth at least 6. $G - V(B_n)$ is a disjoint union of paths, thus does not contain any cycle. Any cycle going through $V(G) \setminus V(B_n)$ has at least two consecutive edges within $G - V(B_n)$. We conclude as no distinct pair of vertices within the same connected component of $V(G) \setminus V(B_n)$ shares a neighbor in $V(B_n)$. □

Lemma 7. *G is $K_6^{(1)}$-induced-minor-free.*

Proof. Assume for the sake of contradiction that G admits $K_6^{(1)}$ as an induced minor. We will then build a minor model of K_6 in B'_n, which, we know, does not exist.

Let $\mathcal{M}$ be an induced minor model of $K_6^{(1)}$ in G such that

- every branch set of a subdivision vertex of $K_6^{(1)}$ is a singleton,
- if such a singleton is on some P_j and its two neighbors on P_j are in the two adjacent branch sets (one in each), then the singleton cannot be a vertex $p_{j,i}$ (it has to be a degree-2 vertex in between some $p_{j,i}$ and $p_{j,i+1}$), and

- each branch set is inclusion-wise minimal.

It is easy to see that this can always be done. Let $X_1, \dots, X_6 \in \mathcal{M}$ be the branch sets corresponding to the branching vertices of $K_6^{(1)}$. We denote by $\{s_{k,k'}\}$ the branch set (of the subdivision vertex) adjacent to X_k and $X_{k'}$, for $k \neq k' \in [6]$. For each $k \in [6]$, let

$$Y_k := (X_k \cap V(B'_n)) \cup \{b_i \ : \ \exists j \in [n],\ p_{j,i} \in X_k \text{ and } \nexists k' < k \in [6], j' \in [n],\ p_{j',i} \in X_{k'}\}, \text{ and}$$

$$Y'_k := Y_k \cup \{s_{k,k'} \in V(B'_n) \setminus (Y_1 \cup \dots \cup Y_6) \ : \ k < k'\}.$$

We now show that $Y'_1, \dots, Y'_6$ is a minor model of K_6 in B'_n.

Claim 1. *The sets $Y'_1, \dots, Y'_6$ are pairwise disjoint.*

Proof of Claim: Suppose there exists some $b_i \in Y'_k \cap Y'_{k'}$ with $k < k'$. From the definition of $Y_1, \dots, Y_6$ and $Y'_1, \dots, Y'_6$, it should be that $b_i \in X_k$ and $p_{j,i} \in X_{k'}$ for some $j \in [n]$ or that $b_i \in X_{k'}$ and $p_{j,i} \in X_k$ for some $j \in [n]$. But that would make X_k and $X_{k'}$ adjacent. ◇

To further show that the sets $Y'_1, \dots, Y'_6$ are connected and pairwise adjacent in B'_n, we need the following notion and claims. An *interval* I of some X_k is a subset of consecutive positive integers such that there is a connected component J of $G[X_k \cap V(P_j)]$ for some $j \in [n]$ such that $\{i \ : \ p_{j,i} \in V(J)\} = I$.

Claim 2. *For any $k \neq k' \in [6]$, any interval I of X_k, and any interval I' of $X_{k'}$, it cannot be that $I \subseteq I'$ (and symmetrically $I' \subseteq I$). Furthermore, at most one vertex of $\{b_i \ : \ i \in I \cap I'\}$ can be in a branch set of $\mathcal{M}$, namely $s_{k,k'}$.*

Proof of Claim: If $I \subseteq I'$, then X_k is a subpath of P_j for some $j \in [n]$, as otherwise X_k and $X_{k'}$ would be adjacent. But then X_k has at most two neighbors that are not neighbors of $X_{k'}$, a contradiction to realize the 4 branch sets adjacent to X_k but not to $X_{k'}$. The rest of the claim follows because $\{s_{k,k'}\}$ is the only branch set adjacent to both X_k and $X_{k'}$, and X_k and $X_{k'}$ are non-adjacent. ◇

We can extend a bit the previous claim.

Claim 3. *For any pairwise distinct $k, k', k'' \in [6]$, any interval I of X_k, any interval I' of $X_{k'}$, and any interval I'' of $X_{k''}$, it cannot be that $I \subseteq I' \cup I''$. Furthermore, if $s_{k',k''} = p_{j,i}$ for some $j \in [n]$, it cannot be that $I \subseteq I' \cup I'' \cup \{i\}$.*

Proof of Claim: Again, any such inclusion would imply that X_k is a subpath of some P_j. But then X_k has at most two neighbors that are not neighbors of $X_{k'} \cup X_{k''}(\cup\{s_{k',k''}\})$, a contradiction to realize the 3 branch sets adjacent to X_k but not to $X_{k'}$ nor $X_{k''}$. ◇

As $\mathcal{M}$ is minimal, Claims 2 and 3 imply in particular that there is at most one pair I, I' of intervals of $X_k, X_{k'}$ with $I \cap I' \neq \emptyset$, per $k \neq k' \in [6]$. As another direct consequence of Claims 2 and 3, we get the following.

Claim 4. *For any pairwise distinct $k, k', k'' \in [6]$, any interval I of X_k and any interval I' of $X_{k'}$ such that $I \cap I' \neq \emptyset$ and $\min(I) < \min(I')$, there is no $i \in [\min(I') - 1, \max(I) + 1]$ such that $p_{j,i} \in X_{k''}$ for some $j \in [n]$ (or $b_i \in X_{k''}$).*

The next two claims complete the proof.

Claim 5. *The sets $Y'_1, \dots, Y'_6$ are connected in B'_n.*

Proof of Claim: For any $k \in [6]$, and any $u, v \in Y'_k$, we exhibit a u–v path P in B'_n such that $V(P) \subseteq Y'_k$. (As we do not need to show that P is a path, we call it so, but only argue that it is a walk, which is sufficient.) Let $u' \in V(G)$ (resp. $v' \in V(G)$) be the vertex in $(V(G) \setminus V(B'_n)) \cap X_k$ causing that $u \in Y'_k$ (resp $v \in Y'_k$) if this applies, or $u' := u$ (resp. $v' := v$), otherwise. Let P' be a u'–v' path in G such that $V(P) \setminus \{u', v'\} \subseteq X_k$. Observe that u' and v' may be equal to some $s_{k,k'}$ with $k < k'$, and thus not be in X_k themselves. In which case, we simply run the following arguments with their neighbors in P' (which are in X_k). Hence, we may as well suppose that $u', v' \in X_k$.

If P' is a subpath of some P_j, we have $u' = p_{j,i}$ and $v' = p_{j,i'}$, no $X_{k'}$ with $k' < k$ contains some vertex $p_{j',i}$ or $p_{j',i'}$, and no other $X_{k'}$ contains $b_{i''}$ for any i'' between i and i'. By Claim 2, it means that for any integer i'' between i and i', no $X_{k'}$ with $k' < k$ contains some vertex $p_{j',i''}$, and no other $X_{k'}$ contains $b_{i''}$. In particular, all such vertices $b_{i''}$ are in Y'_k, and this makes the path P between u and v.

More generally, the path P' alternates between maximal subpaths contained in $V(G) \setminus V(B_n)$ and maximal subpaths contained in $V(B_n)$. The latter are kept to build P. We then mimic each maximal subpath contained in $V(G) \setminus V(B_n)$ with a path of B'_n included in Y'_k, with the appropriate endpoints. By Claim 2, in P', every maximal subpath $p_{j,i} \dots p_{j,i'}$ in $V(G) \setminus V(B_n)$ surrounded by two subpaths in $V(B_n)$ is such that the corresponding vertices $b_i \dots b_{i'}$ are all in Y'_k, hence form the desired subpath of P in B'_n.

We finally move to the case when P' starts with a subpath $u' = p_{j,i} \dots p_{j,i'} \neq v'$ maximal in $V(G) \setminus V(B_n)$; the case when P' ends with such a maximal subpath is dealt with symmetrically. We know that $b_{i'} \in X_k$, no $X_{k'}$ with $k' < k$ contains some vertex $p_{j',i}$, and no other $X_{k'}$ contains some vertex $b_{i''}$ where i'' is between i and i'. Thus by Claim 2, all the vertices $b_i \dots b_{i'}$ are in Y'_k, the desired subpath of P in B'_n. ◇

Claim 6. *The sets $Y'_1, \dots, Y'_6$ are pairwise adjacent in B'_n.*

Proof of Claim: For any $k \neq k' \in [6]$, let $u \in X_k, u' \in X_{k'}$ be such that $us_{k,k'}, u's_{k,k'} \in E(G)$.

Assume first that $s_{k,k'} = b_i$ for some $i \in [|V(B'_n)|]$. If at most one $\ell \in \{k, k'\}$ (thus, at most one $\ell \in [6]$) is such that $p_{j,i} \in X_\ell$ for some $j \in [n]$, then either $s_{k,k'} \in Y'_k$ and $u' \in V(B'_n)$, or $s_{k,k'} \in Y'_{k'}$ and $u \in V(B'_n)$; so Y'_k and $Y'_{k'}$ are adjacent in B'_n. If, instead, there are j, j' such that $p_{j,i} \in X_k$ and $p_{j',i} \in X_{k'}$, consider the intervals I, I' of $X_k, X_{k'}$ associated to $p_{j,i}, p_{j',i}$. Claim 4 implies that there is some i' such that $b_{i'} \in Y'_k$ and $b_{i'+1} \in Y'_{k'}$; so, again, Y'_k and $Y'_{k'}$ are adjacent in B'_n.

We next assume that $s_{k,k'} \in V(G) \setminus V(B'_n)$.

First consider the case both u and u' are also in $V(G) \setminus V(B'_n)$. Let I, I' be their associated interval, and assume without loss of generality that $\max(I) < \min(I')$. By the second item of the conditions satisfied by $\mathcal{M}$, $\min(I') - \max(I) = 1$. By Claim 3, there is no $k'' \in [6] \setminus \{k, k'\}$ such that $X_{k''}$ contains some vertex $p_{j,i}$ or b_i with $i \in [\max(I), \min(I')]$.

Besides, X_k (resp. $X_{k'}$) contains no vertex $p_{j,\min(I')}$ nor $b_{\min(I')}$ (resp. $p_{j,\max(I)}$ nor $b_{\max(I)}$). Therefore, $b_{\max(I)} \in Y'_k$ and $b_{\min(I')} = b_{\max(I)+1} \in Y'_{k'}$, thus Y'_k and $Y'_{k'}$ are adjacent in B'_n.

Finally consider, without loss of generality, that $s_{k,k'} = p_{j,i}$, $p_{j,i-1} \in X_k$, and $b_i \in X_{k'}$. By Claim 2, there is no $\ell \in [6] \setminus \{k\}$ such that X_ℓ contains some vertex $p_{j',i-1}$ nor b_{i-1}. Thus $b_{i-1} \in Y'_k$. As $b_i \in Y'_{k'}$, we have that Y'_k and $Y'_{k'}$ are adjacent. ◇

Claims 1, 5 and 6 imply that $Y'_1, \dots, Y'_6$ is a K_6 minor model in B'_n; a contradiction. □

Lemma 8. *For every K_t-minor-free graph H, G is not a region intersection graph over H.*

Proof. Suppose, for contradiction, that there is a K_t-minor-free graph H for which $G \in \mathrm{RIG}(H)$. Let $\mathcal{R} = (R_v \subseteq H : v \in V(G))$ be a collection of connected subgraphs of H such that $uv \in E(G)$ if and only if $V(R_u) \cap V(R_v) \neq \emptyset$. By Theorem 3, H has a tree-decomposition $\mathcal{T} = (W_x : x \in V(T))$ of adhesion at most ℓ where each torso is ℓ-almost-embeddable.

We claim that there is an $x \in V(T)$ such that the bag W_x intersects $V(R_v)$ for each vertex $v \in V(B_n)$. For each vertex $v \in V(B_n)$, the set $\{x \in V(T) : V(R_v) \cap W_x \neq \emptyset\}$ is a subtree of T. By the Helly property for subtrees, it suffices to show that any two such subtrees meet.

Assume, for contradiction, that there exist $u, v \in V(B_n)$ such that $V(R_u)$ and $V(R_v)$ do not intersect a common bag. Since $\mathcal{T}$ has adhesion at most ℓ, there is a set $S \subseteq V(H)$ with $|S| \leqslant \ell$ whose deletion separates $V(R_u)$ and $V(R_v)$. By construction, G contains n u–v paths $uQ_1v, \dots, uQ_nv$ with $Q_i \subseteq P_i$. So, for each $i \in [n]$, the connected subgraph $Q_i^\star = \bigcup(R_p : p \in V(Q_i))$ of H connects R_u to R_v, hence meets S. Since $Q_1, \dots, Q_n$ are pairwise anti-complete, the subgraphs $Q_1^\star, \dots, Q_n^\star$ are pairwise vertex-disjoint, forcing $|S| \geqslant n \geqslant \ell+1$, a contradiction.

Therefore, there is a bag W_x in $\mathcal{T}$ intersecting all regions R_v for $v \in V(B_n)$. Since every adhesion set is a clique in a torso, $V(R_v) \cap W_x$ induces a connected subgraph R'_v in $H\langle W_x \rangle$ for every $v \in V(B_n)$. However, there may be an edge $uv \in E(B_n)$ for which $V(R'_u) \cap V(R'_v) = \emptyset$. Nevertheless, since $V(R_u) \cap V(R_v) \neq \emptyset$, there is an adhesion set $S = W_x \cap W_y$ (for some edge $xy \in E(T)$) such that $V(R'_u) \cap S \neq \emptyset$ and $V(R'_v) \cap S \neq \emptyset$. Choose vertices $a \in V(R'_u) \cap S$ and $b \in V(R'_v) \cap S$. Then $ab \in E(H\langle W_x \rangle)$. Add a vertex w to $H\langle W_x \rangle$ adjacent to both a and b then include w in the connected subgraphs R'_u and R'_v. Repeating this procedure for every such edge produces a supergraph H' of $H\langle W_x \rangle$ built by performing 2-clique-sums with triangles together with a collection $(R'_v \subseteq H' : v \in V(B_n))$ of connected subgraphs in H' that realizes B_n as a region intersection graph over H'.

Let $Z \subseteq W_x$ be the set of apex vertices in $H\langle W_x \rangle$. Since $|Z| \leqslant \ell$ and B_n consists of $n \geqslant \ell + 1$ anti-complete copies of $A_n^{(1)}$, there exists a copy of $A_n^{(1)}$, denoted as $\widetilde{A}_n^{(1)}$, for which $\bigcup(V(R_x) : x \in V(\widetilde{A}_n^{(1)})) \cap Z = \emptyset$. Let $\widetilde{H}$ be the subgraph of H' induced by $\bigcup(V(R_x) : x \in V(\widetilde{A}_n^{(1)}))$. Then $V(\widetilde{H}) \cap Z = \emptyset$. As such, $\widetilde{H}$ has a tree-decomposition with adhesion at most 2 where one torso is an apex-free ℓ-almost embeddable graph and the other torsos are triangles. By Theorem 4, $\widetilde{H}$ is A_n-minor-free. However, since $\widetilde{A}_n^{(1)} \in \mathrm{RIG}(\widetilde{H})$ and $\widetilde{A}_n^{(1)}$ is isomorphic to $A_n^{(1)}$, Lemma 1 implies A_n is a minor of $\widetilde{H}$, giving us the desired contradiction. □

We now explain how to modify the construction in Theorem 5 to force the girth to be arbitrarily large. Fix positive integer g. Define $B_{g,n}$ to be $nA_n^{(g)}$, that is the disjoint union of n copies of the g-subdivision of A_n. Then $B_{g,n}$ has girth $3(g+1)$. We define a total order $\prec$ of $V(B_{g,n})$ by using the same strategy of that given by Figure 2. Similar to before, we add to $B_{g,n}$ the disjoint union of n paths $P_1, \ldots, P_n$ of length $g|V(B_{g,n})| - 1$ and make, for every $i \in [|V(B_{g,n})|]$ and $j \in [n]$, the $(gi-1)$-st vertex of P_j adjacent to the i-th vertex of $B_{g,n}$ along $\prec$. Call the resulting graph $G_{g,n}$. Since $G_{g,n} - B_{g,n}$ is a disjoint union of paths, it does not contain any cycle. Any cycle going through $V(G_{g,n}) \setminus V(B_{g,n})$ has at least $g-1$ consecutive edges within $G_{g,n} - V(B_{g,n})$. Since no pair of vertices within the same connected component of $V(G_{g,n}) \setminus V(B_{g,n})$ shares a neighbor in $V(B_{g,n})$, we conclude that every cycle in $G_{g,n}$ has length at least g. Since Lemmas 7 and 8 also generalize to $G_{g,n}$, this completes the proof of Theorem 2.

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References

[1] N. Alon, P. D. Seymour, and R. Thomas. A separator theorem for graphs with an excluded minor and its applications. In H. Ortiz, editor, *Proceedings of the 22nd Annual ACM Symposium on Theory of Computing, May 13-17, 1990, Baltimore, Maryland, USA*, pages 293–299. ACM, 1990.

[2] M. Bonamy, É. Bonnet, H. Déprés, L. Esperet, C. Geniet, C. Hilaire, S. Thomassé, and A. Wesolek. Sparse graphs with bounded induced cycle packing number have logarithmic treewidth. *Journal of Combinatorial Theory, Series B*, 167:215–249, 2024.

[3] É. Bonnet, J. Hodor, T. Korhonen, and T. Masařík. Treewidth is polynomial in maximum degree on weakly sparse graphs excluding a planar induced minor. *CoRR*, abs/2312.07962, 2023.

[4] R. Campbell, J. Davies, M. Distel, B. Frederickson, J. P. Gollin, K. Hendrey, R. Hickingbotham, S. Wiederrecht, D. R. Wood, and L. Yepremyan. Treewidth, Hadwiger number, and induced minors, 2024.

[5] C. Dallard, M. Milanic, and K. Storgel. Treewidth versus clique number. III. Tree-independence number of graphs with a forbidden structure. *J. Comb. Theory B*, 167:338–391, 2024.

[6] J. Davies. Oberwolfach report 1/2022. 2022.

[7] J. Davies. Personal communication, 2024.

[8] V. Dujmović, P. Morin, and D. R. Wood. Layered separators in minor-closed graph classes with applications. *J. Combin. Theory Ser. B*, 127:111–147, 2017.

[9] P. Gartland. *Quasi-Polynomial Time Techniques for Independent Set and Beyond in Hereditary Graph Classes*. PhD thesis, UC Santa Barbara, 2023.

[10] A. Georgakopoulos and P. Papasoglu. Graph minors and metric spaces. *CoRR*, 2023.

[11] T. Korhonen and D. Lokshtanov. Induced-minor-free graphs: Separator theorem, subexponential algorithms, and improved hardness of recognition. In D. P. Woodruff, editor, *Proceedings of the 2024 ACM-SIAM Symposium on Discrete Algorithms, SODA 2024, Alexandria, VA, USA, January 7-10, 2024*, pages 5249–5275. SIAM, 2024.

[12] J. R. Lee. Separators in region intersection graphs. In C. H. Papadimitriou, editor, *Proc. 8th Innovations in Theoretical Computer Science Conference*, volume 67 of *LIPIcs*, pages 1:1–1:8. Schloss Dagstuhl, 2017.

[13] R. J. Lipton and R. E. Tarjan. A separator theorem for planar graphs. *SIAM Journal on Applied Mathematics*, 36(2):177–189, 1979.

[14] D. Lokshtanov. Personal communication, 2025.

[15] R. McCarty. Structurally sparse graphs. Lecture series at the Structural Graph Theory Bootcamp, Warsaw, https://sites.google.com/view/strug/main, 2023.

[16] A. C. Pohoata. *Unavoidable induced subgraphs of large graphs*. Senior theses, Department of Mathematics, Princeton University, 2014.

[17] N. Robertson and P. D. Seymour. Graph Minors I–XXIII. *J. Combin. Theory Ser. B (and Journal of Algorithms)*, 1983–2012.

[18] N. Robertson and P. D. Seymour. Graph minors. XVI. Excluding a non-planar graph. *J. Combin. Theory Ser. B*, 89(1):43–76, 2003.

[19] F. W. Sinden. Topology of thin film RC circuits. *Bell System Technical Journal*, 45(9):1639–1662, 1966.

[20] S. Wiederrecht. Graph searching in RIGs. In *Open Problems, GRASTA 2023 Workshop*, Bertinoro, Italy, 2023. Section 1.3, Problem 3: https://www-sop.inria.fr/teams/coati/events/grasta2023/slides/Grasta23_OpenProblems.pdf.