Dual parameterization and parameterized approximability of subset graph problems

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Abstract

We discuss approximability in FPT-time for the class of *subset optimization graph problems* where a feasible solution S is a subset of the vertex set of the input graph. This class encompasses many well-known problems, such as MIN DOMINATING SET, MIN VERTEX COVER, MAX INDEPENDENT SET, MIN FEEDBACK VERTEX SET. We study approximability of such problems with respect to the dual parameter $n - k$ where n is size of the vertex set and k the standard parameter. We show that under such parameterization, many of these problems, while $\mathbf{W}[\cdot]$-hard, admit parameterized approximation schemata.

1 Preliminaries

We say that a minimization (resp., maximization) problem Π, together with a parameter k, is *parameterized r-approximable*, if there exists an FPT-time algorithm which computes a solution of size at most (resp., at least) rk whenever the input instance has a solution of size at most (resp., at least) k, otherwise, it outputs an arbitrary solution. This line of research was initiated by three independent works [13, 8, 10]. For a very interesting overview of older results, see [18].

Here, we handle approximability and inapproximability in FPT-time of a broad class of **NPO** graph problems (see [2] for a formal definition of an **NPO** problem) that we call *subset problems*. They can be defined as follows (for a larger definition of subset problems that goes beyond graphs, one can be referred to [21]).

Definition 1. *Consider an* ***NPO*** *graph problem Π defined on a graph $G = (V, E)$. Then Π is a subset problem, if any feasible solution S for G is a subset of V and if the objective value of S is equal to $|S|$.*

Let us note that the notion of a subset problem can be extended to also capture problems that are not defined by means of graphs. For optimum satisfiability, considering that a feasible solution is a subset of satisfied clauses, maximum or minimum satisfiability can be considered as subset problems. For problems defined on set systems, such as MIN SET COVER or MAX SET PACKING, we can consider that the "reference" set of data is the family $\mathcal{S}$ of subsets of the ground set U (also called universe), and feasible solutions are subfamilies of $\mathcal{S}$.

The class of subset graph problems includes some of the most popular combinatorial problems such as MIN DOMINATING SET, MIN VERTEX COVER, MAX INDEPENDENT SET, MIN INDEPENDENT DOMINATING SET, MIN FEEDBACK VERTEX SET, several fixed cardinality problems (as, for example, MAX and MIN k-VERTEX COVER), etc.

The goal of the paper is to handle *dual parameterization* (defined just below) and parameterized approximation of subset problems. For simplicity, we restrict ourselves in graph problems but our results also apply for satisfiability or set system problems (following the assumptions sketched just above).

Subset graph problems can be optimally solved in time $O^*(2^n)$ (where n is the order of the input graph) by exhausting all the vertex subsets and outputting an optimum feasible subset. Notation $O^*(\cdot)$ suppresses polynomial factors in the size of the input, that is, for any constant c, $n^c f(n)$ is in $O^*(f(n))$.

Given an instance G of a subset problem Π, denoting by n the order of G and by k the standard parameter of Π (i.e., the cardinality of its optimal solution), we call *dual parameter* for Π the parameter $n-k$. Accordingly, we call *dual parametrization of* Π*, denoted by D-*Π, the subset problem that has the same constraints than Π and standard parameter (cardinality of optimal solution) $k_D = n-k$. For example, if Π stands for MIN VERTEX COVER, D-MIN VERTEX COVER is MAX INDEPENDENT SET.

Besides the fact that dual parameterization is interesting per se, for numerous problems has also a natural interpretation, it is even a structural parameter of the instance. For instance:

- for MAX INDEPENDENT SET, the dual parameter $n-\alpha(G)$, where $\alpha(G)$ denotes the stability number of G, is the size of a minimum vertex cover.
- for MIN SET COVER, the dual parameter represents the number of unused sets;
- for MIN FEEDBACK VERTEX SET, the dual parameter is the size of a maximum induced forest;
- for MIN VERTEX COLORING in a graph of order n, the dual parameter $n-\chi$ is the number of unused colors, i.e., the number of available colors not used by the coloring.

Parameterization by the dual parameter has been studied for many classical (and not only subset) problems (see, for example, [7, 12]) and for a lot of them it has been proved that although hard with respect to the standard parameter, they become easy when parameterized by the dual parameter. The most known such problem is the famous MAX INDEPENDENT SET problem when parameterized by the size of a minimum vertex cover (this is a folklore result). This is also the case of MIN VERTEX COLORING problem (which is *not* a subset problem) which, although not in **XP** (unless **P** $\neq$ **NP**) when parameterized by the chromatic number χ, is in **FPT** when parameterized by $n-\chi$ [9, 12].

In Section 2, we show that for another subset graph-problem, the MIN INDEPENDENT DOMINATING SET, its dual parameterization is also in **FPT**. On the contrary, MIN SET COVER parameterized by $n-k$ (where $n = |\mathcal{S}|$) is **W[1]**-hard.

2 Some complexity results for subset problems

As it is done in [6], we first observe that the MIN INDEPENDENT DOMINATING SET problem, while being known to be **W[2]**-hard for the standard parameter, is FPT with respect to the dual parameter.

Proposition 1. MIN INDEPENDENT DOMINATING SET *parameterized by* $n-k$ *is* ***FPT****.*

Proof. For MIN INDEPENDENT DOMINATING SET, $n-k$ is the size of a maximum minimal vertex cover which is at least equal to a minimum vertex cover which is bigger than the treewidth of the input graph [5]. The fact that MIN INDEPENDENT DOMINATING SET parameterized by treewidth is in **FPT** [1], implies that MIN INDEPENDENT DOMINATING SET parameterized by $n-k$ is **FPT**. □

Proposition 2. MIN SET COVER *is* ***W[1]****-hard when parameterized by* $n-k$ *and* ***W[2]****-complete when parameterized by* k*.*

Proof. On the one hand, MIN SET COVER is **W[2]**-complete when parameterized by the standard parameter [12]. On the other hand, when parameterized by the dual parameter, MIN SET COVER is **W[1]**-hard as being a generalization of MAX INDEPENDENT SET. Indeed, MIN VERTEX COVER can be seen as a restriction of MIN SET COVER, where elements are edges, and there is one set per vertex of the graph, which contains all the *edges* incident to this vertex. Besides, the complement of a vertex cover is an independent set. Therefore, this restricted version of set cover is equivalent to finding an independent set of size $n-k$ parameterized by $n-k$ (one may rename $n-k$ by k'), which is known to be **W[1]**-hard. □

Similar negative results hold also for dual parameterizations of several well-known subset graph problems. For instance:

- the dual parameterization of MIN VERTEX COVER is the size of a maximum independent set and MIN VERTEX COVER parameterized by $\alpha(G)$ is **W[1]**-complete [6];
- MIN FEEDBACK VERTEX SET is **W[1]**-complete when parameterized by the dual parameter [9, 14, 17] (the dual parameterization of MIN FEEDBACK VERTEX SET is called MAX VERTEX INDUCED FOREST).

3 Using polynomial approximation to design FPT approximation schemata for W[·]-hard subset problems

For a subset problem Π, D-Π will denote in the sequel the dual parameterization of Π. In the following theorem, we give a sufficient condition under which, given a subset problem Π, problem D-Π admits an approximation schema running in FPT-time with respect to the standard parameter of Π. Informally, the approximation ratio is computed with the dual parameter but the FPT running time is function of the standard parameter.

Theorem 1. *Consider a subset problem Π with standard parameter k and set $k_D = n-k$, where n is the size of an input of Π. Then:*

1. *if Π is a minimization problem and is approximable in polynomial time within ratio at most $(\log n)^{g(k)}$, for some function g, problem D-Π parameterized by k admits a parameterized approximation schema;*
2. *if Π is a maximization problem, problem D-Π parameterized by k admits a parameterized approximation schema.*

Proof. In order to prove Item 1, consider a problem Π satisfying the conditions of the item, its dual version D-Π, a ρ-approximation algorithm A for Π and denote by k' the cardinality of the solution returned by A. The complement of this solution is a solution of size $k'_D = n - k'$ for D-Π, while the size of the optimum is $k_D = n - k$. Thus, the approximation ratio guaranteed for D-Π is (recall that D-Π is a maximization problem):

$$\frac{k'_D}{k_D} = \frac{n-k'}{n-k} \geqslant \frac{n-\rho k}{n-k} \tag{1}$$

Fix some constant $\epsilon > 0$. Then, in order that the last fraction in (1) is greater than $1-\epsilon$, it must hold that:

$$n \geqslant \left(\frac{\rho - 1 + \epsilon}{\epsilon}\right) k \tag{2}$$

If n does not satisfy (2), then the simple $O^*(n^k)$-time algorithm that builds all the subsets of size up to k and chooses the one which constitutes the best solution, runs in FPT time as long as $\rho \leqslant \log^{g(k)} n$. Indeed, in this case, $n \leqslant ((\rho-1+\epsilon)/\epsilon)k \leqslant (((\log n)^{g(k)}-1+\epsilon)/\epsilon)k$. Thus, the $O^*(n^k)$ trivial algorithm for Π is also $O^*((k/\epsilon)^k(\log n)^{kg(k)})$ which is FPT. Indeed, for any function h, $O((\log n)^{h(k)}))$ is $O(F(k)p(n))$, for some function F and polynomial p [19].

Proof of Item 2 is similar. Now, D-Π is a minimization problem. With the same notation as in the proof of Item 1, we distinguish the following two cases.

1. $n \geqslant ((1+\epsilon)k-k')/\epsilon$. This implies:
$$\frac{k'_D}{k_D} = \frac{n-k'}{n-k} \leqslant 1+\epsilon$$
so A yields a $1+\epsilon$-approximation (recall that the objective value, i.e., the value of an optimal solution, is the dual parameter).
2. $n < ((1+\epsilon)k-k')/\epsilon \leqslant (1+\epsilon)k/\epsilon$. In this case, the exhaustive search in $O^*(2^n)$ is also $O^*(2^{(1+\epsilon)k/\epsilon})$, hence FPT.

This completes the proof of the theorem. □

Note that, for minimization problems approximable in polynomial time within ratios that are functions of k (i.e., of the optimum), Item 1 of Theorem 1 applies without any restriction on the values of these ratios. On the other hand, for maximization problems (Item 2), such a case implies the existence of parameterized approximation schemata that are subexponential in k.

Observe that MIN INDEPENDENT DOMINATING SET does not meet the conditions of Theorem 1. Indeed, the best known polynomial time achievable approximation ratio for MIN INDEPENDENT DOMINATING SET is $\Delta + 1$, where Δ is the maximum degree of the input graph and can be aribatrily larger than $O(\log n)$, and it is inapproximable within $\Delta^{1-\epsilon}$, for any $\epsilon > 0$ in polynomial time [15].

The scope of Theorem 1 encompasses more problems than subset problems, for instance, coloring problems. Of course, the classical MIN VERTEX COLORING problem does not meet the conditions of Item 1, since it is inapproximable in polynomial time within better than $n^{1-\epsilon}$, for any $\epsilon > 0$ [22]. Moreover, as it is proved in [3], D-MIN VERTEX COLORING is **APX**-hard. So, a parameterized approximation schema by means of Theorem 1 is impossible for MIN VERTEX COLORING. Consider two edge-coloring problems: the classical MIN EDGE COLORING and the MAX EDGE COLORING[1] problem. The former is polynomially approximable within ratio $4/3$ [20] and the latter one within ratio 2 [16]. Thus, by Theorem 1, both problems admit FPT approximation schemata when parameterized by the dual parameter $m - k$ (where m denotes the size of the edge set of the input graph).

We now give two corollaries of Theorem 1.

Corollary 1. *D-MIN SET COVER, D-MIN DOMINATING SET, D-MIN FEEDBACK VERTEX SET, D-MIN EDGE COLORING, D-MAX EDGE COLORING parameterized by k, admit parameterized approximation schemata.*

Recall that as it has been mentioned in Section 1, even MAX SAT and MIN SAT can be seen as subset problems. Moreover, MIN SAT is polynomially approximable within constant ratio [4]. Thus, the following holds.

Corollary 2. *D-MAX SAT and D-MIN SAT parameterized by k are approximable by parameterized approximation schemata.*

The results of Corollaries 1 and 2 offer interesting insights on the possible relations between classical polynomial approximation and parameterized approximation that deserve further investigation.

4 Some words about "differential" parameterization

Another interesting parameter, not systematically studied yet, is the *differential parameter*. It can be defined by $\omega - k$, where ω is the worst-case solution value of an instance [11]. Informally, given an instance I of a combinatorial problem Π, $\omega(I)$ is the optimal value of a problem Π' defined on the same set of instances and having the same feasibility constraints as Π, but Π' has the opposite goal.

Although for some minimization subset problems, differential and dual parameters coincide (MIN VERTEX COVER, MIN SET COVER, MIN DOMINATING SET, etc., are such problems), this is not the case for any problem. For MIN INDEPENDENT DOMINATING SET, for example, the value of a worst solution on an instance I is the size of a maximum independent set (that is the largest of the independent dominating sets in I). Thus, MIN INDEPENDENT DOMINATING SET, while in **FPT** when parameterized by the dual parameter (Proposition 1), becomes **W[1]**-hard when parameterized by the differential parameter [6].

On the other hand, for many maximization subset problems as MAX INDEPENDENT SET, MAX CLIQUE, KNAPSACK, etc., the wost solution (of value 0) is the empty set. There, the differential parameter coincides with the standard one.

[1]Given an edge-weighted graph G, the weight of a color M (that is a matching of G) is defined as the weight of the "heaviest" edge of M and the objective is to determine a partition of the edges of G into matchings, minimizing the sum of their weights.

In MAX MINIMAL VERTEX COVER, the worst solution value is the size $\tau(G)$ of a minimum vertex cover. Thus, denoting by k its standard parameter, its differential parameter is $k - \tau(G)$. The following easy proposition holds for this problem.

Proposition 3. MAX MINIMAL VERTEX COVER *parameterized by the differential parameter is **W[1]**-hard.*

Proof. Given a graph G, let $\iota(G)$ be the size of a minimum independent dominating set and k the size of a maximum minimal vertex cover. As $k + \iota(G) = n = \alpha(G) + \tau(G)$, it holds that $k - \tau(G) = \alpha(G) - \iota(G)$. Recall that MIN INDEPENDENT DOMINATING SET parameterized by $\alpha(G) - \iota(G)$ is **W[1]**-hard [6]; this implies that it is so when parameterized by $k - \tau(G)$. If MAX MINIMAL VERTEX COVER was in **FPT** when parameterized by $k - \tau(G)$, one by taking the complement of the solution, would be able to determine a minimum independent dominating set in time parameterized by $k - \tau(G) = \alpha(G) - \iota(G)$, a contradiction (unless $\mathbf{FPT} = \mathbf{W[1]}$) since MIN INDEPENDENT DOMINATING SET parameterized by $\alpha(G) - \iota(G)$ is **W[1]**-hard. □

A more systematic study of the complexity of exactly or approximately solving problems parameterized by the differential parameter (when such parameterization makes sense) seems to us an interesting direction of future research.

5 Final remarks

We studied here parameterized approximability of subset problems, which constitute a very natural and popular class of combinatorial problems. We have sketched a systematic approach for approximating subset problems when parameterized by the dual parameter, i.e., parameter "size of the instance minus standard parameter". We showed that such parameterization is able to produce non-trivial parameterized approximation results that, in many cases, can also fit another polynomial-time approximation paradigm: the differential approximation. Studying parameterized approximability of problems with respect to any parameter for which they are known to be hard is very important and adds deeper insights on the parameterized intractability of the world of combinatorial problems.

Finally, a particularly interesting problem is the existence of a dual prameterized approximation schema for MIN VERTEX COLORING. As we have mentioned in Section 3 (above Corollary 1), such a schema is not achievable by application of Theorem 1. Is such schema achievable by ad hoc methods? Can we prove that, under some credible complexity hypothesis, such a result is impossible. This, to our opinion, is a rather difficult open question that deserves further research.

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References

[1] J. Alber and R. Niedermeier. Improved tree decomposition based algorithms for domination-like problems. In S. Rajsbaum, editor, *Proc. atin American Symposium on Theoretical Informatics, LATIN'02*, volume 2286 of *Lecture Notes in Computer Science*, pages 613–628. Springer-Verlag, 2002.

[2] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, and M. Protasi. *Complexity and approximation. Combinatorial optimization problems and their approximability properties.* Springer-Verlag, Berlin, 1999.

[3] C. Bazgan, B. Escoffier, and V. Th. Paschos. Completeness in standard and differential approximation classes: Poly-(D)APX- and (D)PTAS-completeness. *Theoret. Comput. Sci.*, 339:272–292, 2005.

[4] D. Bertsimas, C.-P. Teo, and R. Vohra. On dependent randomized rounding algorithms. In *Proc. International Conference on Integer Programming and Combinatorial Optimization,*

IPCO'96, volume 1084 of *Lecture Notes in Computer Science*, pages 330–344. Springer-Verlag, 1996.

[5] H. L. Bodlaender, B. M. P. Jansen, and S. Kratsch. Preprocessing for treewidth: a combinatorial analysis through kernelization. *CoRR*, abs/1104.4217, 2011.

[6] N. Bourgeois, K. Dabrowski, M. Demange, and V. Th. Paschos. Playing with parameters: structural parameterization in graphs. *CoRR*, abs/1309.6144, 2013.

[7] L. Cai. Parameter complexity of cardinality constrained optimization problems. *The Computer Journal*, 51:102–121, 2008.

[8] L. Cai and X. Huang. Fixed-parameter approximation: conceptual framework and approximability results. In H. L. Bodlaender and M. A. Langston, editors, *Proc. International Workshop on Parameterized and Exact Computation, IWPEC'06*, volume 4169 of *Lecture Notes in Computer Science*, pages 96–108. Springer-Verlag, 2006.

[9] M. Cesati. Compendium of parameterized problems. `http://cesati.sprg.uniroma2.it/research/compendium/`.

[10] Y. Chen, M. Grohe, and M. Grüber. On parameterized approximability. In H. L. Bodlaender and M. A. Langston, editors, *Proc. International Workshop on Parameterized and Exact Computation, IWPEC'06*, volume 4169 of *Lecture Notes in Computer Science*, pages 109–120. Springer-Verlag, 2006.

[11] M. Demange and V. Th. Paschos. On an approximation measure founded on the links between optimization and polynomial approximation theory. *Theoret. Comput. Sci.*, 158:117–141, 1996.

[12] R. G. Downey and M. R. Fellows. *Parameterized complexity.* Monographs in Computer Science. Springer, New York, 1999.

[13] R. G. Downey, M. R. Fellows, and C. McCartin. Parameterized approximation problems. In H. L. Bodlaender and M. A. Langston, editors, *Proc. International Workshop on Parameterized and Exact Computation, IWPEC'06*, volume 4169 of *Lecture Notes in Computer Science*, pages 121–129. Springer-Verlag, 2006.

[14] H. Fernau. Parameterized algorithms: a graph-theoretic approach. Habilitationsschrift. Universität Tübingen, 2005.

[15] M. M. Halldórsson. Approximating the minimum maximal independence number. *Inform. Process. Lett.*, 46:169–172, 1993.

[16] A. Kesselman and K. Kogan. Nonpreemptive scheduling of optical switches. *IEEE Transactions on Communications*, 55:1212–1219, 2007.

[17] S. Khot and V. Raman. Parameterized complexity of finding subgraphs with hereditary properties. *Theoret. Comput. Sci.*, 289(2):997–1008, 2002.

[18] D. Marx. Parameterized complexity and approximation algorithms. *The Computer Journal*, 51(1):60–78, 2008.

[19] C. Sloper and J. A. Telle. An overview of techniques for designing parameterized algorithms. *The Comput. J.*, 51(1):122–136, 2008.

[20] V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz.*, 3:25–30, 1964.

[21] G. J. Woeginger. Exact algorithms for NP-hard problems: a survey. In M. Juenger, G. Reinelt, and G. Rinaldi, editors, *Combinatorial Optimization - Eureka! You shrink!*, volume 2570 of *Lecture Notes in Computer Science*, pages 185–207. Springer-Verlag, 2003.

[22] D. Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. In *Proc. STOC'06*, pages 681–690, 2006.