TEMPORAL VALUED CONSTRAINT SATISFACTION PROBLEMS

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Abstract. We study the complexity of the *valued* constraint satisfaction problem (VCSP) for every valued structure with the domain $\mathbb{Q}$ that is preserved by all order-preserving bijections. Such VCSPs will be called *temporal*, in analogy to the (classical) constraint satisfaction problem: a relational structure is preserved by all order-preserving bijections if and only if all its relations have a first-order definition in $(\mathbb{Q};<)$, and the CSPs for such structures are called *temporal* CSPs. Many optimization problems that have been studied intensively in the literature can be phrased as a temporal VCSP. We prove that a temporal VCSP is in P, or NP-complete. Our analysis uses the concept of *fractional polymorphisms*; this is the first dichotomy result for VCSPs over infinite domains which is complete in the sense that it treats *all* valued structures with a given automorphism group.

1. Introduction

Valued constraint satisfaction problems (VCSPs) form a large class of computational optimization problems. A VCSP is parameterized by a *valued structure* (sometimes called the *template*), which consists of a *domain* D and cost functions, each defined on D^k for some k. The input to the VCSP consists of a finite set of variables, a finite sum of cost functions applied to these variables, and a threshold u, and the task is to find an assignment to the variables so that the sum of the costs is at most u. The computational complexity of such problems has been studied depending on the valued structure that parameterizes the problem. VCSPs generalize constraint satisfaction problems (CSPs), which can be viewed as a variant of VCSPs with costs from the set $\{0,\infty\}$: every constraint is either satisfied or surpasses every finite threshold. VCSPs also generalize min-CSPs, which are the natural variant of CSPs where, instead of asking whether all constraints can be satisfied at once, we search for an assignment that minimizes the number of unsatisfied constraints. Such problems can be modeled as VCSPs with costs from the set $\{0,1\}$.

A major achievement of the field is that if the domain of the valued structure $\mathfrak{A}$ is finite, then the computational complexity of VCSP($\mathfrak{A}$) is in P, or NP-complete. This result has an interesting history. The classification task was first considered in [21] with important first results that indicated that we might expect a good systematic theory for such VCSPs. A milestone was reached by Thapper and Živný with the proof of a complexity dichotomy for the case where the cost functions never take value ∞ [38]. On the hardness side, Kozik and Ochremiak [29] formulated a condition that implies hardness for VCSP($\mathfrak{A}$) and found equivalent characterisations that suggested that this condition characterises NP-hardness (unless P=NP, of course). Kolmogorov, Krokhin, and Rolínek [28] then showed that if the hardness condition from [29] does not apply, linear programming relaxation in combination with algorithms for classical CSPs can be used to solve VCSP($\mathfrak{A}$), conditional on the tractability conjecture for (classical) CSPs. Finally, this conjecture about CSPs has been confirmed [41] (the result has been announced independently by Bulatov [17] and by Zhuk [42]), thus

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completing the complexity dichotomy for VCSP($\mathfrak{A}$) for finite-domain templates $\mathfrak{A}$ as well. A key tool for distinguishing the tractable VCSPs from the NP-hard ones are *fractional polymorphisms*, which, in some sense, capture the symmetries of the VCSP template.

Many important optimization problems in the literature cannot be modeled as VCSPs if we restrict to valued structures on a finite domain; VCSPs that require an infinite domain are, for example, the min-correlation-clustering problem with partial information [1, 39], ordering min-CSPs [27], phylogeny min-CSPs [19], VCSPs with semilinear constraints [10], and the class of resilience problems from database theory [15, 24, 25, 30].

For VCSPs with infinite templates we cannot hope for general classification results, since this is already out of reach for the special case of CSPs over infinite domains [6]. However, a powerful algebraic machinery was developed to study CSPs of structures with a rich automorphism group, which has led to classification results for many concrete automorphism groups: we list [4,9,11,12,31] as a representative sample. Some part of this machinery has also been developed for VCSPs in [15], inspired by the concepts from infinite-domain CSPs and finite-domain VCSPs [26,29]. Nevertheless, no complexity classification such as for the classes of CSPs discussed above was obtained so far.

In this article we provide the first VCSP dichotomy result for a class of templates which consists of *all* valued structures preserved by some fixed permutation group. Concretely, we prove that the VCSP for every valued structure with the domain $\mathbb{Q}$ that is preserved by all order-preserving bijections is in P or NP-complete. We call such valued structures *temporal*, in analogy to temporal relational structures, i.e., structures with the domain $\mathbb{Q}$ preserved by all order-preserving bijections – these are precisely the structures with a first-order definition over $(\mathbb{Q}; <)$. We also provide disjoint algebraic conditions that characterize the tractable and the NP-complete case, in analogy to similar classifications for classes of (V)CSPs. The result confirms the dichotomy conjecture from [15, Conjecture 9.3] for the special case of temporal valued structures.

Apart from the relevance of temporal CSPs as a test case for understanding VCSPs on infinite domains, they constitute an important class for several reasons:

- Temporal CSPs encompass many natural optimization problems[1] such as *Directed Feedback Arc Set*, *Directed Subset Feedback Arc Set*, *Edge Multicut* (aka Min-Correlation-Clustering with Partial Information), *Symmetric Directed Multicut*, *Steiner Multicut*, *Disjunctive Multicut*, and many more [16, 23, 32–34].
- The complete classification for temporal CSPs has been the basis to obtain other complexity classifications in temporal and spatial reasoning via complexity classification transfer techniques [7]; we expect that temporal valued CSPs play a similar role for the corresponding optimisation problems.

Our tractability condition for temporal VCSPs is based on fractional polymorphisms, that is, probability distributions on operations with particular properties. Surprisingly, the fractional polymorphisms that appear in this classification are of a very particular shape: there is always a single operation with probability 1. The operations that appear in this context are the same operations that were already essential for classification of temporal CSPs. The hardness condition is based on the notion of *expressibility*, which generalizes primitive positive definitions, and on the notion of

[1]It should be mentioned here that these problems come in two flavours: one is where the input is a graph, or more generally a structure; the other, which we adopt here, is that the input consists of a finite sum of cost functions, which in particular allows that the sum contains identical summands. In the setting of Min-CSPs for graphs, this corresponds to considering *multigraphs* in the input. However, often (but not always, see Theorems 8.6 and 8.7 in [30]) the two variants of the problem have the same complexity (see, e.g., [22, Section 2.3] for relevant techniques in this context).

generalized *pp-constructions* [15], which provide polynomial-time reductions between VCSPs. Interestingly enough, it is not known whether preservation by fractional polymorphisms characterises expressibility in our setting; this is known for valued structures with a finite domain [20, 26]. Our classification proof, however, does not rely on such a characterisation.

1.1. **Related Work.** VCSPs on infinite domain have been studied in [37, 39, 40]. Nevertheless, the valued structures considered in these articles typically do not have an oligomorphic automorphism group, a property that is essential for applying our techniques for classifying the complexity of VCSPs. The foundations of the theory for VCSPs of valued structures with an oligomorphic automorphism group were laid in [15] with the motivation to study *resilience problems* from database theory. Concrete subclasses of temporal VCSPs have been studied in the context of min-CSPs from the parameterized complexity perspective: the complexity of *equality min-CSPs* has been classified in [35] and parameterized complexity of min-CSPs over the Point Algebra has been classified in [33]. The authors mention a classification of first-order generalizations of the Point Algebra as a natural continuation of their research [33, Section 5]. The present paper contains a classification of VCSPs over all temporal structures and thus lays the foundations for classifying the parameterized complexity of min-CSPs for such structures, including algebraic techniques that we expect to be useful in this context as well.

1.2. **Outline.** The article is organized as follows. Section 2 contains preliminaries on VCSPs in general, with some notation and properties specific to temporal VCSPs. Section 3 contains several new facts about VCSPs that have been used in the classification. Section 4 contains the classification of *equality* VCSPs, that is, VCSPs of valued structures with an automorphism group equal to the full symmetric group; on the one hand, this serves as a warm-up, on the other hand it is a building block for the general case. Section 5 contains the full classification of temporal VCSPs, which is the main contribution of the paper. We conclude with some promising questions for future research in Section 6.

2. Preliminaries

Let $\mathbb{N} := \{0, 1, 2, \dots\}$ be the set of natural numbers. For $k \in \mathbb{N}$ the set $\{1, \dots, k\}$ will be denoted by $[k]$. The set of rational numbers is denoted by $\mathbb{Q}$ and the standard strict linear order of $\mathbb{Q}$ by $<$. We also need an additional value ∞; all we need to know about ∞ is that

- $a < \infty$ for every $a \in \mathbb{Q}$,
- $a + \infty = \infty + a = \infty$ for all $a \in \mathbb{Q} \cup \{\infty\}$, and
- $0 \cdot \infty = \infty \cdot 0 = 0$ and $a \cdot \infty = \infty \cdot a = \infty$ for $a > 0$.

If A is a set and $t \in A^k$, then we implicitly assume that $t = (t_1, \dots, t_k)$, where $t_1, \dots, t_k \in A$. If $f \colon A^\ell \to A$ is an operation on A and $t^1, \dots, t^\ell \in A^k$, then we denote

$$(f(t_1^1, t_1^2, \dots, t_1^\ell), \dots, f(t_k^1, t_k^2, \dots, t_k^\ell))$$

by $f(t^1, \dots, t^\ell)$ and say that f *is applied componentwise.*

2.1. **Valued structures.** Let A be a set and let $k \in \mathbb{N}$. A *valued relation of arity k over A* is a function $R \colon A^k \to \mathbb{Q} \cup \{\infty\}$. We write $\mathscr{R}_A^{(k)}$ for the set of all valued relations over A of arity k, and define

$$\mathscr{R}_A := \bigcup_{k \in \mathbb{N}} \mathscr{R}_A^{(k)}.$$

A valued relation is called *finite-valued* if it takes values only in $\mathbb{Q}$.

Usual relations will also be called *crisp* relations. A valued relation $R \in \mathscr{R}_A^{(k)}$ that only takes values from $\{0, \infty\}$ will be identified with the crisp relation $\{t \in A^k \mid R(t) = 0\}$. A valued relation is called *essentially crisp* if it attains at most one finite value. For $R \in \mathscr{R}_A^{(k)}$ the *feasibility relation of* R is defined as

$$\mathrm{Feas}(R) := \{t \in A^k \mid R(t) < \infty\}.$$

For $S \subseteq A^k$ and $a, b \in \mathbb{Q} \cup \{\infty\}$, we denote by S_a^b the valued relation such that $S_a^b(t) = a$ if $t \in S$, and $S_a^b(t) = b$ otherwise. We often write S_0^∞ to stress that S is a crisp relation.

Example 2.1. *On the domain* $\mathbb{Q}$*, the valued relation* $(=)_0^\infty$ *denotes the crisp equality relation, while* $(<)_0^1$ *denotes the valued relation* $(<)_0^1(x, y) = 0$ *if* $x < y$ *and* $(<)_0^1(x, y) = 1$ *if* $x \geq y$*.* $(\emptyset)_0^\infty$ *is the unary empty relation (where every element of* $\mathbb{Q}$ *evaluates to* ∞*).*

A *(relational) signature* τ is a set of *relation symbols*, each of them equipped with an arity from $\mathbb{N}$. A *valued* τ*-structure* $\mathfrak{A}$ consists of a set A, which is also called the *domain* of $\mathfrak{A}$, and a valued relation $R^{\mathfrak{A}} \in \mathscr{R}_A^{(k)}$ for each relation symbol $R \in \tau$ of arity k. All valued structures in this article have countable domains. We often write R instead of $R^{\mathfrak{A}}$ if the valued structure is clear from the context. When not specified, we assume that the domains of valued structures $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ are denoted $A, B, C, \dots$, respectively. If $\mathcal{R}$ is a set of valued relations over a common domain A, we write $(A; \mathcal{R})$ for a valued structure $\mathfrak{A}$ whose relations are precisely the relations from $\mathcal{R}$; we only use this notation if the precise choice of the signature does not matter. A valued τ-structure where all valued relations only take values from $\{0, \infty\}$ may then be viewed as a *relational* or *crisp* τ-structure in the classical sense. A valued structure is called *essentially crisp* if all of its valued relations are essentially crisp. If $\mathfrak{A}$ is a valued τ-structure on the domain A, then $\mathrm{Feas}(\mathfrak{A})$ denotes the relational τ-structure $\mathfrak{A}'$ on the domain A where $R^{\mathfrak{A}'} = \mathrm{Feas}(R^{\mathfrak{A}}) \in \langle \mathfrak{A} \rangle$ for every $R \in \tau$. If $\sigma \subseteq \tau$ and $\mathfrak{A}'$ is a valued σ-structure such that $R^{\mathfrak{A}'} = R^{\mathfrak{A}}$ for every $R \in \sigma$, then we call $\mathfrak{A}'$ a *reduct of* $\mathfrak{A}$.

2.2. **Valued constraint satisfaction problems.** Let τ be a relational signature. An *atomic* τ*-expression* is an expression of the form $R(x_1, \dots, x_k)$ for $R \in \tau$, $(=)_0^\infty(x_1, x_2)$, or $(\emptyset)_0^\infty(x_1)$ where $x_1, \dots, x_k$ are (not necessarily distinct) variable symbols. A τ*-expression* is an expression ϕ of the form $\sum_{i \leq m} \phi_i$ where $m \in \mathbb{N}$ and ϕ_i for $i \in \{1, \dots, m\}$ is an atomic τ-expression. Note that the same atomic τ-expression might appear several times in the sum. We write $\phi(x_1, \dots, x_n)$ for a τ-expression where all the variables are from the set $\{x_1, \dots, x_n\}$. If $\mathfrak{A}$ is a valued τ-structure, then a τ-expression $\phi(x_1, \dots, x_n)$ defines over $\mathfrak{A}$ a member of $\mathscr{R}_A^{(n)}$, which we denote by $\phi^{\mathfrak{A}}$. If ϕ is the empty sum then $\phi^{\mathfrak{A}}$ is constant 0.

Let $\mathfrak{A}$ be a valued structure over a finite signature τ. The *valued constraint satisfaction problem for* $\mathfrak{A}$, denoted by $\mathrm{VCSP}(\mathfrak{A})$, is the computational problem to decide for a given τ-expression $\phi(x_1, \dots, x_n)$ and a given $u \in \mathbb{Q}$ whether there exists $t \in A^n$ such that $\phi^{\mathfrak{A}}(t) \leq u$. We refer to ϕ as an *instance* of $\mathrm{VCSP}(\mathfrak{A})$, and to u as the *threshold*. We also refer to the pair (ϕ, u) as a (positive or negative) *instance* of $\mathrm{VCSP}(\mathfrak{A})$. Tuples $t \in A^n$ such that $\phi^{\mathfrak{A}}(t) \leq u$ are called a *solution for* (ϕ, u). The *cost* of ϕ (with respect to $\mathfrak{A}$) is defined to be

$$\inf_{t \in A^n} \phi^{\mathfrak{A}}(t).$$

In some contexts, it will be beneficial to consider only a given τ-expression ϕ to be the input of $\mathrm{VCSP}(\mathfrak{A})$ (rather than ϕ and the threshold u) and a tuple $t \in A^n$ will then be called a *solution for* ϕ if the cost of ϕ equals $\phi^{\mathfrak{A}}(t)$. Note that in general there might not be any solution. If there exists a tuple $t \in A^n$ such that $\phi^{\mathfrak{A}}(t) < \infty$ then ϕ is called *satisfiable*.

Example 2.2. *Let $\tau = \{E\}$ and let $\mathfrak{A}$ be a valued τ-structure on the domain $\mathbb{Q}$ where $E^{\mathfrak{A}} = (<)^1_0$. Then every τ-expression can be interpreted as a (not necessarily simple) digraph with the edge relation E and every digraph corresponds to a τ-expression. The cost of every instance of* VCSP($\mathfrak{A}$) *is equal to the number of edges that have to be removed to make the digraph acyclic. Therefore,* VCSP($\mathfrak{A}$) *is the minimum feedback arc set problem.*

Remark 2.3. *If $\mathfrak{A}$ be a relational τ-structure, then* CSP($\mathfrak{A}$) *is the problem of deciding satisfiability of conjunctions of atomic τ-formulas in $\mathfrak{A}$. Note that every τ-expression $\phi(x_1, \dots, x_k)$ defines a crisp relation and can be viewed as an existentially quantified conjunction of atomic formulas, i.e., primitive positive formula, which defines the same relation. Therefore,* VCSP($\mathfrak{A}$) *and* CSP($\mathfrak{A}$) *are essentially the same problem.*

2.3. **Automorphisms.** Let $k \in \mathbb{N}$, let $R \in \mathscr{R}_A^{(k)}$, and let α be a permutation of A. Then α *preserves* R if for all $t \in A^k$ we have $R(\alpha(t)) = R(t)$. If $\mathfrak{A}$ is a valued structure with domain A, then an *automorphism* of $\mathfrak{A}$ is a permutation of A that preserves all valued relations of R. The set of all automorphisms of $\mathfrak{A}$ is denoted by $\mathrm{Aut}(\mathfrak{A})$, and forms a group with respect to composition. If $\mathfrak{B}$ is a valued structure and we write $\mathrm{Aut}(\mathfrak{B}) \subseteq \mathrm{Aut}(\mathfrak{A})$ or $\mathrm{Aut}(\mathfrak{B}) = \mathrm{Aut}(\mathfrak{A})$, we implictly assume that $\mathfrak{A}$ and $\mathfrak{B}$ have the same domain.

Let $k \in \mathbb{N}$. An *orbit of k-tuples* of a permutation group G on a set A is a set of the form $\{\alpha(t) \mid \alpha \in G\}$ for some $t \in A^k$. A permutation group G on a countable set is called *oligomorphic* if for every $k \in \mathbb{N}$ there are finitely many orbits of k-tuples in G [18]. For example, $\mathrm{Aut}(\mathbb{Q}; <)$ and therefore every permutation group on $\mathbb{Q}$ that contains $\mathrm{Aut}(\mathbb{Q}; <)$ is oligomorphic. If $\mathfrak{A}$ is a relational structure with an oligomorphic automorphism group and $R \subseteq A^k$, then R is first-order definable over $\mathfrak{A}$ if and only if R is preserved by $\mathrm{Aut}(\mathfrak{A})$, see, e.g., [3, Theorem 4.2.9].

Let $\mathfrak{A}$ be a valued τ-structure and $\mathfrak{B}$ a relational structure. Suppose that $\mathrm{Aut}(\mathfrak{B})$ is oligomorphic and $\mathrm{Aut}(\mathfrak{B}) \subseteq \mathrm{Aut}(\mathfrak{A})$ (and hence $\mathrm{Aut}(\mathfrak{A})$ is oligomorphic). Let $R \in \tau$ be of arity k. Then $R^{\mathfrak{A}}$ attains only finitely many values by the oligomorphicity of $\mathrm{Aut}(\mathfrak{A})$. Moreover, if for some $s, t \in A^k$ we have $R^{\mathfrak{A}}(s) \neq R^{\mathfrak{A}}(t)$, then s and t lie in a different orbit of $\mathrm{Aut}(\mathfrak{B})$. Therefore, for every value $a \in \mathbb{Q} \cup \{\infty\}$, there is a union U_a of orbits of k-tuples under the action of $\mathrm{Aut}(\mathfrak{B})$ such that $R^{\mathfrak{A}}(t) = a$ if and only if $t \in U_a$. Since U_a is preserved by $\mathrm{Aut}(\mathfrak{B})$, it is first-order definable over $\mathfrak{B}$ by a formula ϕ_a. Hence, R can be given by a list of values a in the range of R and first-order formulas ϕ_a over $\mathfrak{B}$. Such a collection

$$((R, a, \phi_a) \mid R \in \tau, \exists t \in A^k (R(t) = a))$$

will be called a *first-order definition* of $\mathfrak{A}$ in $\mathfrak{B}$. Clearly, if a valued structure $\mathfrak{A}$ has a first-order definition in a relational structure $\mathfrak{B}$, then $\mathrm{Aut}(\mathfrak{B}) \subseteq \mathrm{Aut}(\mathfrak{A})$. Note that for some structures $\mathfrak{B}$ such as $(\mathbb{Q}; =)$ and $(\mathbb{Q}; <)$, the formulas ϕ_a can be chosen to be quantifier-free, and hence as disjunctions of conjunctions of atomic formulas over $\mathfrak{B}$ (in fact, this is the case for every homogeneous structure with a finite relational signature). We will use first-order definitions of valued structures to be able to give valued structures as an input to decision problems (see Remark 4.7 and Proposition 5.25).

2.4. **Expressive power.** We define generalizations of the concepts of *primitive positive definitions* and *relational clones*. The motivation is that relations with a primitive positive definition can be added to the structure without changing the complexity of the respective CSP.

Definition 2.4. *Let A be a set and $R, R' \in \mathscr{R}_A$. We say that R' can be obtained from R by*

- projecting *if R' is of arity k, R is of arity $k + n$ and for all $s \in A^k$*

$$R'(s) = \inf_{t \in A^n} R(s, t).$$

- non-negative scaling *if there exists* $r \in \mathbb{Q}_{\geq 0}$ *such that* $R = rR'$*;*
- shifting *if there exists* $s \in \mathbb{Q}$ *such that* $R = R' + s$*.*

If R is of arity k, then the relation that contains all minimal-value tuples of R is

$$\mathrm{Opt}(R) := \{t \in \mathrm{Feas}(R) \mid R(t) \leq R(s) \text{ for every } s \in A^k\}.$$

Note that $\inf_{t \in A^n} R(s,t)$ in item (1) might be irrational or $-\infty$. If this is the case, then $\inf_{t \in A^n} R(s,t)$ does not express a valued relation because valued relations must have weights from $\mathbb{Q} \cup \{\infty\}$. However, if R is preserved by all permutations of an oligomorphic automorphism group, then R attains only finitely many values and therefore this is never the case.

If $S \subseteq \mathscr{R}_A$, then an atomic expression over S is an atomic τ-expression where $\tau = S$. We say that S is *closed under forming sums of atomic expressions* if it contains all valued relations defined by sums of atomic expressions over S.

Definition 2.5 (valued relational clone)**.** *A* valued relational clone (over A) *is a subset of* $\mathscr{R}_A$ *that is closed under forming sums of atomic expressions, projecting, shifting, non-negative scaling,* Feas*, and* Opt*. For a valued structure $\mathfrak{A}$ with the domain A, we write $\langle\mathfrak{A}\rangle$ for the smallest valued relational clone that contains the valued relations of $\mathfrak{A}$. If $R \in \langle\mathfrak{A}\rangle$, we say that $\mathfrak{A}$* expresses *R.*

Remark 2.6. *Note that if a valued relational clone $\mathscr{C}$ contains a set $S \subseteq \mathscr{R}_A$ of crisp relations, then every relation which is primitively positively definable from S is in $\mathscr{C}$ by forming a sum of the corresponding atomic expressions and projecting on the variables that are not existentially quantified. Therefore, valued relational clones are a generalization of relational clones. Moreover, if $\mathfrak{A}$ is a relational structure and $R \in \langle\mathfrak{A}\rangle$, then R is essentially crisp and* $\mathrm{Feas}(R)$ *is primitively positively definable from $\mathfrak{A}$; this is easily verified by induction.*

The following lemma is the main motivation for the concept of expressibility.

Lemma 2.7 (Lemma 4.6 in [15])**.** *Let $\mathfrak{A}$ be a valued structure on a countable domain with an oligomorphic automorphism group and a finite signature. Suppose that $\mathfrak{B}$ is a valued structure with a finite signature over the same domain A such that every valued relation of $\mathfrak{B}$ is from $\langle\mathfrak{A}\rangle$. Then there is a polynomial-time reduction from* $\mathrm{VCSP}(\mathfrak{B})$ *to* $\mathrm{VCSP}(\mathfrak{A})$*.*

We now introduce notation that enables us to talk about the crisp relations expressible in a valued structure, which turn out to be essential to understanding temporal VCSPs.

Definition 2.8. *Let $\mathfrak{A}$ be a valued structure. Then $\langle\mathfrak{A}\rangle_0^\infty$ denotes the set of valued relations*

$$\{R \in \langle\mathfrak{A}\rangle \mid R \text{ of arity } k, \forall a \in A^k \colon R(a) \in \{0, \infty\}\}.$$

In words, $\langle\mathfrak{A}\rangle_0^\infty$ contains all crisp relations that can be expressed in $\mathfrak{A}$. We finish this section with a remark on expressibility in essentially crisp valued structures.

Remark 2.9. *Let $\mathfrak{A}$ be an essentially crisp valued τ-structure. For every $R \in \tau$, let $a_R \in \mathbb{Q}$ be such that $R^{\mathfrak{A}}$ only attains values in $\{a_R, \infty\}$; such an a_R exists because $\mathfrak{A}$ is essentially crisp. Then $R^{\mathfrak{A}} = \mathrm{Feas}(R^{\mathfrak{A}}) + a_R$. Therefore, $\langle\mathfrak{A}\rangle = \langle\mathrm{Feas}(\mathfrak{A})\rangle$ and, by Remark 2.6, $\langle\mathfrak{A}\rangle_0^\infty$ consists of precisely those relations that are primitively positively definable in* $\mathrm{Feas}(\mathfrak{A})$*. By Lemma 2.7 and Remark 2.3, there is a polynomial-time reduction from* $\mathrm{VCSP}(\mathfrak{A})$ *to* $\mathrm{CSP}(\mathrm{Feas}(\mathfrak{A}))$ *and vice versa.*

2.5. **Pp-constructions.** Next, we introduce a concept of pp-constructions which give rise to polynomial-time reductions between VCSPs. The acronym 'pp' stands for primitive positive, since the concept of pp-constructions for relational structures is a generalization of primitive positive definitions used for reductions between CSPs.

Definition 2.10 (pp-power)**.** *Let $\mathfrak{A}$ be a valued structure with domain A and let $d \in \mathbb{N}$. Then a (d-th)* pp-power *of $\mathfrak{A}$ is a valued structure $\mathfrak{B}$ with domain A^d such that for every valued relation R of $\mathfrak{B}$ of arity k there exists a valued relation S of arity kd in $\langle\mathfrak{A}\rangle$ such that*

$$R((a_1^1, \dots, a_d^1), \dots, (a_1^k, \dots, a_d^k)) = S(a_1^1, \dots, a_d^1, \dots, a_1^k, \dots, a_d^k).$$

Let A and B be sets and $f : B \to A$. If $k \in \mathbb{N}$ and $s \in B^k$, then by $f(s)$ we mean the tuple $(f(s_1), \dots, f(s_k)) \in A^k$. We equip the space A^B of functions from B to A with the topology of pointwise convergence, where A is taken to be discrete. In this topology, a basis of open sets is given by

$$\mathscr{S}_{s,t} := \{f \in A^B \mid f(s) = t\}$$

for $s \in B^k$ and $t \in A^k$ for some $k \in \mathbb{N}$. For any topological space T, we denote by $\mathcal{B}(T)$ the Borel σ-algebra on T, i.e., the smallest subset of the powerset $\mathcal{P}(T)$ which contains all open sets and is closed under countable intersection and complement. We write $[0,1]$ for the set $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.

Definition 2.11 (fractional map)**.** *Let A and B be sets. A* fractional map *from B to A is a probability distribution*

$$(A^B, \mathcal{B}(A^B), \omega \colon \mathcal{B}(A^B) \to [0,1]),$$

that is, $\omega(A^B) = 1$ and ω is countably additive: if $S_1, S_2, \dots \in \mathcal{B}(A^B)$ are disjoint, then

$$\omega(\bigcup_{i \in \mathbb{N}} S_i) = \sum_{i \in \mathbb{N}} \omega(S_i).$$

We often use ω for both the entire fractional map and for the map $\omega \colon \mathcal{B}(A^B) \to [0,1]$.

The set $[0,1]$ carries the topology inherited from the standard topology on $\mathbb{R}$. We also view $\mathbb{R} \cup \{\infty\}$ as a topological space with a basis of open sets given by all open intervals (a,b) for $a, b \in \mathbb{R}$, $a < b$ and additionally all sets of the form $\{x \in \mathbb{R} \mid x > a\} \cup \{\infty\}$.

A *(real-valued) random variable* is a *measurable function* $X \colon T \to \mathbb{R} \cup \{\infty\}$, i.e., pre-images of elements of $\mathcal{B}(\mathbb{R} \cup \{\infty\})$ under X are in $\mathcal{B}(T)$. If X is a real-valued random variable, then the *expected value of X (with respect to a probability distribution ω)* is denoted by $E_\omega[X]$ and is defined via the Lebesgue integral

$$E_\omega[X] := \int_T X d\omega.$$

Let A and B be sets. In the rest of the paper, we will work exclusively on a topological space A^B and the special case where $B = A^\ell$ for some $\ell \in \mathbb{N}$ and A^B is the set of ℓ-ary operations on A, we denote this set by $\mathscr{O}_A^{(\ell)}$.

Definition 2.12 (fractional homomorphism)**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be valued τ-structures with domains A and B, respectively. A* fractional homomorphism *from $\mathfrak{B}$ to $\mathfrak{A}$ is a fractional map ω from B to A such that for every $R \in \tau$ of arity k and every tuple $t \in B^k$ it holds for the random variable $X \colon A^B \to \mathbb{R} \cup \{\infty\}$ given by*

$$f \mapsto R^{\mathfrak{A}}(f(t))$$

that $E_\omega[X]$ exists and that

$$E_\omega[X] \leq R^{\mathfrak{B}}(t).$$

We refer to [15] for a detailed introduction to fractional homomorphisms in full generality. If $\mathfrak{A}$ is a countable valued structure, we have the following handy expression for $E_\omega[X]$, which we sometimes use in the proofs:

$$E_\omega[X] = \sum_{s \in A^k} R(t)\omega(\mathscr{S}_{t,s}). \tag{1}$$

All concrete fractional homomorphisms ω that appear in this paper are of a very special form, namely, there is a single $f \in A^B$ such that $\omega(\{f\}) = 1$. In this case, we also write f instead of ω. If ω is of this form, then for every $R \in \tau$ of arity k and $t \in B^k$, the expected value in Definition 2.12 always exists and is equal to $R^{\mathfrak{A}}(f(t))$. If additionally $\mathfrak{A}$ and $\mathfrak{B}$ are crisp structures, then we call f a *homomorphism*. It is easy to see that there are *valued* structures $\mathfrak{A}$ and $\mathfrak{B}$ with a fractional homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$, but no fractional homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ of the form $f \in A^B$.

Lemma 2.13. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be valued τ-structures on countable domains such that* $\mathrm{Aut}(\mathfrak{A})$ *is oligomorphic. If there exists a fractional homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$, then there also exists a homomorphism from* $\mathrm{Feas}(\mathfrak{B})$ *to* $\mathrm{Feas}(\mathfrak{A})$*. In particular, if $\mathfrak{A}$ and $\mathfrak{B}$ are crisp, then there is a fractional homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ if and only if there is a homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$.*

Proof. Suppose that there exists a fractional homomorphism ω from $\mathfrak{B}$ to $\mathfrak{A}$. Since B is countable and $\mathrm{Aut}(\mathfrak{A}) \subseteq \mathrm{Aut}(\mathrm{Feas}(\mathfrak{A}))$ is oligomorphic, it suffices to show that every finite substructure $\mathfrak{F}$ of $\mathrm{Feas}(\mathfrak{B})$ has a homomorphism to $\mathrm{Feas}(\mathfrak{A})$ (see, e.g., [3, Lemma 4.1.7]). Let $b_1, \dots, b_n$ be the elements of $\mathfrak{F}$ and $b := (b_1, \dots, b_n)$. By the (countable) additivity of probability distributions, there exists $a \in A^n$ such that $\omega(\mathscr{S}_{b,a}) > 0$. Let f be the map that takes b to a. Suppose that there exists $R \in \tau$ of arity k and $t \in F^k$ such that $R^{\mathfrak{A}}(f(t)) = \infty$. Since $\omega(\mathscr{S}_{b,a}) > 0$ and $\mathscr{S}_{b,a} \subseteq \mathscr{S}_{t,f(t)}$, we have $\omega(\mathscr{S}_{t,f(t)}) > 0$, and thus $E_\omega[g \mapsto R^{\mathfrak{A}}(g(t))] = \infty$ by (1). Then $R^{\mathfrak{B}}(t) = \infty$, because ω is a fractional polymorphism. Hence, for every $R \in \tau$ of arity k and $t \in F^k$ we have $R^{\mathfrak{A}}(f(t)) < \infty$ whenever $R^{\mathfrak{B}}(t) < \infty$. Therefore, f is a homomorphism from $\mathfrak{F}$ to $\mathrm{Feas}(\mathfrak{A})$.

The final statement follows from the fact that every homomorphism is a fractional homomorphism and that $\mathfrak{C} = \mathrm{Feas}(\mathfrak{C})$ for every crisp structure $\mathfrak{C}$. □

We say that two valued τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *fractionally homomorphically equivalent* if there exists a fractional homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ and from $\mathfrak{B}$ to $\mathfrak{A}$. Clearly, fractional homomorphic equivalence is indeed an equivalence relation on valued structures of the same signature.

Definition 2.14 (pp-construction)**.** *Let $\mathfrak{A}, \mathfrak{B}$ be valued structures. Then $\mathfrak{B}$ has a* pp-construction *in $\mathfrak{A}$ if $\mathfrak{B}$ is fractionally homomorphically equivalent to a structure $\mathfrak{B}'$ which is a pp-power of $\mathfrak{A}$.*

Remark 2.15. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be relational structures, and suppose that $\mathfrak{A}$ pp-constructs $\mathfrak{B}$, that is, there is a pp-power $\mathfrak{A}'$ of $\mathfrak{A}$ which is fractionally homomorphically equivalent to $\mathfrak{B}$. It follows from Remark 2.6 that $\mathfrak{A}'$ is essentially crisp and* $\mathrm{Feas}(\mathfrak{A}')$ *is a pp-power of $\mathfrak{A}$ with all relations primitively positively definable in $\mathfrak{A}$ when viewed over the domain A. By Lemma 2.13,* $\mathrm{Feas}(\mathfrak{A}')$ *is homomorphically equivalent to $\mathfrak{B}$. Hence, our definition of pp-constructability between two relational structures with oligomorphic automorphism groups coincides with the definition from [2].*

The relation of pp-constructability is transitive: if $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ are valued structures such that $\mathfrak{A}$ pp-constructs $\mathfrak{B}$ and $\mathfrak{B}$ pp-constructs $\mathfrak{C}$, then $\mathfrak{A}$ pp-constructs $\mathfrak{C}$ [15, Lemma 5.14].

By K_3 we denote the complete graph on 3 vertices. The following is a direct consequence of [15, Corollary 5.13] and [3, Corollary 6.7.13].

Lemma 2.16. *Let $\mathfrak{A}$ be a valued structure with an oligomorphic automorphism group. If $\mathfrak{A}$ pp-constructs K_3, then $\mathfrak{A}$ has a reduct $\mathfrak{A}'$ over a finite signature such that* $\mathrm{VCSP}(\mathfrak{A}')$ *is NP-hard.*

It is well-known that K_3 pp-constructs all finite relational structures (see, e.g, [3, Corollary 6.4.4]). Hence, by the transitivity of pp-constructability, every valued structure that pp-constructs K_3 pp-constructs all finite relational structures.

2.6. **Fractional polymorphisms.** Let A be a set and $R \subseteq A^k$. An operation $f\colon A^\ell \to A$ on the set A *preserves* R if $f(t^1, \ldots, t^\ell) \in R$ for every $t^1, \ldots, t^\ell \in R$. If $\mathfrak{A}$ is a relational structure and f preserves all relations of $\mathfrak{A}$, then f is called a *polymorphism* of $\mathfrak{A}$. The set of all polymorphisms of $\mathfrak{A}$ is denoted by $\mathrm{Pol}(\mathfrak{A})$ and is closed under composition. We write $\mathrm{Pol}^{(\ell)}(\mathfrak{A})$ for the set $\mathrm{Pol}(\mathfrak{A}) \cap \mathscr{O}_A^{(\ell)}$, $\ell \in \mathbb{N}$. Unary polymorphisms are called *endomorphisms* and $\mathrm{Pol}^{(1)}(\mathfrak{A})$ is also denoted by $\mathrm{End}(\mathfrak{A})$.

Let $\mathfrak{A}$ be a relational structure and $\ell \in \mathbb{N}$. An operation $f \in \mathrm{Pol}^{(\ell)}(\mathfrak{A})$ is called a *pseudo weak near unanimity (pwnu)* polymorphism if there exist $e_1, \ldots, e_\ell \in \mathrm{End}(\mathfrak{A})$ such that for every $x, y \in A$

$$e_1 f(y, x, \ldots, x) = e_2 f(x, y, x, \ldots, x) = \cdots = e_\ell f(x, \ldots, x, y).$$

We now introduce *fractional polymorphisms* of valued structures, which generalize polymorphisms of relational structures. Similarly as polymorphisms, fractional polymorphisms are an important tool for formulating tractability results and complexity classifications of VCSPs. For valued structures with a finite domain, our definition specialises to the established notion of a fractional polymorphism which has been used to study the complexity of VCSPs for valued structures over finite domains (see, e.g. [38]). Our definition is taken from [15] and allows arbitrary probability spaces in contrast to [37, 39, 40].

Definition 2.17 (fractional operation)**.** *Let $\ell \in \mathbb{N}$. A* fractional operation on A of arity ℓ *is a probability distribution*

$$\big(\mathscr{O}_A^{(\ell)}, \mathcal{B}(\mathscr{O}_A^{(\ell)}), \omega\colon \mathcal{B}(\mathscr{O}_A^{(\ell)}) \to [0,1]\big).$$

The set of all fractional operations on A of arity ℓ is denoted by $\mathscr{F}_A^{(\ell)}$.

Definition 2.18. *A fractional operation $\omega \in \mathscr{F}_A^{(\ell)}$* improves *a valued relation $R \in \mathscr{R}_A^{(k)}$ if for all $t^1, \ldots, t^\ell \in A^k$*

$$E := E_\omega[f \mapsto R(f(t^1, \ldots, t^\ell))]$$

exists, and

$$E \leq \frac{1}{\ell} \sum_{j=1}^{\ell} R(t^j). \tag{2}$$

Note that (2) has the interpretation that the expected value of $R(f(t^1, \ldots, t^\ell))$ is at most the average of the values $R(t^1), \ldots, R(t^\ell)$.

Definition 2.19 (fractional polymorphism)**.** *If a fractional operation ω improves every valued relation in a valued structure $\mathfrak{A}$, then ω is called a* fractional polymorphism of $\mathfrak{A}$*; the set of all fractional polymorphisms of $\mathfrak{A}$ is denoted by* $\mathrm{fPol}(\mathfrak{A})$.

Remark 2.20. *A fractional polymorphism of arity ℓ of a valued τ-structure $\mathfrak{A}$ might also be viewed as a fractional homomorphism from a specific ℓ-th pp-power of $\mathfrak{A}$, which we denote by $\mathfrak{A}^\ell$, to $\mathfrak{A}$: the domain of $\mathfrak{A}^\ell$ is A^ℓ, and for every $R \in \tau$ of arity k we have*

$$R^{\mathfrak{A}^\ell}((a_1^1, \ldots, a_\ell^1), \ldots, (a_1^k, \ldots, a_\ell^k)) := \frac{1}{\ell} \sum_{i=1}^{\ell} R^{\mathfrak{A}}(a_i^1, \ldots, a_i^k).$$

Example 2.21. *Let A be a set and $\pi_i^\ell \in \mathscr{O}_A^{(\ell)}$ be the i-th projection of arity ℓ, which is given by $\pi_i^\ell(x_1, \ldots, x_\ell) = x_i$. The fractional operation Id_ℓ of arity ℓ such that $\mathrm{Id}_\ell(\pi_i^\ell) = \frac{1}{\ell}$ for every $i \in \{1, \ldots, \ell\}$ is a fractional polymorphism of every valued structure with domain A.*

As mentioned above, all concrete fractional polymorphisms ω that we need in this article are such that there exists an operation $f \in \mathscr{O}_A^{(\ell)}$ such that $\omega(\{f\}) = 1$.

Remark 2.22. *Let $\mathfrak{A}$ be a* relational *τ-structure on the domain A. Then $\mathrm{Pol}(\mathfrak{A}) \subseteq \mathrm{fPol}(\mathfrak{A})$ (using the convention introduced above that an operation on A can be viewed as a fractional operation). More concretely, for every $\ell \in \mathbb{N}$ we have that $\mathrm{fPol}^{(\ell)}(\mathfrak{A})$ consists of precisely the fractional operations ω of arity ℓ such that $\omega(\mathrm{Pol}^{(\ell)}(\mathfrak{A})) = 1$. To see this, note that*

$$\mathscr{O}_A^{(\ell)} \setminus \mathrm{Pol}^{(\ell)}(\mathfrak{A}) = \bigcup_{t^1,\dots,t^\ell \in R, s \in A^k \setminus R} \mathscr{S}_{(t^1,\dots,t^\ell),s} \tag{3}$$

and therefore a measurable set. Hence, $\mathrm{Pol}^{(\ell)}(\mathfrak{A})$ is also measurable. Let $R \in \tau$ be of arity k and let $t^1, \dots, t^\ell \in A^k$. Note that by (1)

$$E_\omega[f \mapsto R(f(t^1,\dots,t^\ell))] = \sum_{s \in A^k} \omega(\mathscr{S}_{(t^1,\dots,t^\ell),s}) R(s).$$

Therefore, if $\omega(\mathrm{Pol}^{(\ell)}(\mathfrak{A})) = 1$, either $R(t^j) = \infty$ for some j and (2) holds trivially, or $t^1, \dots, t^\ell \in R$ and $\omega(\mathscr{S}_{(t^1,\dots,t^\ell),s}) = 0$ whenever $s \notin R$. In this case (2) holds because both sides of the inequality are equal to 0. On the other hand, if $\omega(\mathscr{O}_A^{(\ell)} \setminus \mathrm{Pol}^{(\ell)}(\mathfrak{A})) > 0$, then by (3) there exist $t^1, \dots, t^\ell \in R$ and $s \in A^k \setminus R$ such that $\omega(\mathscr{S}_{(t^1,\dots,t^\ell),s}) > 0$. Then $E_\omega[f \mapsto R(f(t^1,\dots,t^\ell))] = \infty$, which contradicts (2) since $R(t^j) = 0$ for all j.

Lemma 2.23 (Lemma 6.8 in [15])**.** *Let $\mathfrak{A}$ be a valued τ-structure $\mathfrak{A}$ over a countable domain A. Then every valued relation $R \in \langle \mathfrak{A} \rangle$ is improved by all fractional polymorphisms of $\mathfrak{A}$.*

Remark 2.24. *Recall that $\langle \mathfrak{A} \rangle = \langle \mathrm{Feas}(\mathfrak{A}) \rangle$ for essentially crisp valued structures $\mathfrak{A}$. Therefore, Lemma 2.23 implies that $\mathrm{fPol}(\mathfrak{A}) = \mathrm{fPol}(\mathrm{Feas}(\mathfrak{A}))$.*

2.7. **Temporal valued structures.** Let $\mathfrak{A}$ be a valued structure such that $\mathrm{Aut}(\mathbb{Q}; <) \subseteq \mathrm{Aut}(\mathfrak{A})$. Then every τ-expression ϕ (in particular, every valued relation of $\mathfrak{A}$) attains only finitely many values in $\mathfrak{A}$: if $\phi = \phi(x_1, \dots, x_k)$, we may have at most one value for every orbit of the action of $\mathrm{Aut}(\mathfrak{A})$ on $\mathbb{Q}^k$. Since the group $\mathrm{Aut}(\mathbb{Q}; <)$ has only finitely many orbits of k-tuples for every k, so does $\mathrm{Aut}(\mathfrak{A})$. In particular, if $k = 2$ and $a, b \in \mathbb{Q}$, $a < b$, then the values $\phi(a,a), \phi(a,b)$ and $\phi(b,a)$ do not depend on the choice of a and b and if $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Aut}(\mathbb{Q}; =)$, then $\phi(a,b) = \phi(b,a)$. As a consequence, for every τ-expression $\phi(x_1, \dots, x_k, y_1, \dots, y_\ell)$ and $b \in A^\ell$ there exists $a^* \in A^k$ such that

$$\inf_{a \in A^k} \phi(a,b) = \min_{a \in A^k} \phi(a,b) = \phi(a^*, b).$$

The following theorem is a special case of [15, Theorem 3.4].

Theorem 2.25. *Let $\mathfrak{A}$ be a valued structure over a finite signature such that $\mathrm{Aut}(\mathbb{Q}; <) \subseteq \mathrm{Aut}(\mathfrak{A})$. Then $\mathrm{VCSP}(\mathfrak{A})$ is in NP.*

We will often use the following notation in the proofs.

Definition 2.26 (E_t, N_t, O_t)**.** *If $t \in \mathbb{Q}^k$ for some $k \in \mathbb{N}$, we define*

$$E_t := \{(p,q) \in [k]^2 \mid t_p = t_q\},$$
$$N_t := \{(p,q) \in [k]^2 \mid t_p \neq t_q\}, \text{ and}$$
$$O_t := \{(p,q) \in [k]^2 \mid t_p < t_q\},$$

where $<$ is the natural order over $\mathbb{Q}$.

As we already alluded to, we will repeatedly use the fact that in temporal VCSPs, any valued relation R is such that $R(t)$ only depends on the *order type* of the tuple t.

Observation 2.27. *Let $\mathfrak{A}$ be valued structure with finite signature τ such that* $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$ *and let $R \in \tau$ be any valued relation of arity k. Then for every $t, t' \in \mathbb{Q}^k$ such that $E_t = E_{t'}$ and $O_t = O_{t'}$ it holds that $R(t) = R(t')$.*

3. General Facts

In this section we formulate and prove some relatively easy, but very general and useful facts for analysing the computational complexity of VCSPs.

Lemma 3.1. *Let $\mathfrak{A}$ be a valued structure with domain A and finite relational signature τ such that there exists a unary constant operation $c \in \mathrm{fPol}(\mathfrak{A})$. Then* $\mathrm{VCSP}(\mathfrak{A})$ *is in P.*

Proof. Suppose that there exists $b \in A$ such that the unary operation c defined by $c(a) = b$ for all $a \in A$ is a fractional polymorphism of $\mathfrak{A}$. Then for every $R \in \tau$ of arity k and $t \in A^k$, we have $R(b, \dots, b) = R(c(t)) \leq R(t)$. Let $\phi(x_1, \dots, x_n) = \sum_i \phi_i$ be an instance of $\mathrm{VCSP}(\mathfrak{A})$, where each ϕ_i is an atomic τ-expression. Then the minimum of ϕ equals $\sum_i \phi_i(b, \dots, b)$ and hence $\mathrm{VCSP}(\Gamma)$ is in P. □

The following proposition relates pp-constructability in a valued structure $\mathfrak{A}$ with pp-constructability in the relational structure $(\mathbb{Q}; \langle \mathfrak{A} \rangle_0^\infty)$.

Proposition 3.2. *Let $\mathfrak{A}$ be a valued structure and let $\mathfrak{B}$ be a relational τ-structure on countable domains A and B, respectively. Then $\mathfrak{A}$ pp-constructs $\mathfrak{B}$ if and only if $(A; \langle \mathfrak{A} \rangle_0^\infty)$ pp-constructs $\mathfrak{B}$.*

Proof. Clearly, whenever $(A; \langle \mathfrak{A} \rangle_0^\infty)$ pp-constructs $\mathfrak{B}$, then $\mathfrak{A}$ pp-constructs $\mathfrak{B}$. Suppose that $\mathfrak{A}$ pp-constructs $\mathfrak{B}$. Then there exists $d \in \mathbb{N}$ and a pp-power $\mathfrak{C}$ on the domain $C = A^d$ of $\mathfrak{A}$ which is fractionally homomorphically equivalent to $\mathfrak{B}$. We claim that $\mathrm{Feas}(\mathfrak{C})$ is fractionally homomorphically equivalent to $\mathfrak{B}$ as well, witnessed by the same fractional homomorphisms.

Let ω_1 be a fractional homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$ and ω_2 be a fractional homomorphism from $\mathfrak{B}$ to $\mathfrak{C}$. Let $R \in \tau$ be of arity k and $t \in C^k$. By the definition of a fractional homomorphism,

$$E_{\omega_1}[f \mapsto R^{\mathfrak{B}}(f(t))] \leq R^{\mathfrak{C}}(t).$$

We claim that

$$E_{\omega_1}[f \mapsto R^{\mathfrak{B}}(f(t))] \leq \mathrm{Feas}(R^{\mathfrak{C}})(t). \tag{4}$$

This is clear if $\mathrm{Feas}(R^{\mathfrak{C}})(t) = \infty$. Otherwise, $\mathrm{Feas}(R^{\mathfrak{C}})(t) = 0$, and therefore $R^{\mathfrak{C}}(t)$ is finite. Hence,

$$E_{\omega_1}[f \mapsto R^{\mathfrak{B}}(f(t))] = \sum_{s \in B^k} \omega_1(\mathscr{S}_{t,s}) R^{\mathfrak{B}}(s)$$

is finite. Since $R^{\mathfrak{B}}$ attains only values 0 and ∞, it follows that $E_{\omega_1}[f \mapsto R^{\mathfrak{B}}(f(t))] = 0$. Therefore (4) holds. Since R and t were arbitrary, it follows that ω_1 is a fractional homomorphism from $\mathrm{Feas}(\mathfrak{C})$ to $\mathfrak{B}$. The proof that ω_2 is a fractional homomorphism from $\mathfrak{B}$ to $\mathrm{Feas}(\mathfrak{C})$ is similar.

Note that $\mathrm{Feas}(\mathfrak{C})$ is a pp-power of $(A; \langle \mathfrak{A} \rangle_0^\infty)$: every relation R of $\mathfrak{C}$ of arity k lies in $\langle \mathfrak{A} \rangle$ when viewed as a relation of arity dk and therefore $\mathrm{Feas}(R) \in \langle \mathfrak{A} \rangle_0^\infty$. Hence, $(A; \langle \mathfrak{A} \rangle_0^\infty)$ pp-constructs $\mathfrak{B}$ as we wanted to prove. □

Let A be a set, $k \in \mathbb{N}$, and $i \in [k]$. It is easy to see that the k-ary i-th projection $\pi_i^k \in \mathrm{Pol}(\mathfrak{A})$ for every $k \in \mathbb{N}$, $i \in [k]$, and every relational structure $\mathfrak{A}$ on the domain A.

Lemma 3.3. *Let $\mathfrak{A}$ be a valued structure. Then* $\text{fPol}(\mathfrak{A})$ *contains* π_1^2 *if and only if* $\mathfrak{A}$ *is essentially crisp.*

Proof. Suppose that $\mathfrak{A}$ contains a valued relation R of arity k which takes two finite values a and b with $a < b$. Let $s \in A^k$ be such that $R(s) = a$ and $t \in A^k$ be such that $R(t) = b$. Then $R(\pi_1^2(t,s)) > \frac{R(s)+R(t)}{2}$, and hence $\pi_1^2 \notin \text{fPol}(\mathfrak{A})$.

Conversely, suppose that $\mathfrak{A}$ is essentially crisp. Let R be a valued relation of arity k in $\mathfrak{A}$ and let $s,t \in A^k$. If $R(s) = \infty$ or $R(t) = \infty$ then $R(\pi_1^2(s,t)) \leq \frac{R(s)+R(t)}{2} = \infty$. Otherwise, $R(s) = R(t)$ and the inequality holds trivially. □

4. Equality VCSPs

An *equality structure* is a relational structure whose automorphism group is the group of all permutations of its domain [8]; we define an *equality valued structure* analogously. In this section, we prove that for every equality valued structure $\mathfrak{A}$, VCSP($\mathfrak{A}$) is in P or NP-complete. This generalises the P versus NP-complete dichotomy for equality min-CSPs from [34].

If the domain of $\mathfrak{A}$ is finite, then this is already known (see the discussion in the introduction). It is easy to see that classifying the general infinite case reduces to the countably infinite case. For notationally convenient use in the later sections, we work with the domain $\mathbb{Q}$, but we could have used any other countably infinite set instead. We will need the following relation:

$$\text{Dis} := \{(x,y,z) \in \mathbb{Q}^3 \mid (x = y \neq z) \vee (x \neq y = z)\}. \tag{5}$$

It is known that $(\mathbb{Q}; \text{Dis})$ pp-constructs K_3, see, e.g., Theorem 7.4.1 and Corollary 6.1.23 in [3]. Let $\text{const}\colon \mathbb{Q} \to \mathbb{Q}$ be the constant zero operation, given by $\text{const}(x) := 0$ for all $x \in \mathbb{Q}$. Let $\text{inj}\colon \mathbb{Q}^2 \to \mathbb{Q}$ be injective. A tuple is called *injective* if it has pairwise distinct entries.

Theorem 4.1 ([3,8])**.** *If $\mathfrak{A}$ is a relational structure such that* $\text{Aut}(\mathfrak{A}) = \text{Sym}(\mathbb{Q})$*, then exactly one of the following cases applies.*

- $\text{const} \in \text{fPol}(\mathfrak{A})$ *or* $\text{inj} \in \text{fPol}(\mathfrak{A})$*. In this case, for every reduct $\mathfrak{A}'$ of $\mathfrak{A}$ with a finite signature,* CSP($\mathfrak{A}'$) *is in P.*
- *The relation* Dis *has a primitive positive definition in $\mathfrak{A}$. In this case, $\mathfrak{A}$ pp-constructs K_3 and $\mathfrak{A}$ has a reduct $\mathfrak{A}'$ with a finite signature such that* CSP($\mathfrak{A}$) *is NP-complete.*

We prove the following general lemma that assumes only $\text{Aut}(\mathbb{Q};<) \subseteq \text{Aut}(\mathfrak{A})$ to avoid repeating the proof in Section 5. The case $\text{Aut}(\mathfrak{A}) = \text{Sym}(\mathbb{Q})$ is a special case.

Lemma 4.2. *Let $\mathfrak{A}$ be a valued structure such that* $\text{Aut}(\mathbb{Q};<) \subseteq \text{Aut}(\mathfrak{A})$*. If* $\text{const} \notin \text{fPol}(\mathfrak{A})$*, then* $(\neq)_0^\infty \in \langle\mathfrak{A}\rangle$ *or* $(<)_0^\infty \in \langle\mathfrak{A}\rangle$*. In particular, if* $\text{Aut}(\mathfrak{A}) = \text{Sym}(\mathbb{Q})$*, then* $(\neq)_0^\infty \in \langle\mathfrak{A}\rangle$*.*

Proof. By assumption, there exists $R \in \tau$ of arity k and $t \in A^k$ such that $m := R(t) < R(0,\dots,0)$. For $i \in \{1,\dots,k\}$, define $\psi_i(x_1,\dots,x_i)$ to be $R(x_1,\dots,x_i,x_i,\dots,x_i)$. Choose t and i such that i is minimal with the property that $\psi_i(t_1,\dots,t_i) < R(0,\dots,0)$. Note that such an i exists, because for $i = k$ we have $\psi_i(t_1,\dots,t_i) = R(t_1,\dots,t_k) < R(0,\dots,0)$. Moreover, $i > 1$, since for every $a \in A$ there exists $\alpha \in \text{Aut}(\mathfrak{A})$ such that $\alpha(a) = 0$ and hence $R(a,\dots,a) = R(\alpha(a),\dots,\alpha(a)) = R(0,\dots,0)$. Also note that $t_{i-1} \neq t_i$, by the minimality of i.

From all the such pairs (t,i) that minimise i, choose a pair (t,i) where $\psi_i(t_1,\dots,t_i)$ is minimal. Such a t exists because R attains only finitely many values. Define

$$\psi(x_{i-1},x_i) := \min_{x_1,\dots,x_{i-2}} \psi_i(x_1,\dots,x_{i-2},x_{i-1},x_i).$$

Let $a \in A$ and note that, by Observation 2.27, value $\psi(a,a)$ does not depend on the choice of a. By our choice of i, $\psi(a,a) > \psi(t_{i-1},t_i)$; otherwise, there are $a_1,\dots,a_{i-2} \in A$ such that $\psi_i(a_1,\dots,a_{i-2},a,a) \leq \psi_i(t_1,\dots,t_i)$, in contradiction to the choice of (t,i) such that i is minimal. We distinguish three cases (recall that $t_{i-1} \neq t_i$):

(1) $\psi(t_{i-1},t_i) = \psi(t_i,t_{i-1})$,
(2) $\psi(t_{i-1},t_i) < \psi(t_i,t_{i-1})$ and $t_{i-1} < t_i$,
(3) $\psi(t_{i-1},t_i) < \psi(t_i,t_{i-1})$ and $t_i < t_{i-1}$.

Note that for $a,b \in A$ such that $a < b$, the values $\psi(a,b)$ and $\psi(b,a)$ do not depend on the choice of a, b. In case (1), $\mathrm{Opt}(\psi)$ expresses $(\neq)_0^\infty$. In case (2), $\mathrm{Opt}(\psi)$ expresses $(<)_0^\infty$. Finally, in case (3) $\mathrm{Opt}(\psi)$ expresses $(>)_0^\infty$, which expresses $(<)_0^\infty$ by exchanging the input variables.

The last statement follows from the fact that $\mathrm{Sym}(\mathbb{Q})$ does not preserve $(<)_0^\infty$. □

Lemma 4.3 (see, e.g., [15, Example 4.9])**.** *Let $\mathfrak{A}$ be a valued structure such that $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Sym}(\mathbb{Q})$. If $\langle\mathfrak{A}\rangle$ contains $(=)_0^1$ and $(\neq)_0^\infty$, then $\mathrm{Dis} \in \langle\mathfrak{A}\rangle$. In particular, $\mathfrak{A}$ pp-constructs K_3, and, if the signature of $\mathfrak{A}$ is finite, $\mathrm{VCSP}(\mathfrak{A})$ is NP-complete.*

Lemma 4.4. *Let $\mathfrak{A}$ be a valued structure such that $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Sym}(\mathbb{Q})$. Suppose that $\mathrm{fPol}(\mathfrak{A})$ contains inj. Then $\mathrm{VCSP}(\mathfrak{A})$ is in P.*

Proof. Let (ϕ,u) be an instance of $\mathrm{VCSP}(\mathfrak{A})$ with variable set V. We first check whether ϕ contains summands with at most one variable that evaluate to ∞ for some (equivalently, for all) assignment; in this case, the minimum of ϕ is above every rational threshold and the algorithm rejects. Otherwise, we propagate (crisp) forced equalities: if ϕ contains a summand $R(x_1,\dots,x_k)$ and for all $f\colon V \to A$ we have that if $R(f(x_1),\dots,f(x_k))$ is finite, then $f(x_i) = f(x_j)$ for some $i < j$, then we say that $x_i = x_j$ is *forced*. In this case, we replace all occurrences of x_j in ϕ by x_i and repeat this process (including the check for unary summands that evaluate to ∞); clearly, this procedure must terminate after finitely many steps. Let V' be the resulting set of variables, and let ϕ' be the resulting instance of $\mathrm{VCSP}(\mathfrak{A})$. Clearly, the minimum for ϕ' equals the minimum for ϕ. Fix any injective $g\colon V' \to \mathbb{Q}$; we claim that g minimises ϕ'. To see this, let $f\colon V' \to \mathbb{Q}$ be any assignment and let $\psi(x) = \psi(x_1,\dots,x_k)$ be a summand of ϕ'. We show that $\psi(g(x)) \leq \psi(f(x))$. The statement is trivially true if $k = 1$ by the transitivity of $\mathrm{Aut}(\mathfrak{A})$. Assume therefore that $k \geq 2$.

We first prove that $\psi(g(x))$ is finite. Let $u^1,\dots,u^n$ be an enumeration of representatives of all orbits of k-tuples such that $\psi(u^i) < \infty$. If for some distinct $p,q \in \{1,\dots,k\}$ we have $(u^i)_p = (u^i)_q$ for all $i \in \{1,\dots,n\}$, then the algorithm would have replaced all occurrences of x_p by x_q or vice versa. So for all distinct $p,q \in \{1,\dots,k\}$ there exists $i \in \{1,\dots,n\}$ such that $(u^i)_p \neq (u^i)_q$. Therefore, since inj is injective, the tuple $\mathrm{inj}(u^1,\mathrm{inj}(u^2,\dots,\mathrm{inj}(u^{n-1},u^n)\dots))$ lies in the same orbit as $g(x)$. Since $\mathrm{inj} \in \mathrm{fPol}(\mathfrak{A})$, we have $\psi(g(x)) < \infty$.

Note that

- $2\psi(\mathrm{inj}(g(x),f(x)) \leq \psi(g(x)) + \psi(f(x))$, because $\mathrm{inj} \in \mathrm{fPol}(\mathfrak{A})$, and
- $\mathrm{inj}(g(x),f(x))$ lies in the same orbit of $\mathrm{Aut}(\mathfrak{A})$ as $g(x)$, and thus $\psi(\mathrm{inj}(g(x),f(x)) = \psi(g(x))$.

Combining, we obtain that $\psi(g(x)) \leq \psi(f(x))$. It follows that g minimises ϕ'. Recall that the minimum of ϕ and ϕ' are equal. Therefore, the algorithm accepts if the evaluation of ϕ' under g is at most u and rejects otherwise. Since checking whether a summand forces an equality can be done in constant time, and there is a linear number of variables, the propagation of forced equalities can be done in polynomial time. It follows that $\mathrm{VCSP}(\mathfrak{A})$ is in P. □

Lemma 4.5. *Let $\mathfrak{A}$ be a valued structure such that $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Sym}(\mathbb{Q})$. Suppose that $\mathrm{inj} \notin \mathrm{fPol}(\mathfrak{A})$. Then $(=)_0^1 \in \langle(\mathfrak{A},(\neq)_0^\infty)\rangle$ or $\mathrm{Dis} \in \langle(\mathfrak{A},(\neq)_0^\infty)\rangle$.*

Proof. By assumption, there exists $R \in \tau$ with arity k which is not improved by inj, that is, there exist $s, t \in \mathbb{Q}^k$ such that

$$R(s) + R(t) < 2R(\text{inj}(s,t)). \tag{6}$$

Note that, in particular, $R(s), R(t) < \infty$. Suppose first that $\text{inj}(s,t) = \infty$. In this case, the inequality above implies that $\text{Feas}(R)$ is not improved by inj. It follows that $\text{Pol}(\mathbb{Q}; \text{Feas}(R), (\neq)_0^\infty)$ contains neither const nor inj. Hence, by Theorem 4.1, $(\mathbb{Q}; \text{Feas}(R), (\neq)_0^\infty)$ primitively positively defines Dis and thus $\text{Dis} \in \langle(\mathfrak{A}, (\neq)_0^\infty\rangle$. We may therefore assume in the rest of the proof that $R(\text{inj}(s,t)) < \infty$.

Inequality (6) implies that $R(s) < R(\text{inj}(s,t))$ or $R(t) < R(\text{inj}(s,t))$. Without loss of generality, assume $R(t) < R(\text{inj}(s,t))$. Since $R(\text{inj}(s,t))$ is finite, this implies

$$R(\text{inj}(s,t)) + R(t) < 2R(\text{inj}(s,t)) = 2R(\text{inj}(\text{inj}(s,t),t)),$$

where the last equality follows from the fact that $\text{inj}(s,t)$ and $\text{inj}(\text{inj}(s,t),t)$ lie in the same orbit of $\text{Aut}(\mathfrak{A}) = \text{Sym}(\mathbb{Q})$. This is an inequality of the same form as (6) (with $\text{inj}(s,t)$ in the role of s), which implies that we can assume without loss of generality that s and $\text{inj}(s,t)$ lie in the same orbit of $\text{Aut}(\mathfrak{A})$. Then (6) implies that $R(t) < R(\text{inj}(s,t)) = R(s)$. We show that in this case $(=)_0^1 \in \langle(\mathfrak{A}, (\neq)_0^\infty)\rangle$.

Out of all pairs $(s,t) \in (A^k)^2$ such that s and $\text{inj}(s,t)$ lie in the same orbit and $R(t) < R(s)$, we choose (s,t) such that $t_p \neq t_q$ holds for as many pairs (p,q) as possible. Note that s and t cannot lie in the same orbit, and by the injectivity of inj there exist $i, j \in \{1, \ldots, k\}$ such that $t_i = t_j$ and $s_i \neq s_j$. Note that since s and $\text{inj}(s,t)$ lie in the same orbit, we have $E_s \subseteq E_t$ and $N_t \subseteq N_s$ (recall Definition 2.26). For the sake of the notation, assume that $i = 1$ and $j = 2$. Consider the expression

$$\phi(x_1, x_2) := \min_{x_3,\ldots,x_k} R(x_1, \ldots, x_k) + \sum_{(p,q)\in E_s} (=)_0^\infty(x_p, x_q) + \sum_{(p,q)\in N_t} (\neq)_0^\infty(x_p, x_q).$$

Then $\phi(x,y)$ attains at most two values by the 2-transitivity of $\text{Aut}(\mathfrak{A})$. For every $x \in \mathbb{Q}$, we have $\phi(x,x) =: m \leq R(t)$. Let $\ell := \phi(x,y)$ for some distinct $x, y \in \mathbb{Q}$; this value does not depend on the choice of x and y by the 2-transitivity of $\text{Aut}(\mathfrak{A})$. Suppose for contradiction that $\ell \leq m$. Then there exists a tuple $u \in \mathbb{Q}^k$ such that

(i) $R(u) \leq R(t)$,
(ii) $u_i \neq u_j$,
(iii) $E_s \subseteq E_u$, and
(iv) $N_t \subseteq N_u$.

By (iii), $\text{inj}(s,u)$ lies in the same orbit as s. By (i), we get that $R(u) \leq R(t) < R(s)$. By (ii) and (iv), u satisfies $u_p \neq u_q$ for more pairs (p,q) than t, which contradicts our choice of t. Therefore, $m < \ell$. It follows that $\phi(x_1,x_2)$ is equivalent to $(=)_m^\ell$ with $m < \ell$. Recall that $(=)_0^\infty \in \langle\mathfrak{A}\rangle$ by definition and therefore $(=)_m^\ell \in \langle\mathfrak{A}, (\neq)_0^\infty\rangle$. Note that $\ell \leq R(s) < \infty$. Hence, shifting ϕ by $-m$ and scaling it by $1/(\ell - m)$ shows that $(=)_0^1 \in \langle\mathfrak{A}, (\neq)_0^\infty\rangle$, as we wanted to prove. □

Theorem 4.6. *Let $\mathfrak{A}$ be a valued structure with a countably infinite domain $\mathbb{Q}$ over a finite relational signature such that $\text{Aut}(\mathfrak{A}) = \text{Sym}(\mathbb{Q})$. Then exactly one of the following two cases applies.*

(1) $(\neq)_0^\infty \in \langle\mathfrak{A}\rangle$ and $\text{Dis} \in \langle\mathfrak{A}\rangle$. In this case, $\mathfrak{A}$ pp-constructs K_3, and $\text{VCSP}(\mathfrak{A})$ is NP-complete.
(2) $\text{const} \in \text{fPol}(\mathfrak{A})$ or $\text{inj} \in \text{fPol}(\mathfrak{A})$. In both of these cases, $\text{VCSP}(\mathfrak{A})$ is in P.

Proof. If $\text{const} \in \text{fPol}(\mathfrak{A})$, then Lemma 3.1 implies that $\text{VCSP}(\Gamma)$ is in P. If $\text{const} \notin \text{fPol}(\mathfrak{A})$, then $\mathfrak{A}$ can express $(\neq)_0^\infty$ by Lemma 4.2.

If inj $\in$ fPol($\mathfrak{A}$), then Lemma 4.4 implies that VCSP($\mathfrak{A}$) is in P. If inj $\notin$ fPol($\mathfrak{A}$), then $\mathfrak{A}$ can express Dis or $(=)^1_0$ by Lemma 4.5. If Dis $\in \langle\mathfrak{A}\rangle$, then the statement follows from Theorem 4.1. If $(=)^1_0 \in \langle\mathfrak{A}\rangle$, then we obtain that Dis $\in \langle\mathfrak{A}\rangle$ by Lemma 4.3 and again the statement follows. Note that neither const nor inj improves Dis. Therefore, the two cases in the statement of the theorem are disjoint. □

Remark 4.7. *If $\mathfrak{A}$ is as in Theorem 4.6, then $\mathfrak{A}$ has a first-order definition in $(\mathbb{Q};=)$ as introduced in Section 2.3, where the defining formulas can be chosen as disjunctions of conjunctions of atomic formulas over $(\mathbb{Q};=)$. If $\mathfrak{A}$ is given by such a first-order definition over $(\mathbb{Q};=)$, then it is decidable which of the conditions (1) and (2) in Theorem 4.6 applies: we can just test whether all valued relations of $\mathfrak{A}$ are improved by* const*, and we can test whether all of them are improved by* inj.

Remark 4.8. *The complexity classification of equality minCSPs from [35], which can be viewed as VCSPs of valued structures where each relation attains only values* 0 *and* 1*, can be obtained as a special case of Theorem 4.6. Suppose that $\mathfrak{A}$ is such a valued structure. If* const $\in$ fPol($\mathfrak{A}$)*, then $\mathfrak{A}$ is constant (in the terminology of [35]) and* VCSP($\mathfrak{A}$) *is in P. If* inj $\in$ fPol($\mathfrak{A}$)*, then it is immediate that $\mathfrak{A}$ is Horn (in the terminology of [35]) and even strictly negative: otherwise, by [35, Lemma 16] we have $(=)^1_0 \in \langle\mathfrak{A}\rangle$. But this is in contradiction to the assumption that* inj $\in$ fPol($\mathfrak{A}$)*, since* inj *applied to a pair of equal elements and pair of distinct elements yields a pair of distinct elements, increasing the cost to* 1 *compared to the average cost* 1/2 *of the input tuples. Otherwise, it follows from Theorem 4.6 that* VCSP($\mathfrak{A}$) *is NP-hard.*

5. Temporal VCSPs

In this section we generalise the classification result from equality VCSPs to temporal VCSPs, which is the main result of this paper.

5.1. **Preliminaries on temporal CSPs.** We first define several important relations on $\mathbb{Q}$ that already played a role in the classification of temporal CSPs [9].

Definition 5.1. *Let*

$$
\begin{aligned}
\mathrm{Betw} &:= \{(x,y,z) \in \mathbb{Q}^3 \mid (x<y<z) \vee (z<y<x)\},\\
\mathrm{Cycl} &:= \{(x,y,z) \in \mathbb{Q}^3 \mid (x<y<z) \vee (y<z<x) \vee (z<x<y)\},\\
\mathrm{Sep} &:= \{(x_1,y_1,x_2,y_2) \in \mathbb{Q}^4 \mid \quad (x_1<x_2<y_1<y_2) \vee (x_1<y_2<y_1<x_2) \vee (y_1<x_2<x_1<y_2)\\
&\qquad \vee (y_1<y_2<x_1<x_2) \vee (x_2<x_1<y_2<y_1) \vee (x_2<y_1<y_2<x_1)\\
&\qquad \vee (y_2<x_1<x_2<y_1) \vee (y_2<y_1<x_2<x_1)\},\\
T_3 &:= \{(x,y,z) \in \mathbb{Q}^3 \mid (x=y<z) \vee (x=z<y)\}.
\end{aligned}
$$

Theorem 5.2 (Theorem 20 in [9])**.** *Let $\mathfrak{A}$ be a relational structure with a finite signature such that* $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$*. Then it satisfies at least one of the following:*

- *$\mathfrak{A}$ primitively positively defines* Betw*,* Cycl*, or* Sep*.*
- const $\in$ Pol($\mathfrak{A}$).
- $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Sym}(\mathbb{Q})$.
- *There is a primitive positive definition of $<$ in $\mathfrak{A}$.*

We need the following operations on $\mathbb{Q}$. By min and max we refer to the binary minimum and maximum operation on the set $\mathbb{Q}$, respectively.

Definition 5.3. *Let $e_{<0}, e_{>0}$ be any endomorphisms of* $(\mathbb{Q};<)$ *satisfying* $e_{<0}(x) < 0$ *and* $e_{>0}(x) > 0$ *for every* $x \in \mathbb{Q}$. *We denote by* $\pi\pi$ *the binary operation on* $\mathbb{Q}$ *defined by*

$$\pi\pi(x,y) = \begin{cases} e_{<0}(x) & x \leq 0, \\ e_{>0}(y) & x > 0. \end{cases}$$

The operation lex *is any binary operation on* $\mathbb{Q}$ *satisfying* $\mathrm{lex}(x,y) < \mathrm{lex}(x',y')$ *iff* $x < x'$, *or* $x = x'$ *and* $y < y'$ *for all* $x, x', y, y' \in \mathbb{Q}$. *We denote by* ll *the binary operation on* $\mathbb{Q}$ *defined by*

$$\mathrm{ll}(x,y) = \begin{cases} \mathrm{lex}(e_{<0}(x), e_{<0}(y)) & x \leq 0, \\ \mathrm{lex}(e_{>0}(y), e_{>0}(x)) & x > 0. \end{cases}$$

Definition 5.4. *Let* $e_<$, $e_=$ *and* $e_>$ *be any endomorphisms of* $(\mathbb{Q};<)$ *satisfying for all* $x, \varepsilon \in \mathbb{Q}$, $\varepsilon > 0$,

$$e_=(x) < e_>(x) < e_<(x) < e_=(x+\varepsilon).$$

We denote by mi *the binary operation on* $\mathbb{Q}$ *defined by*

$$\mathrm{mi}(x,y) = \begin{cases} e_<(x) & x < y, \\ e_=(x) & x = y, \\ e_>(y) & x > y. \end{cases}$$

Definition 5.5. *Let* $e_=$ *and* $e_{\neq}$ *be any endomorphisms of* $(\mathbb{Q};<)$ *satisfying for all* $x, \varepsilon \in \mathbb{Q}$, $\varepsilon > 0$,

$$e_{\neq}(x) < e_=(x) < e_{\neq}(x+\varepsilon).$$

We denote by mx *the binary operation on* $\mathbb{Q}$ *defined by*

$$\mathrm{mx}(x,y) = \begin{cases} e_{\neq}(\min(x,y)) & x \neq y, \\ e_=(x) & x = y. \end{cases}$$

The construction of endomorphisms that appear in Definitions 5.4 and 5.5 can be found for example in [3, Section 12.5]. The following was observed and used in [9].

Lemma 5.6. *If* $\mathfrak{A}$ *is a relational structure such that* $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$ *and* $\mathfrak{A}$ *is preserved by a binary injective operation* f, *then it is also preserved by the operation defined by one of* $\mathrm{lex}(x,y)$, $\mathrm{lex}(-x,y)$, $\mathrm{lex}(x,-y)$, *or* $\mathrm{lex}(-x,-y)$. *In particular, if* f *preserves* $\leq$ *(for example,* ll*), then* $\mathfrak{A}$ *is preserved by* lex.

Definition 5.7. *The* dual *of an operation* $g\colon \mathbb{Q}^k \to \mathbb{Q}$ *is the operation*

$$g^*\colon (x_1,\dots,x_k) \mapsto -g(-x_1,\dots,-x_k).$$

The dual *of a relation* $R \subseteq \mathbb{Q}^\ell$ *is the relation*

$$-R = \{(-a_1,\dots,-a_\ell) \mid (a_1,\dots,a_\ell) \in R\}.$$

Note that $\min^* = \max$ and the relation $-(>)$ is equal to $<$. Statements about operations and relations on $\mathbb{Q}$ can be naturally dualized and we will often refer to the dual version of a statement.

By combining Theorem 50, Corollary 51, Corollary 52 and the accompanying remarks in [9], we obtain the following; see also Theorem 12.10.1 in [3].

Theorem 5.8. *Let* $\mathfrak{A}$ *be a relational structure such that* $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$. *Then exactly one of the following is true.*

(1) *At least one of the operations* const, min, mx, mi, ll, *or one of their duals lies in* $\mathrm{Pol}(\mathfrak{A})$. *In this case, for every reduct* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *with a finite signature,* $\mathrm{CSP}(\mathfrak{A}')$ *is P.*

(2) $\mathfrak{A}$ *primitively positively defines one of the relations* Betw, Cycl, Sep, T_3, $-T_3$, *or* Dis. *In this case,* $\mathfrak{A}$ *has a reduct* $\mathfrak{A}'$ *with a finite signature such that* $\mathrm{CSP}(\mathfrak{A}')$ *is NP-complete.*

Moreover, it is decidable whether (1) or (2) holds.

We also need an alternative version of the classification theorem above.

Theorem 5.9 (Theorem 12.0.1 in [3]; see also Theorem 7.24 in [14])**.** *Let* $\mathfrak{A}$ *be a relational structure such that* $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$. *Then exactly one of the following is true:*

(1) $\mathrm{Pol}(\mathfrak{A})$ *contains a pwnu polymorphism. In this case, for every reduct* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *with a finite signature,* $\mathrm{CSP}(\mathfrak{A}')$ *is in P.*

(2) $\mathrm{Pol}(\mathfrak{A})$ *pp-constructs* K_3. *In this case, there exists a reduct* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *with a finite signature such that* $\mathrm{CSP}(\mathfrak{A}')$ *is NP-complete.*

Proposition 5.10. *Each of the relational structures* $(\mathbb{Q};\mathrm{Betw}), (\mathbb{Q};\mathrm{Cycl}), (\mathbb{Q};\mathrm{Sep}), (\mathbb{Q};T_3)$, *and* $(\mathbb{Q};-T_3)$ *pp-constructs* K_3.

Proof. This is proved in the proof of [3, Theorem 12.0.1]. In fact, the proof shows that each of these structures pp-interprets all finite structures. Since K_3 is finite and a pp-interpretation is a special case of a pp-construction, the statement follows. □

Proposition 5.11 (Proposition 25, 27, and 29 in [9])**.** *Let* $\mathfrak{A}$ *be a relational structure such that* $\mathrm{Pol}(\mathfrak{A})$ *contains* min, mi, *or* mx. *Then* $\mathrm{Pol}(\mathfrak{A})$ *contains* $\pi\pi$.

Proposition 5.12 (Lemma 12.4.4 in [3])**.** *Let* $\mathfrak{A}$ *be a relational structure such that* $\mathrm{Pol}(\mathfrak{A})$ *contains* lex *and* $\pi\pi$. *Then* $\mathrm{Pol}(\mathfrak{A})$ *contains* ll.

Note that there exists $\alpha \in \mathrm{Aut}(\mathbb{Q};<)$ such that, for all $x, y \in \mathbb{Q}$, $\mathrm{lex}^*(x,y) = \alpha(\mathrm{lex}(x,y))$. Hence, whenever $\mathfrak{A}$ is a relational structure such that $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$, then $\mathrm{Pol}(\mathfrak{A})$ contains lex if and only if it contains lex^*. This is relevant for dualising statements like Proposition 5.12. We also need the following result from [5].

Theorem 5.13 (Theorem 5.1 in [5])**.** *Let* $\mathfrak{A}$ *be a relational structure such that* $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$. *If* $\pi\pi \in \mathrm{Pol}(\mathfrak{A})$ *and* $\mathrm{ll} \notin \mathrm{Pol}(\mathfrak{A})$, *then the relation*

$$R^{\mathrm{mix}} = \{(x,y,z) \in \mathbb{Q}^3 \mid (x = y) \vee (z < x \wedge z < y)\}$$

has a primitive positive definition in $\mathfrak{A}$.

5.2. **Expressibility of Valued Relations.** In this section we consider *valued* structures $\mathfrak{A}$ such that $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$, and study expressibility of valued relations in $\mathfrak{A}$. For $\alpha, \beta, \gamma \in \mathbb{Q} \cup \{\infty\}$, define the binary valued relation $R_{\alpha,\beta,\gamma}$ on $\mathbb{Q}$:

$$R_{\alpha,\beta,\gamma}(x,y) := \begin{cases} \alpha & x = y \\ \beta & x < y \\ \gamma & x > y \end{cases}$$

Note that $R_{0,1,1}$ is equal to $(=)^1_0$, $R_{1,0,0}$ is equal to $(\neq)^1_0$, $R_{1,0,1}$ is equal to $(<)^1_0$, and $R_{0,0,1}$ is equal to $(\leq)^1_0$.

Lemma 5.14. *Let* $\mathfrak{A}$ *be a valued structure such that* $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$ *and* $\alpha > \frac{1}{3}$. *If* $\langle\mathfrak{A}\rangle$ *contains* $R_{\alpha,0,1}$, *then* $\mathrm{Cycl} \in \langle\mathfrak{A}\rangle$.

Proof. Note that $\mathrm{Cycl}(x,y,z) = \mathrm{Opt}(R_{\alpha,0,1}(x,y) + R_{\alpha,0,1}(y,z) + R_{\alpha,0,1}(z,x))$. Therefore, $\mathrm{Cycl} \in \langle\mathfrak{A}\rangle$. □

Lemma 5.15. *Let $\mathfrak{A}$ be a valued structure such that $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$. Let $(<)_0^\infty \in \langle\mathfrak{A}\rangle$. Let $\alpha,\beta,\gamma \in \mathbb{Q} \cup \{\infty\}$ be such that*

- *$\alpha < \min(\beta,\gamma) < \infty$, or*
- *$\beta \neq \gamma$ and $\beta,\gamma < \infty$.*

If $R_{\alpha,\beta,\gamma} \in \langle\mathfrak{A}\rangle$, then $\mathrm{Cycl} \in \langle\mathfrak{A}\rangle$.

Proof. Without loss of generality, we may assume that $\beta \leq \gamma$, because otherwise we may consider $R_{\alpha,\gamma,\beta}(x,y) = R_{\alpha,\beta,\gamma}(y,x)$. In the first case of the statement we have that $\alpha < \min(\beta,\gamma) = \beta < \infty$. Then

$$(<)_0^1(x,y) = \frac{1}{\beta-\alpha} \min_z \left(R_{\alpha,\beta,\gamma}(z,x) + (<)_0^\infty(z,y) - \alpha\right).$$

Suppose now that we are not in the first case, i.e., $\alpha \geq \min(\beta,\gamma) = \beta$, and additionally $\beta < \gamma < \infty$ as in the second case of the statement. Then for every $x,y \in \mathbb{Q}$

$$(<)_0^1(x,y) = \frac{1}{\gamma-\beta} \min_z \left(R_{\alpha,\beta,\gamma}(x,z) + (<)_0^\infty(z,y) - \beta\right).$$

Therefore, in both cases, $(<)_0^1 \in \langle\mathfrak{A}\rangle$. Since $(<)_0^1$ is equal to $R_{1,0,1}$, the statement follows from Lemma 5.14. □

Lemma 5.16. *Let $\mathfrak{A}$ be a valued structure such that $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A}) \neq \mathrm{Sym}(\mathbb{Q})$. If $\mathrm{const} \notin \mathrm{fPol}(\mathfrak{A})$, then $\langle\mathfrak{A}\rangle$ contains Betw, Cycl, Sep, or $(<)_0^\infty$.*

Proof. By Lemma 4.2, $\langle\mathfrak{A}\rangle$ contains $(<)_0^\infty$ or $(\neq)_0^\infty$. If $(<)_0^\infty \in \langle\mathfrak{A}\rangle$, then we are done. Assume therefore $(\neq)_0^\infty \in \langle\mathfrak{A}\rangle$. Let R be a valued relation of $\mathfrak{A}$ of arity k such that there exists an orbit O of the action of $\mathrm{Sym}(\mathbb{Q})$ on $\mathbb{Q}^k$ and $s,t \in O$ with $R(s) < R(t)$. Let $s \in O$ be such that $R(s)$ is minimal. Note that O is not the orbit of constant tuples, because $\mathrm{Sym}(\mathbb{Q})$ is transitive.

Consider the crisp relation $S \in \langle\mathfrak{A}\rangle_0^\infty$ defined by

$$S(x_1,\dots,x_k) := \mathrm{Opt}\left(R(x_1,\dots,x_k) + \sum_{(p,q)\in E_s} (=)_0^\infty(x_p,x_q) + \sum_{(p,q)\in N_s} (\neq)_0^\infty(x_p,x_q)\right).$$

Clearly, $s \in S$. Note that a tuple $u \in \mathbb{Q}^k$ lies in O if and only if $E_s \subseteq E_u$ and $N_s \subseteq N_u$. In particular, $S \subseteq O$. Since $R(s) < R(t)$, we have $t \notin S$. It follows that S is not preserved by $\mathrm{Sym}(\mathbb{Q})$. Moreover, S is not preserved by const, because O is not the orbit of constant tuples. By Theorem 5.2, the relational structure $(\mathbb{Q};S)$ admits a primitive positive definition of Betw, Cycl or Sep, or a primitive positive definition of $<$. Since $S \in \langle\mathfrak{A}\rangle$, the statement of the lemma follows. □

In Lemma 5.19 below we present a polynomial-time algorithm for VCSPs of valued structures $\mathfrak{A}$ improved by lex provided that $\mathfrak{A}$ cannot express any crisp relation that prevents tractability. In fact, to check whether the algorithm can be applied, it suffices to check whether a certain structure $\hat{\mathfrak{A}}$ with a finite signature has a tractable CSP, instead of considering all relations in $\langle\mathfrak{A}\rangle_0^\infty$. We define $\hat{\mathfrak{A}}$ below.

Definition 5.17. *Let A be a set and let R be a valued relation on A of arity k. Let $\ell \in \mathbb{N}$, $\ell \leq k$, and let $\sigma\colon [k] \to [\ell]$ be a map. Then R_σ is the valued relation on A of arity ℓ defined by $R_\sigma(x_1,\dots,x_\ell) = R(x_{\sigma(1)},\dots,x_{\sigma(k)})$ for all $x_1,\dots,x_\ell \in A$. If S is a valued relation of some arity $\ell \leq k$ such that there exists $\sigma\colon [k] \to [\ell]$ and $S = R_\sigma$, we call S a* minor *of R.*

Let $\mathfrak{A}$ be a valued τ-structure such that $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$*. Then $\hat{\mathfrak{A}}$ denotes the relational structure with domain $\mathbb{Q}$ which contains the relations* $\mathrm{Feas}(R^{\mathfrak{A}})$ *and* $\mathrm{Opt}((R^{\mathfrak{A}})_\sigma)$ *for every $R \in \tau$ of arity k, $\ell \leq k$, and $\sigma\colon [k] \to [\ell]$.*

Note that $R_\sigma \in \langle (A; R)\rangle$ for every valued relation R of arity k and every $\sigma\colon [k] \to [\ell]$.

Remark 5.18. *Note that we do not need to include relations of the form* $\mathrm{Feas}((R^{\mathfrak{A}})_\sigma)$ *in $\hat{\mathfrak{A}}$, because for every valued relation R on A of arity k and $\sigma\colon [k] \to [\ell]$, we have*

$$\mathrm{Feas}(R_\sigma) = \mathrm{Feas}(R)_\sigma$$

and therefore $\mathrm{Feas}(R_\sigma) \in \langle (A; \mathrm{Feas}(R))\rangle$.

Lemma 5.19. *Let $\mathfrak{A}$ be a valued structure over a finite signature such that* $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$*. Suppose that* $\mathrm{lex} \in \mathrm{fPol}(\mathfrak{A})$ *and that $\hat{\mathfrak{A}}$ is preserved by one of the operations* const*,* min*,* mx*,* mi*,* ll *or one of their duals. Then* $\mathrm{VCSP}(\mathfrak{A})$ *is in P.*

Proof. If $\mathrm{const} \in \mathrm{fPol}(\mathfrak{A})$, then $\mathrm{VCSP}(\mathfrak{A})$ is in P by Lemma 3.1. We may therefore assume that $\mathrm{const} \notin \mathrm{fPol}(\mathfrak{A})$. Let $R \in \langle \mathfrak{A}\rangle$ be of arity k. Since $\mathrm{lex} \in \mathrm{fPol}(\mathfrak{A})$, it improves R. Therefore, for every injective tuple $s \in \mathbb{Q}^k$ and any $t \in \mathbb{Q}^k$, it holds that

$$R(s) = R(\mathrm{lex}(s,t)) \leq 1/2 \cdot (R(s) + R(t)),$$

where the first equality follows from s and $\mathrm{lex}(s,t)$ being in the same orbit of $\mathrm{Aut}(\mathfrak{A})$. Therefore, if $R(s) < \infty$, then $R(s) \leq R(t)$. In particular, there is $m_R \in \mathbb{Q}$ such that for every injective tuple $s \in \mathbb{Q}^k$, we have $R(s) = m_R$ or $R(s) = \infty$. Note that if there is at least one injective tuple s with $R(s) = m_R$, then $\mathrm{Opt}(R)$ is the crisp relation that consists of all the tuples t such that $R(t) = m_R$.

Let (ϕ, u) be an instance of $\mathrm{VCSP}(\mathfrak{A})$ with variable set $V = \{v_1, \dots, v_N\}$. Note that ϕ interpreted over $\mathrm{Feas}(\mathfrak{A})$ can be seen as an instance of $\mathrm{CSP}(\mathrm{Feas}(\mathfrak{A}))$ where each summand $R(x_1, \dots, x_k)$ of ϕ is interpreted as $\mathrm{Feas}(R^{\mathfrak{A}})(x_1, \dots, x_k)$. By the assumption on $\hat{\mathfrak{A}}$, $\mathrm{Feas}(\mathfrak{A})$ is preserved by one of the operations const, min, mx, mi, ll, or one of their duals. Since $\mathrm{lex} \in \mathrm{fPol}(\mathfrak{A})$, by Lemma 2.23 and Remark 2.22, $\mathrm{lex} \in \mathrm{Pol}(\mathrm{Feas}(\mathfrak{A}))$. Since $\mathrm{const} \notin \mathrm{fPol}(\mathfrak{A})$, by Lemma 4.2, $\mathrm{const} \notin \mathrm{Pol}(\mathrm{Feas}(\mathfrak{A}))$. Then min, mx, mi, or one of their duals preserves $\mathrm{Feas}(\mathfrak{A})$, and, by Proposition 5.11, $\pi\pi \in \mathrm{Pol}(\mathrm{Feas}(\mathfrak{A}))$ or $\pi\pi^* \in \mathrm{Pol}(\mathrm{Feas}(\mathfrak{A}))$. Therefore, by Proposition 5.12, we always have that ll or ll^* preserves $\mathrm{Feas}(\mathfrak{A})$. Hence, by Theorem 5.8, $\mathrm{CSP}(\mathrm{Feas}(\mathfrak{A}))$ is solvable in polynomial time and we can use the polynomial-time algorithm from [9] based on the operation ll or ll^* to solve $\mathrm{CSP}(\mathrm{Feas}(\mathfrak{A}))$. If ϕ, viewed as a primitive positive formula, is not satisfiable over $\mathrm{Feas}(\mathfrak{A})$, then the minimum of ϕ is above every rational threshold and (ϕ, u) is rejected. Otherwise, we may compute the set $E \subseteq V^2$ of all pairs (x, y) such that $f(x) = f(y)$ in every solution of $f\colon V \to \mathbb{Q}$ of ϕ over $\mathrm{Feas}(\mathfrak{A})$ (we may assume without loss of generality that $\mathrm{Feas}(\mathfrak{A})$ contains the relation $(\neq)_0^\infty$; since $\mathrm{Feas}(\mathfrak{A})$ is preserved by lex it suffices to test the unsatisfiability of $\phi \wedge x \neq y$ for each of these pairs). It follows from the definition of Feas that for every $g\colon V \to \mathbb{Q}$, if ϕ evaluates to a finite value in $\mathfrak{A}$ under the assignment g, then $g(x) = g(y)$ for every $(x, y) \in E$. Moreover, for every $(x, y) \in V^2 \setminus E$, there exists $g\colon V \to \mathbb{Q}$ such that ϕ evaluates to a finite value under g and $g(x) \neq g(y)$.

We create a new τ-expression ϕ' from ϕ by replacing each occurrence of v_j by v_i for every $(v_i, v_j) \in E$ such that $i < j$. Let V' be the set of variables of ϕ'. By the discussion above, the minimum for ϕ' over $\mathfrak{A}$ equals the minimum for ϕ. Moreover, for every $(x, y) \in (V')^2$, there exists $g'\colon V' \to \mathbb{Q}$ such that ϕ' evaluates to a finite value under g' and $g'(x) \neq g'(y)$. Let $\phi' := \phi'_1 + \dots + \phi'_n$ where for every $j \in [n]$ the summand ϕ'_j is an atomic τ-expression. We execute the following procedure for each $j \in [n]$. Let $\phi'_j = R(x_1, \dots, x_k)$. Let $y^j_1, \dots, y^j_{\ell_j}$ be an enumeration of all distinct variables that appear in $\{x_1, \ldots, x_k\}$ and let S_j be a valued relation of arity ℓ defined by $S_j(y_1^j, \ldots, y_{\ell_j}^j) = R(x_1, \ldots, x_k)$. Clearly, S_j is a minor of R. Note that the relation S_j might be different for every summand, even if they contain the same relation symbol R, due to possibly different variable identifications. Observe that, by the properties of ϕ', there exists an injective tuple $s^j \in \mathbb{Q}^{\ell_j}$ such that $S_j(s^j)$ is finite. Note that $S_j \in \langle \mathfrak{A} \rangle$, and let $m_j := m_{S_j}$. By the discussion in the beginning of the proof, $S_j(s^j) = m_j$ and $\mathrm{Opt}(S_j) \in \langle \mathfrak{A} \rangle_0^\infty$ consists of all tuples that evaluate to m_j in S_j. Since S_j attains only finitely many values, we can identify m_j in polynomial time for every j.

Let $\mathfrak{B}$ be the relational structure with domain $\mathbb{Q}$ and relations $\mathrm{Opt}(S_1), \ldots, \mathrm{Opt}(S_n)$. Let ψ be the instance of CSP($\mathfrak{B}$) obtained from ϕ' by replacing the summand ϕ'_j by $\mathrm{Opt}(S_j)(y_1^j, \ldots, y_{\ell_j}^j)$ for all $j \in [n]$; all relations in ψ are crisp and hence it can be seen as a primitive positive formula. Note that the variable set of ψ is equal to V'. By assumption, $\hat{\mathfrak{A}}$ is preserved by one of the operations const, min, mx, mi, ll, or one of their duals and, in particular, $\mathfrak{B}$ is preserved by one of them. Hence, CSP($\mathfrak{B}$) is in P by Theorem 5.8. Therefore, the satisfiability of ψ over $\mathfrak{B}$ can be tested in polynomial time. We claim that if ψ is unsatisfiable, then the minimum of ϕ is above every rational threshold and the algorithm rejects.

We prove the claim by contraposition. Suppose that the minimum of ϕ over $\mathfrak{A}$ is finite. Then the minimum of ϕ' over $\mathfrak{A}$ is finite and hence there exists $f' \colon V' \to \mathbb{Q}$ such that ϕ' evaluates to a finite value under f'. From all f' with this property, choose f' with the property that $f'(x) \neq f'(y)$ holds for as many pairs $(x, y) \in (V')^2$ as possible. We first show that f' is in fact injective. Suppose that there are $v, w \in V'$ such that $f'(v) = f'(w)$. Let $g' \colon V' \to \mathbb{Q}$ be such that ϕ' evaluates to a finite value under g' and $g'(v) \neq g'(w)$; recall that such g' must exist by the construction of ϕ'. Consider the assignment $\mathrm{lex}(f', g') \colon V' \to \mathbb{Q}$ and note that $\mathrm{lex}(f', g')(x) \neq \mathrm{lex}(f', g')(y)$ holds for all pairs (x, y) such that $f'(x) \neq f'(y)$ and also $\mathrm{lex}(f', g')(v) \neq \mathrm{lex}(f', g')(w)$. Moreover, ϕ' evaluates to a finite value under $\mathrm{lex}(f', g')$: for every $j \in [n]$, if ϕ'_j is of the form $R(x_1, \ldots, x_k)$, then, since $\mathrm{lex} \in \mathrm{fPol}(\mathfrak{A})$,

$$R(\mathrm{lex}(f', g')(x_1, \ldots, x_k)) \leq 1/2 \cdot (R(f'(x_1, \ldots x_k)) + R(g'(x_1, \ldots, x_k))) < \infty.$$

This contradicts our choice of f'. Therefore, f' is injective.

Note that for every $j \in [n]$ we have $S_j(f'(y_1^j), \ldots f'(y_{\ell_j}^j)) < \infty$, because ϕ'_j evaluates to a finite value under f'. Since $(f'(y_1^j), \ldots f'(y_{\ell_j}^j))$ is an injective tuple, this implies $S_j(f'(y_1^j), \ldots f'(y_{\ell_j}^j)) = m_j$ and $(f'(y_1^j), \ldots f'(y_{\ell_j}^j)) \in \mathrm{Opt}(S_j)$ for every $j \in [n]$. It follows that f' is a satisfying assignment to ψ. Therefore, we proved that whenever ψ unsatisfiable, the algorithm correctly rejects, because there is no assignment to ϕ of finite cost.

Finally, suppose that there exists a solution $h' \colon V' \to \mathbb{Q}$ to the instance ψ of CSP($\mathfrak{B}$). Then, for every $j \in [n]$, ϕ'_j takes under h' the value $S_j(h'(y_1^j), \ldots, h'(y_{\ell_j}^j))$. By the definition of Opt, $(h'(y_1^j), \ldots, h'(y_{\ell_j}^j))$ minimizes S_j and therefore h' minimizes ϕ'_j. It follows that h' minimizes ϕ' and that the cost of ϕ' under h' is equal to $m_1 + \cdots + m_n$. Since the minimum of ϕ' is equal to the minimum of ϕ, the algorithm accepts if $m_1 + \cdots + m_n \leq u$ and rejects otherwise. This completes the algorithm and its correctness proof. It follows that VCSP($\mathfrak{A}$) is in P. □

Lemma 5.20. *Let $\mathfrak{A}$ be a valued τ-structure such that* $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Aut}(\mathbb{Q}; <)$ *and* $(<)_0^\infty \in \langle \mathfrak{A} \rangle$. *Suppose that $\mathfrak{A}$ is not essentially crisp. Then one of the following holds:*

- $\mathrm{Cycl} \in \langle \mathfrak{A} \rangle$,
- $(\neq)_0^1 \in \langle \mathfrak{A} \rangle$, *or*

- $R_{1,0,\infty} \in \langle \mathfrak{A} \rangle$.

Proof. Let R be a valued relation of $\mathfrak{A}$ of arity k that attains at least two finite values. Let $m, \ell \in \mathbb{Q}$, with $m < \ell$, be the two smallest finite values attained by R. Let $t \in \mathbb{Q}^k$ be such that $R(t) = \ell$. Choose $s \in \mathrm{Opt}(R)$ so that $|(E_s \cap E_t) \cup (O_s \cap O_t)|$ is maximal (recall Definition 2.26). Clearly, $R(s) = m$.

Let $\sim \subseteq (\mathbb{Q}^2)^2$ be the equivalence relation with the classes $=$, $<$, and $>$. Since $R(s) \neq R(t)$, there exist distinct i, j such that $(s_i, s_j) \not\sim (t_i, t_j)$. For the sake of notation, assume that $(i, j) = (1, 2)$. Let

$$\phi(x_1, x_2) := \min_{x_3,\dots,x_k} \left(R(x_1, \dots, x_k) + \sum_{(p,q) \in E_s \cap E_t} (=)_0^\infty(x_p, x_q) + \sum_{(p,q) \in O_s \cap O_t} (<)_0^\infty(x_p, x_q) \right).$$

Observe that $\phi(x, y) \geq m$ for all $x, y \in \mathbb{Q}$ and hence whenever $(x, y) \sim (s_1, s_2)$ we have $\phi(x, y) = m$. Let $(x, y) \sim (t_1, t_2)$. Then $\phi(x, y) \leq \ell$. By the choice of s, there is no $s' \in \mathrm{Opt}(R)$ that satisfies $(s'_1, s'_2) \sim (t_1, t_2)$, $(E_s \cap E_t) \subseteq E_{s'}$ and $(O_s \cap O_t) \subseteq O_{s'}$. Therefore, $\phi(x, y) > m$. It follows that $\phi(x, y) = \ell$.

Let

$$S(x, y) := \frac{1}{\ell - m}(\phi(x, y) - m).$$

By the construction, $S \in \langle \mathfrak{A} \rangle$, $S(x, y) = 0$ for $(x, y) \sim (s_1, s_2)$, and $S(x, y) = 1$ for $(x, y) \sim (t_1, t_2)$. Note that $\mathrm{Aut}(\mathbb{Q}; <)$ has three orbits of pairs, two of which are represented by (s_1, s_2) and (t_1, t_2). Let $(u_1, u_2) \in \mathbb{Q}^2$ be a representative of the third orbit and let $\alpha = S(u_1, u_2)$. It follows that S is equal to one of the relations $R_{0,1,\alpha}$, $R_{0,\alpha,1}$, $R_{1,0,\alpha}$, $R_{1,\alpha,0}$, $R_{\alpha,0,1}$ or $R_{\alpha,1,0}$. By the choice of m and ℓ, we have that $\alpha = 0$ or $\alpha \geq 1$. By Lemma 5.15, this implies that $\mathrm{Cycl} \in \langle \mathfrak{A} \rangle$ unless $S = R_{1,0,0}$, $S = R_{1,0,\infty}$, or $S = R_{1,\infty,0}$. Since $R_{1,0,0}$ is equal to $(\neq)_0^1$ and $R_{1,0,\infty}(x, y) = R_{1,\infty,0}(y, x)$, the statement follows. ☐

Lemma 5.21. *Let $\mathfrak{A}$ be a valued structure with* $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Aut}(\mathbb{Q}; <)$ *and* $(<)_0^\infty \in \langle \mathfrak{A} \rangle$. *Suppose that* $\mathrm{lex} \notin \mathrm{Pol}(\mathbb{Q}; \langle \mathfrak{A} \rangle_0^\infty)$ *and that $\mathfrak{A}$ is not essentially crisp. Then $\langle \mathfrak{A} \rangle$ contains one of the relations* Betw*,* Cycl*,* Sep*,* T_3*,* $-T_3$*, or* Dis*.*

Proof. Let $\mathfrak{A}' := (\mathbb{Q}; \langle \mathfrak{A} \rangle_0^\infty)$. Note that $\mathrm{const} \notin \mathrm{Pol}(\mathfrak{A}')$, because const does not preserve $(<)_0^\infty$. If $\langle \mathfrak{A} \rangle_0^\infty$ contains one of the relations Betw, Cycl, Sep, T_3, $-T_3$, or Dis, then we are done. Suppose that this is not the case. Then $\mathrm{Pol}(\mathfrak{A}')$ contains min, mx, mi, ll, or one of their duals by Theorem 5.8. Suppose first that $\mathrm{Pol}(\mathfrak{A}')$ contains min, mx, mi, or ll. Since $\mathrm{lex} \notin \mathrm{Pol}(\mathfrak{A}')$, we have $\mathrm{ll} \notin \mathrm{Pol}(\mathfrak{A}')$ (Lemma 5.6). By Proposition 5.11, $\mathrm{Pol}(\mathfrak{A}')$ contains $\pi\pi$. By Theorem 5.13, $\mathfrak{A}'$ primitively positively defines, equivalently, contains the relation R^{mix}. By Lemma 5.20, we have that $\langle \mathfrak{A} \rangle$ contains Cycl, $(\neq)_0^1$, or $R_{1,0,\infty}$. If $\mathrm{Cycl} \in \langle \mathfrak{A} \rangle$, then we are done. Suppose therefore that $(\neq)_0^1 \in \langle \mathfrak{A} \rangle$ or $R_{1,0,\infty} \in \langle \mathfrak{A} \rangle$. Note that for every $x, y \in \mathbb{Q}$, we have

$$(<)_0^1(x, y) = \min_z \left(R^{\mathrm{mix}}(y, z, x) + (\neq)_0^1(y, z) \right) = \min_z \left(R^{\mathrm{mix}}(y, z, x) + R_{1,0,\infty}(y, z) \right).$$

Indeed, if $x < y$, then by choosing $z > y$ we get $R^{\mathrm{mix}}(y, z, x) + (\neq)_0^1(y, z) = R^{\mathrm{mix}}(y, z, x) + R_{1,0,\infty}(y, z) = 0$, which is clearly the minimal value that can be obtained. If $x \geq y$, then by choosing $z = y$ we get $R^{\mathrm{mix}}(y, z, x) + (\neq)_0^1(y, z) = R^{\mathrm{mix}}(y, z, x) + R_{1,0,\infty}(y, z) = 1$, which is clearly the minimal value, because if $z \neq y$ we obtain $R^{\mathrm{mix}}(y, z, x) + (\neq)_0^1(y, z) = R^{\mathrm{mix}}(y, z, x) + R_{1,0,\infty}(y, z) = \infty$.

It follows that $(<)_0^1 \in \langle \mathfrak{A} \rangle$. Observe that $(<)_0^1$ equals $R_{1,0,1}$. Therefore, $\mathrm{Cycl} \in \langle \mathfrak{A} \rangle$ by Lemma 5.14, as we wanted to prove. If $\mathrm{Pol}(\mathfrak{A}')$ contains min^*, mx^*, mi^*, or ll^*, we use the dual versions of Proposition 5.11 and Theorem 5.13 to analogously prove that $(>)^1_0 \in \langle \mathfrak{A} \rangle$. Since $(<)^1_0(x,y) = (>)^1_0(y,x)$ we obtain Cycl $\in \langle \mathfrak{A} \rangle$ by Lemma 5.14. □

Lemma 5.22. *Let $\mathfrak{A}$ be a valued structure with* $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Aut}(\mathbb{Q}; <)$ *and* $(<)^\infty_0 \in \langle \mathfrak{A} \rangle$. *Suppose that* lex $\notin \mathrm{fPol}(\mathfrak{A})$ *and* lex $\in \mathrm{Pol}(\mathbb{Q}; \langle \mathfrak{A} \rangle^\infty_0)$. *Then* Cycl $\in \langle \mathfrak{A} \rangle$.

Proof. Let R be a valued relation of $\mathfrak{A}$ of arity k that is not improved by lex. Then there exist $s, t \in \mathbb{Q}^k$ such that

$$R(s) + R(t) < 2R(\mathrm{lex}(s,t)).$$

In particular, $R(s), R(t) < \infty$. Since $\mathrm{Feas}(R) \in \langle \mathfrak{A} \rangle^\infty_0$ is improved by lex, we have $R(\mathrm{lex}(s,t)) < \infty$. Let $u := \mathrm{lex}(s,t)$. Note that we must have $R(s) < R(u)$ or $R(t) < R(u)$. Moreover, $E_u = E_s \cap E_t$. Let $v \in \{s,t\}$ be such that $R(v) < R(u) < \infty$. Note that we have $E_u \subseteq E_v$. Let O be a maximal subset of O_u such that there exists $w \in \mathbb{Q}^k$ satisfying

- $R(w) \leq R(v)$,
- $E_u \subseteq E_w$, and
- $O \subseteq O_w$,

and let w be any such witness for O. Such a maximal set O must exist, because v satisfies these conditions for $O = \emptyset$.

Since $R(w) \neq R(u)$ and $E_u \subseteq E_w$, there exist $i, j \in [k]$ such that $w_i \leq w_j$ and $u_i > u_j$. Without loss of generality me may assume $(i,j) = (1,2)$, because otherwise we permute the entries of R. Let

$$\phi(x_1, x_2) := \min_{x_3,\dots,x_k} \left(R(x_1,\dots,x_k) + \sum_{(p,q) \in E_u} (=)^\infty_0(x_p, x_q) + \sum_{(p,q) \in O} (<)^\infty_0(x_p,x_q) \right). \tag{7}$$

Let $a, b \in \mathbb{Q}$ such that $a < b$. Then $\phi(b,a) = \phi(u_1,u_2) \leq R(u)$, because $O \subseteq O_u$. Suppose that $\phi(b,a) \leq R(w)$. Then there exists $w' \in \mathbb{Q}^k$ that realizes the minimum in (7) and hence $\phi(b,a) = R(w') \leq R(w) \leq R(v)$ and $w'_1 > w'_2$. In particular, the sums in (7) are finite. Therefore, $O \cup \{(2,1)\} \subseteq O_{w'}$ and $E_u \subseteq E_{w'}$. Since $(2,1) \in O_u \setminus O$, this contradicts the choice of O and w. Therefore, $\phi(b,a) > R(w)$. Note that $\phi(w_1,w_2) \leq R(w)$. If $w_1 < w_2$, then $\phi(a,b) \leq R(w)$ and ϕ expresses $R_{\alpha,\beta,\gamma}$ where $\beta = \phi(a,b)$ and $\gamma = \phi(b,a)$. In particular, $\beta < \gamma < \infty$. Therefore, by Lemma 5.15, Cycl $\in \langle \mathfrak{A} \rangle$. Otherwise we have $w_1 = w_2$. Then $\phi(a,a) \leq R(w)$ and ϕ expresses $R_{\alpha,\beta,\gamma}$ where $\alpha = \phi(a,a) \leq R(w) < \phi(b,a) = \gamma$. If $\beta \geq \gamma$, then $\alpha < \min(\beta,\gamma)$, and otherwise $\beta < \gamma < \infty$. In both cases, Cycl $\in \langle \mathfrak{A} \rangle$ by Lemma 5.15. □

5.3. **Classification.** We are now ready to prove the complexity dichotomy for temporal VCSPs. We first phrase the classification with 4 cases, where we distinguish between the tractable cases that are based on different algorithms. As a next step, we formulate two corollaries each of which provides two concise mutually disjoint conditions that correspond to NP-completeness and polynomial-time tractability, respectively.

Theorem 5.23. *Let $\mathfrak{A}$ be a valued structure such that* $\mathrm{Aut}(\mathbb{Q}; <) \subseteq \mathrm{Aut}(\mathfrak{A})$. *Then at least one of the following holds:*

(1) $\langle \mathfrak{A} \rangle$ *contains one of the relations* Betw, Cycl, Sep, T_3 *(see Definition 5.1),* $-T_3$, *or* Dis *(see* (5)*). In this case, $\mathfrak{A}$ has a reduct $\mathfrak{A}'$ over a finite signature such that* VCSP$(\mathfrak{A}')$ *is NP-complete.*
(2) const $\in \mathrm{fPol}(\mathfrak{A})$. *In this case, for every reduct $\mathfrak{A}'$ of $\mathfrak{A}$ over a finite signature,* VCSP$(\mathfrak{A}')$ *is in P.*

(3) lex $\in$ fPol($\mathfrak{A}$) *and* Pol($\hat{\mathfrak{A}}$) *contains* min, mx, mi, ll, *or one of their duals. In this case, for every reduct* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *over a finite signature,* VCSP($\mathfrak{A}'$) *is in P.*

(4) $\pi_1^2 \in$ fPol($\mathfrak{A}$) *and* fPol($\mathfrak{A}$) *contains* min, mx, mi, ll, *or one of their duals. In this case, for every reduct* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *over a finite signature,* VCSP($\mathfrak{A}'$) *is in P.*

Proof. Note that for every reduct $\mathfrak{A}'$ of $\mathfrak{A}$, the automorphism group Aut($\mathfrak{A}'$) contains Aut($\mathfrak{A}$) and hence is oligomorphic. If $\langle\mathfrak{A}\rangle$ contains one of the relations Betw, Cycl, Sep, T_3, $-T_3$, or Dis, then there is a reduct $\mathfrak{A}'$ of $\mathfrak{A}$ over a finite signature such that VCSP($\mathfrak{A}'$) is NP-hard by Lemma 2.7 and Theorem 5.8. By Theorem 2.25, VCSP($\mathfrak{A}'$) is in NP, therefore it is NP-complete. If const $\in$ fPol($\mathfrak{A}$), then const $\in$ fPol($\mathfrak{A}'$) for every reduct $\mathfrak{A}'$ of $\mathfrak{A}$ over a finite signature, and VCSP($\mathfrak{A}'$) is in P by Lemma 3.1. Suppose therefore that const $\notin$ fPol($\mathfrak{A}$) and that $\langle\mathfrak{A}\rangle$ does not contain any of the relations Betw, Cycl, Sep, T_3, $-T_3$, or Dis. By Lemma 4.2, $\langle\mathfrak{A}\rangle$ contains $(\neq)_0^\infty$ or $(<)_0^\infty$, and hence const $\notin$ Pol($\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty$). Recall that $\langle\mathfrak{A}\rangle_0^\infty$ contains all relations primitively positively definable in ($\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty$) (Remark 2.9). By Theorem 5.8, Pol($\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty$) (and thus Pol($\hat{\mathfrak{A}}$)) contains min, mx, mi, ll, or one of their duals.

Let $\mathfrak{A}'$ be a reduct of $\mathfrak{A}$ with a finite signature. If Aut($\mathfrak{A}$) = Sym($\mathbb{Q}$), then by Theorem 4.6, inj $\in$ fPol($\mathfrak{A}$) $\subseteq$ fPol($\mathfrak{A}'$) and VCSP($\mathfrak{A}'$) is in P. By Lemma 5.6, and since Aut($\mathfrak{A}$) contains $x \mapsto -x$, we have lex $\in$ fPol($\mathfrak{A}$) and therefore satisfy (3).

Finally, suppose that Aut($\mathfrak{A}$) $\neq$ Sym($\mathbb{Q}$). By Lemma 5.16 we have $(<)_0^\infty \in \langle\mathfrak{A}\rangle$, and hence Aut($\mathfrak{A}$) = Aut($\mathbb{Q}; <$). By Lemma 5.21 we have that $\mathfrak{A}$ is essentially crisp or lex $\in$ Pol($\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty$). If $\mathfrak{A}$ is not essentially crisp, we have lex $\in$ Pol($\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty$), and lex $\in$ fPol($\mathfrak{A}$) $\subseteq$ fPol($\mathfrak{A}'$) by Lemma 5.22. Then VCSP($\mathfrak{A}'$) is in P by Lemma 5.19 and Condition (3) holds. Suppose that $\mathfrak{A}$ is essentially crisp. Then by Lemma 3.3 we have $\pi_1^2 \in$ fPol($\mathfrak{A}$). Since const $\notin$ fPol($\mathfrak{A}$), we have const $\notin$ Pol(Feas($\mathfrak{A}$)) (see Remark 2.24). Since $\langle\mathfrak{A}\rangle$ does not contain any of the relations Betw, Cycl, Sep, T_3, $-T_3$, or Dis, none of these relations are primitively positively definable in Feas($\mathfrak{A}$). By Theorem 5.8, Pol(Feas($\mathfrak{A}$)) $\subseteq$ Pol(Feas($\mathfrak{A}'$)) contains min, mx, mi, ll, or one of their duals and CSP(Feas($\mathfrak{A}'$)) is in P. By Remark 2.9, VCSP($\mathfrak{A}'$) is in P. By Remark 2.24, fPol($\mathfrak{A}$) contains min, mx, mi, ll, or one of their duals. Therefore, (4) holds. ☐

Recall from Section 2.3 that a valued structure $\mathfrak{A}$ with Aut($\mathbb{Q}; <$) $\subseteq$ Aut($\mathfrak{A}$) has a (quantifier-free) first-order definition in Aut($\mathbb{Q}; <$) with the defining formulas being disjunctions of conjunctions of atomic formulas over ($\mathbb{Q}; <$). We continue by proving that the complexity dichotomy we gave in Theorem 5.23 is decidable, using the representation of $\mathfrak{A}$ by a first-order definition in ($\mathbb{Q}; <$) of the form described above.

Remark 5.24. *We also obtain decidability if arbitrary first-order formulas may be used for defining the valued relations, because every first-order formula can be effectively transformed into such a formula. This holds more generally over so-called* finitely bounded homogeneous *structures; see, e.g., [36, Proposition 7]. Without the finite boundedness assumption, the problem can become undecidable [13].*

Proposition 5.25. *Given a first-order definition of a valued structure* $\mathfrak{A}$ *with a finite signature in* ($\mathbb{Q}; <$)*, it is decidable whether* VCSP($\mathfrak{A}$) *is in P or NP-complete.*

Proof. Recall that if $\mathfrak{A}$ has a first-order definition in ($\mathbb{Q}; <$), then Aut($\mathbb{Q}; <$) $\subseteq$ Aut($\mathfrak{A}$) and, in particular, VCSP($\mathfrak{A}$) is in NP by Theorem 2.25. If P = NP, then the decision problem is trivial. Suppose that P $\neq$ NP. Then in the statement of Theorem 5.23, item (1) and the union of (2)–(4) is disjoint. Since $\mathfrak{A}$ has a finite signature, we can decide whether const improves $\mathfrak{A}$, i.e., whether (2) holds. Similarly, we can decide whether lex improves $\mathfrak{A}$. By the last sentence of Theorem 5.8

applied to $\hat{\mathfrak{A}}$ we can decide whether one of the operations min, mx, mi, ll, or one of their duals preserves $\hat{\mathfrak{A}}$ (recall that by Lemma 4.2 we have that const improves $\mathfrak{A}$ if and only if it preserves $\hat{\mathfrak{A}}$). Therefore, we can decide whether (3) holds. Finally, we can decide whether π_1^2 improves $\mathfrak{A}$. If yes, $\mathfrak{A}$ is essentially crisp by Lemma 3.3. In this case fPol($\mathfrak{A}$) contains min, mx, mi, ll, or one of their duals if and only if Pol(Feas($\mathfrak{A}$)) does (Remark 2.22 and 2.24), which can be decided by Theorem 5.8. It follows that we can decide whether union of (2)–(4) holds, which implies the statement. □

We reformulate Theorem 5.23 with two mutually exclusive cases that capture the respective complexities of the VCSPs.

Corollary 5.26. *Let* $\mathfrak{A}$ *be a valued structure with a finite signature such that* $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$. *Then exactly one of the following holds.*

(1) $\langle\mathfrak{A}\rangle$ *contains one of the relations* Betw, Cycl, Sep, T_3, $-T_3$, *or* Dis. *In this case,* VCSP($\mathfrak{A}$) *is NP-complete.*
(2) $(\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty)$ *is preserved by one of the operations* const, min, mx, mi, ll, *or one of their duals. In this case,* VCSP($\mathfrak{A}$) *is in P.*

Proof. Let $\mathfrak{A}' := (\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty)$. Theorem 5.8 states that either $\mathfrak{A}'$ primitively positively defines one of the relations Betw, Cycl, Sep, T_3, $-T_3$, Dis, or Pol($\mathfrak{A}'$) contains const, min, mx, mi, ll, or one of their duals. Clearly, $\mathfrak{A}'$ primitively positively defines a relation R is and only if $R \in \langle\mathfrak{A}\rangle_0^\infty$, which is the case if and only if $R \in \langle\mathfrak{A}\rangle$.

It remains to discuss the implications for the complexity of VCSP($\mathfrak{A}$). If (1) holds, then VCSP($\mathfrak{A}$) is NP-complete by Theorem 5.23. On the other hand, if (1) does not hold, one of the cases (2)–(4) in Theorem 5.23 applies and VCSP($\mathfrak{A}$) is in P. □

Note that the corollary above implies that if $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$, then the complexity of VCSP($\mathfrak{A}$) is up to polynomial-time reductions determined by the complexity of the crisp relations $\mathfrak{A}$ can express. Loosely speaking, the complexity of such a VCSP is determined solely by the CSPs that can be encoded in this VCSP. We formulate an alternative and more concise variant of the previous result.

Corollary 5.27. *Let* $\mathfrak{A}$ *be a valued structure with a finite signature such that* $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$. *Then exactly one of the following holds.*

(1) $\mathfrak{A}$ *pp-constructs* K_3. *In this case,* VCSP($\mathfrak{A}$) *is NP-complete.*
(2) Pol$(\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty)$ *contains a pwnu operation. In this case,* VCSP($\mathfrak{A}$) *is in P.*

Proof. Let $\mathfrak{A}' := (\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty)$. By Theorem 2.25, VCSP($\mathfrak{A}$) is in NP. By Proposition 3.2, $\mathfrak{A}$ pp-constructs K_3 if and only if $\mathfrak{A}'$ pp-constructs K_3 and in this case, VCSP($\mathfrak{A}$) is NP-complete by Lemma 2.16. Hence, it follows from Theorem 5.9 applied on $\mathfrak{A}'$ that either (1) holds or Pol$(\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty)$ contains a pwnu operation. Hence, if Pol$(\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty)$ contains a pwnu operation, then $\mathfrak{A}$ does not pp-construct K_3. By Proposition 5.10 and Theorem 4.1, Betw, Cycl, Sep, T_3, $-T_3$, Dis $\notin \langle\mathfrak{A}\rangle$ and therefore, item (2) from Corollary 5.26 applies and VCSP($\mathfrak{A}$) is in P. □

Conjecture 9.3 in [15] states that, under some structural assumptions on $\mathfrak{A}$, VCSP($\mathfrak{A}$) is in P whenever $\mathfrak{A}$ does not pp-construct K_3 (and is NP-hard otherwise)[2]. All temporal structures satisfy the assumptions of the conjecture and hence Corollary 5.27 confirms the conjecture for the class of temporal VCSPs.

[2]The original formulation uses the structure $(\{0,1\}; \mathrm{OIT})$, but it well-known that this structure pp-constructs K_3 and vice versa [3].

6. Concluding Remarks

We proved a complexity dichotomy for temporal VCSPs: $\mathrm{VCSP}(\mathfrak{A})$ is in P or NP-complete for every valued structure $\mathfrak{A}$ such that $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$. Moreover, we showed that the meta-problem of deciding whether $\mathrm{VCSP}(\mathfrak{A})$ is in P or NP-complete for a given $\mathfrak{A}$ is decidable. As a side product of our proof, we obtain that the complexity of every such VCSP is captured by the classical relations that it can express, in other words, by the CSPs that are encoded in this VCSP. Our results confirm [15, Conjecture 9.3] for all temporal valued structures.

The proof of our decidability result (Proposition 5.25) is based on the distinction of two cases depending on whether P=NP. The typical results on decidability of these meta-problems in the theory of (V)CSPs are rather formulated by deciding the algebraic conditions that imply the respective complexities, more concretely, deciding the presence of certain (fractional) polymorphisms. This can often be checked by the naive approach, as long as the signature of the structure is finite. However, if we wanted to do so in our case, we would have to check for polymorphisms of the structure $(\mathbb{Q}; \langle\mathfrak{A}\rangle_0^\infty)$, which has an infinite signature by definition. This motivates the following question.

Question 6.1. *Let $\mathfrak{A}$ be a valued structure with a finite signature such that $\mathrm{Aut}(\mathbb{Q};<) \subseteq \mathrm{Aut}(\mathfrak{A})$. Given a first-order definition of a valued structure $\mathfrak{A}$ with a finite signature in $(\mathbb{Q};<)$, is it decidable whether $\mathfrak{A}$ pp-constructs K_3, equivalently, whether item (2) in Corollary 5.26 holds?*

In analogy to the development of the results on infinite-domain CSPs, we propose the class of valued structures that are preserved by all automorphisms of the countable random graph as a natural next step in the complexity classification of VCSPs on infinite domains.

Question 6.2. *Does the class of VCSPs of all valued structures $\mathfrak{A}$ over a finite signature such that $\mathrm{Aut}(\mathfrak{A})$ contains the automorphism group of the countable random graph exhibit a P vs. NP-complete dichotomy? In particular, is $\mathrm{VCSP}(\mathfrak{A})$ in P whenever $\mathfrak{A}$ does not pp-construct K_3?*

A positive answer to the second question in Question 6.2 would confirm [15, Conjecture 9.3] for valued structures preserved by all automorphisms of the countable random graph.

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