On the Twin-Width of Smooth Manifolds

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Abstract

Building on Whitney's classical method of triangulating smooth manifolds, we show that every compact d-dimensional smooth manifold admits a triangulation with dual graph of twin-width at most $d^{O(d)}$. In particular, it follows that every compact 3-manifold has a triangulation with dual graph of bounded twin-width. This is in sharp contrast to the case of treewidth, where for any natural number n there exists a closed 3-manifold such that every triangulation thereof has dual graph with treewidth at least n. To establish this result, we bound the twin-width of the incidence graph of the d-skeleton of the second barycentric subdivision of the $2d$-dimensional hypercubic honeycomb. We also show that every compact, piecewise-linear (hence smooth) d-dimensional manifold has triangulations where the dual graph has an arbitrarily large twin-width.

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1 Introduction

Structural graph parameters have become increasingly important in computational topology in the past two decades. This is mainly due to the emergence of fixed-parameter tractable (FPT) algorithms for problems on knots, links [12, 34, 35, 37] and 3-manifolds [14, 16, 17, 18, 19],[1] most of which are known to be NP-hard in general. Although these FPT algorithms may have exponential worst-case running time, on inputs with bounded *treewidth* they are guaranteed to terminate in polynomial (or even in linear) time.[2] In addition, some of these algorithms have been implemented in software packages such as `Regina`, providing practical tools for researchers in topology [11, 13].[3]

The success of the above algorithms naturally leads to the following question. Given a 3-manifold $\mathcal{M}$ (resp. knot $\mathcal{K}$), what is the smallest treewidth that the dual graph of a triangulation of $\mathcal{M}$ (resp. a diagram of $\mathcal{K}$) may have?[4] Motivated by this challenge, in recent years several results have been obtained that reveal quantitative connections between *topological invariants* of knots and 3-manifolds, and *width parameters* associated with their diagrams [21, 33] and triangulations [25, 26, 27, 28, 29, 38], respectively. It turns out that

[1] Also see [2] for an FPT algorithm checking tightness of (weak) pseudomanifolds in arbitrary dimensions.
[2] Here the term *input* refers either to a link diagram $\mathcal{D}$, or to a 3-manifold triangulation $\mathcal{T}$. In the first case the treewidth means the treewidth of $\mathcal{D}$ considered as a 4-regular graph, in the second case it means the treewidth of the dual graph $\Gamma(\mathcal{T})$ of $\mathcal{T}$. The running times are measured in terms of the *size* of the input. The size of a link diagram is defined as the number of its crossings, and the size of a 3-manifold triangulation is the number of its tetrahedra. More definitions are given in Section 2.
[3] See [3] for an implemented algorithm to effectively compute certain Khovanov homology groups of knots. This algorithm is conjectured to be FPT in the *cutwidth* of the input knot diagram, cf. [3, Section 6].
[4] For links and knots, this question was respectively asked in [35, Section 4] and [15, p. 2694].

topological properties of 3-manifolds may prohibit the existence of “thin” triangulations.[5,6] At the same time, geometric or topological descriptions of 3-manifolds can also give strong hints on how to triangulate them so that their dual graphs have constant pathwidth or treewidth, or at least bounded in terms of a topological invariant of these 3-manifolds.[7]

In this work we establish similar results for another graph parameter called *twin-width*.[8] Introduced in [10], this notion has been subject of growing interest and found many algorithmic applications in recent years.[9] Namely, on classes of effectively[10] bounded twin-width, first-order properties can be decided efficiently[11] [10] (also see [8] for improved running times on specific problems definable in first-order logic), first-order queries can be enumerated fast [23], and enhanced approximation algorithms can be designed for several graph optimization problems [4]. Besides, classes of bounded twin-width are fairly general and broad. They for instance include classes of bounded tree-width, and even its dense analogue, clique-width, classes excluding a minor, proper permutation classes, d-dimensional grid graphs [10], some classes of cubic expanders [9], segment intersection graphs without biclique subgraphs of a fixed size [7], and modifications definable in first-order logic (called first-order *transductions*) of all these classes [10]. We observe that the definition of twin-width can readily be lifted to binary structures (i.e., edge-colored multigraphs).

Our first result shows that a compact d-dimensional smooth manifold always has a triangulation with dual graph of twin-width bounded in terms of d.

▶ **Theorem 1.** *Any compact d-dimensional smooth manifold admits a triangulation with dual graph of twin-width at most $d^{O(d)}$.*

Since every 3-manifold is smooth [39] (also see [36]), the following corollary is immediate. We recall that $\mathrm{tww}(G)$ denotes the twin-width of a graph G, and $\Gamma(\mathcal{T})$ denotes the dual graph of a triangulation $\mathcal{T}$.

▶ **Corollary 2.** *There exists a universal constant $C > 0$ such that every compact 3-dimensional manifold $\mathcal{M}$ admits a triangulation $\mathcal{T}$ with $\mathrm{tww}(\Gamma(\mathcal{T})) \leqslant C$.*

This is in sharp contrast to the case of treewidth, for which it is known that for every $n \in \mathbb{N}$ there are infinitely many 3-manifolds where the smallest treewidth of the dual graph of *every* triangulation is at least n [28, 29]. Complementing Theorem 1, we also show that for any fixed $d \geqslant 3$, the d-dimensional triangulations of large twin-width are abundant (Theorem 21). Moreover we show that any piecewise-linear (hence smooth) manifold of dimension at least three admits triangulations with dual graph of arbitrarily large twin-width.

▶ **Theorem 3.** *Let $d \geqslant 3$ be an integer. For every compact d-dimensional piecewise-linear manifold $\mathcal{M}$ and natural number $n \in \mathbb{N}$, there is a triangulation $\mathcal{T}$ of $\mathcal{M}$ with $\mathrm{tww}(\Gamma(\mathcal{T})) \geqslant n$.*

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5 In particular, for non-Haken 3-manifolds of large Heegaard genus [29] or Haken 3-manifolds with a “complicated” JSJ decomposition [28], the dual graph of *any* triangulation must also have large treewidth.

6 For results where the treewidth of a knot diagram is bounded below by topological properties of the underlying knot, see [21, 33].

7 This is case with Seifert fibered spaces [27] or hyperbolic 3-manifolds [26, 38].

8 We will denote by $\mathrm{tww}(G)$ the twin-width of a graph G.

9 For an introduction to twin-width and an overview of its applications, see [6] and the references therein.

10 A class has *effectively bounded twin-width* if it has bounded twin-width, and contraction sequences of width $O(1)$ (objects witnessing the twin-width upper bound) can be found in polynomial time; see Section 2.1 for the definitions of *contraction sequences* and *twin-width*.

11 More precisely, there is a *fixed-parameter tractable* algorithm that, given a first-order sentence φ and an n-vertex graph G with a contraction sequence of width d, decides if G satisfies φ in time $f(\varphi, d) \cdot n$, for some computable function f.

▶ Remark 4. The assumption of $d \geqslant 3$ in Theorem 3 is essential. Indeed, the dual graph of any triangulation of the genus-g surface S_g is, in particular, a graph that embeds into S_g and such graphs are known to have twin-width bounded above by $c(\sqrt{g}+1)$ for some universal constant $c > 0$ [31].[12]

Outline of the paper. In Section 2 we review the relevant notions from graph theory and topology. In Section 3 we recall Whitney's seminal work on triangulating smooth manifolds. This is followed by a detailed proof of Theorem 1 in Section 4. Finally, in Section 5 we prove the complementary results about triangulations with dual graph of large twin-width.

2 Preliminaries

Basic notation. For a finite set S we let $|S|$ denote its cardinality, while for a real number x we let $|x|$ denote its absolute value. For a positive integer $k \leqslant |S|$, we let $\binom{S}{k}$ denote the set of k-element subsets of S. For a positive integer n, we let $[n]$ denote the set of all positive integers up to n, and for two real numbers $a \leqslant b$, we use $[a, b]$ to denote the closed interval $\{x \in \mathbb{R} : a \leqslant x \leqslant b\}$. For a vector $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ we use $\|y\|$ to denote its Euclidean norm, i.e, $\|y\|^2 = \sum_{i=1}^{d} y_i^2$. However, if $\mathcal{X}$ is a (cubical or simplicial) complex, then $\|\mathcal{X}\|$ refers to its geometric realization.

2.1 Trigraphs, contraction sequences and twin-width

Following [10, Sections 3 and 4], in this section we review the graph-theoretic notions central to our work and collect some basic, yet important facts about twin-width.

Trigraphs. A *trigraph* G is a triple $G = (V, E, R)$, where V is a finite set of *vertices*, and $E, R \subseteq \binom{V}{2}$ are two disjoint subsets of pairs of vertices called *black edges* and *red edges*, respectively. We also refer to the sets of vertices, black edges and red edges of a given trigraph G as $V(G)$, $E(G)$ and $R(G)$, respectively. Any simple graph $G = (V, E)$ may be regarded as a trigraph (V, E, R) with $R = \emptyset$. For a vertex $v \in V(G)$ the *degree* $\deg(v)$ *of* v is the number of edges incident to it, i.e., $\deg(v) = |\{e \in E \cup R : v \in e\}|$. Additionally, the *red degree* $\deg_R(v)$ *of* v is the number of red edges incident to it, i.e., $\deg_R(v) = |\{e \in R : v \in e\}|$. A trigraph G for which $\deg_R(G) = \max_{v \in V(G)} \deg_R(v) \leqslant b$ is called a *b-trigraph*. Given two trigraphs $G = (V, E, R)$ and $G' = (V', E', R')$, we say that G' is a *subtrigraph* of G, if $V' \subseteq V$, $E' \subseteq E \cap \binom{V'}{2}$ and $R' \subseteq R \cap \binom{V'}{2}$.[13] In addition, if $E' = E \cap \binom{V'}{2}$ and $R' = R \cap \binom{V'}{2}$, then we say that G' is an *induced subtrigraph* of G. For a trigraph G and a subset $S \subseteq V(G)$ of its vertices, $G - S$ denotes the induced subtrigraph of G with vertex set $V(G) \setminus S$.

Contraction sequences and twin-width. Let $G = (V, E, R)$ be a trigraph and $u, v \in V$ be two arbitrary distinct vertices of G. We say that the trigraph $G/u, v = (V', E', R')$ is obtained from G by *contracting* u and v into a new vertex w if **1.** $V' = (V \setminus \{u, v\}) \cup \{w\}$, **2.** $G - \{u, v\} = (G/u, v) - \{w\}$ and **3.** for any $x \in V' \setminus \{w\} = V \setminus \{u, v\}$ we have

- $\{w, x\} \in E'$ if and only if $\{u, x\} \in E$ and $\{v, x\} \in E$,
- $\{w, x\} \notin E' \cup R'$ if and only if $\{u, x\} \notin E \cup R$ and $\{v, x\} \notin E \cup R$, and

[12] This bound is sharp up to a constant multiplicative factor [31]. For $g = 0$, we know that planar graphs have twin-width at most eight [24], and there are planar graphs with twin-width equal to seven [30].
[13] As usual, subtrigraphs of graphs (those without any red edges) will also be called subgraphs.

- $\{w, x\} \in R'$ otherwise.

We call the trigraph $G/u, v$ a *contraction* of G. A sequence $\mathfrak{S} = (G_1, \ldots, G_m)$ of trigraphs is a *contraction sequence* if G_{i+1} is a contraction of G_i for every $1 \leqslant i \leqslant m-1$. Note that $|V(G_{i+1})| = |V(G_i)| - 1$. We use the notation "$\mathfrak{S}\colon G_1 \rightsquigarrow G_m$" to indicate that the trigraphs G_1 and G_m are initial and terminal entries of the contraction sequence $\mathfrak{S}$. The *width* $\mathrm{w}(\mathfrak{S})$ of a contraction sequence $\mathfrak{S} = (G_1, \ldots, G_m)$ is defined as $\mathrm{w}(\mathfrak{S}) = \max_{1 \leqslant i \leqslant m} \deg_R(G_i)$, i.e., the largest red degree of any vertex of any trigraph in $\mathfrak{S}$. Now, the *twin-width* $\mathrm{tww}(G)$ of a trigraph G is defined as the smallest width of any contraction sequence $(G_1, \ldots, G_{|V(G)|})$ with $G_1 = G$ and $G_{|V(G)|} = \bullet$, where $\bullet$ denotes the trigraph consisting of a single vertex.

Some properties of twin-width; grid graphs

We conclude this section by collecting some properties of twin-width that we will rely on later. The first one states that twin-width is monotonic under taking induced subtrigraphs and is a simple consequence of the definitions (cf. [10, Section 4.1]).

▶ **Proposition 5.** *If H is an induced subtrigraph of a trigraph G, then* $\mathrm{tww}(H) \leqslant \mathrm{tww}(G)$.

Smallness. An infinite class $\mathcal{G}$ of graphs is *small* if there exists a constant $c > 1$ such that for every $n \in \mathbb{N}$ the class $\mathcal{G}$ contains at most $n!c^n$ labeled graphs on n vertices. The next theorem says that every graph class of bounded twin-width—i.e., for which there exists a constant $C > 0$, such that $\mathrm{tww}(G) \leqslant C$ for every graph G in the class—is small.

▶ **Theorem 6** ([9, Theorem 2.5])**.** *Every graph class with bounded twin-width is small.*

Now let s be a non-negative integer. The *s-subdivision* of G is the graph $\mathrm{subd}_s(G)$ obtained from G by subdividing each edge in $E(G)$ exactly s times. A simple counting argument together with Theorem 6 yields the following:

▶ **Proposition 7.** *For any fixed integers $k \geqslant 4$ and $s \geqslant 0$, the class $\mathrm{subd}_s(\mathcal{G}_k)$ of s-subdivisions of k-regular*[14] *simple graphs is not small, hence has unbounded twin-width.*

This proposition follows from the adaptation of an argument given in the first paragraph of [22, Section 3]. For completeness, we spell out this proof below.

Proof of Proposition 7. Let $N_k(m)$ be the number of labeled k-regular simple graphs on m vertices. Note that if $N_k(m) > 0$, then km is even; which we now assume. It is known (cf. [40, Section 6.4.1]) that, asymptotically

$$N_k(m) \sim \exp\left(\frac{1-k^2}{4}\right) \frac{(km)!}{(km/2)! \cdot 2^{km/2} \cdot (k!)^m} = \Omega\left(\frac{(km/2)!}{(k!)^m}\right). \tag{1}$$

Further, let $N_k^{(s)}(n)$ be the number of n-vertex graphs in the class $\mathrm{subd}_s(\mathcal{G}_k)$ of s-subdivisions of k-regular simple graphs. If a graph $G \in \mathrm{subd}_s(\mathcal{G}_k)$ has n vertices, then $n = m + \frac{km}{2}s$, where m is the number of vertices of G of degree k. Note that such a graph G can be obtained by first choosing a k-regular labeled graph H on m vertices, then ordering the remaining

[14] A simple graph $G = (V, E)$ is *k-regular* if every vertex $v \in V$ has degree $\deg(v) = k$.

$n - m = kms/2$ vertices arbitrarily and evenly distributing them to the edges of H according to some fixed ordering of $E(H)$. It follows that

$$N_k^{(s)}(n) \geqslant \binom{n}{m} \cdot N_k(m) \cdot (n-m)! = n! \frac{N_k(m)}{m!} \stackrel{(1)}{=} n! \cdot \Omega\left(\frac{(km/2)!}{m! \cdot (k!)^m}\right). \quad (2)$$

Since $k \geqslant 4$, we have $km/2 \geqslant 2m$. This, together with $(2m)! \geqslant 2^m (m!)^2$ and (2) implies

$$N_k^{(s)}(n) \geqslant n! \cdot \Omega\left(\frac{m!}{(2 \cdot k!)^m}\right). \quad (3)$$

Recall that $m = 2n/(2+ks)$. In particular, m is proportional to n. Hence $m!/(2 \cdot k!)^m$ grows faster than c^n for any fixed constant $c > 1$. Thus the graph class $\mathrm{subd}_s(\mathcal{G}_k)$ is not small. ◀

Grid graphs. The *d-dimensional n-grid* P_n^d is the graph with vertex set $V(P_n^d) = [n]^d$, and $\{u,v\} \in E(P_n^d)$ for two vertices $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ if and only if $\sum_{i=1}^d |u_i - v_i| = 1$. The next result states that $\mathrm{tww}(P_n^d) \leqslant 3d$ irrespective of the value of n.

▶ **Theorem 8** (Theorem 4 in [10]). *For every positive d and n, the d-dimensional n-grid P_n^d has twin-width at most $3d$.*

The *d-dimensional n-grid with diagonals* $D_{n,d}$ is the graph with $V(D_{n,d}) = [n]^d$, and $\{u,v\} \in E(D_{n,d})$ for two vertices $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ if and only if $\max_{i=1}^d |u_i - v_i| \leqslant 1$. Now, for a given trigraph $G = (V, E, R)$ we set $\mathrm{red}(G) = (V, \emptyset, E \cup R)$. In words, $\mathrm{red}(G)$ is the trigraph obtained from G by replacing every black edge of G by a red edge between the same vertices. With this notation $\mathrm{red}(D_{n,d})$ is the *d-dimensional **red** n-grid with diagonals*, i.e., $\mathrm{red}(D_{n,d}) = ([n]^d, \emptyset, E(D_{n,d}))$. Clearly, $\mathrm{tww}(D_{n,d}) \leqslant \mathrm{tww}(\mathrm{red}(D_{n,d}))$.

▶ **Theorem 9** (Lemma 4.4 in [10]). *For every positive d and n, every subtrigraph of the d-dimensional red n-grid with diagonals* $\mathrm{red}(D_{n,d})$ *has twin-width at most* $2(3^d - 1)$.

2.2 Background in topology

For general background in (combinatorial and differential) topology we refer to [41].

2.2.1 Simplicial and cubical complexes

Abstract simplicial complexes. Given a finite ground set $\mathcal{S}$, an *abstract simplicial complex* (or *simplicial complex*, for short) $\mathcal{X}$ over $\mathcal{S}$ is a downward closed subset of the power set $2^{\mathcal{S}}$, i.e., $\mathcal{F} \in \mathcal{X}$ and $\mathcal{F}' \subset \mathcal{F}$ imply $\mathcal{F}' \in \mathcal{X}$. Any element of $\mathcal{X}$ is called a *face* or *simplex* of $\mathcal{X}$, and for $\sigma \in \mathcal{X}$ the *dimension of* σ is defined as $\dim \sigma = |\sigma| - 1$. The *dimension of* $\mathcal{X}$, denoted by $\dim \mathcal{X}$, is then the maximum dimension of a face of $\mathcal{X}$. If $\dim \mathcal{X} = d$, we also say that $\mathcal{X}$ is a *simplicial d-complex*. For $0 \leqslant i \leqslant \dim \mathcal{X}$, we let $\mathcal{X}(i) = \{\sigma \in \mathcal{X} : \dim \sigma = i\}$ denote the set of i-dimensional faces (or i-faces, or i-simplices) of $\mathcal{X}$. The *i-skeleton* $\mathcal{X}_i = \bigcup_{j=0}^i \mathcal{X}(j)$ of $\mathcal{X}$ is the union of all faces of $\mathcal{X}$ up to dimension i. Note that any simple graph $G = (V, E)$ can be regarded as a 1-dimensional simplicial complex $\mathcal{X}_G$ with $\mathcal{X}_G(0) = \{\{v\} : v \in V\}$ and $\mathcal{X}_G(1) = E$. For $0 \leqslant i \leqslant 3$ the i-simplices of a simplicial complex $\mathcal{X}$ are respectively called the *vertices*, *edges*, *triangles*, and *tetrahedra* of $\mathcal{X}$.

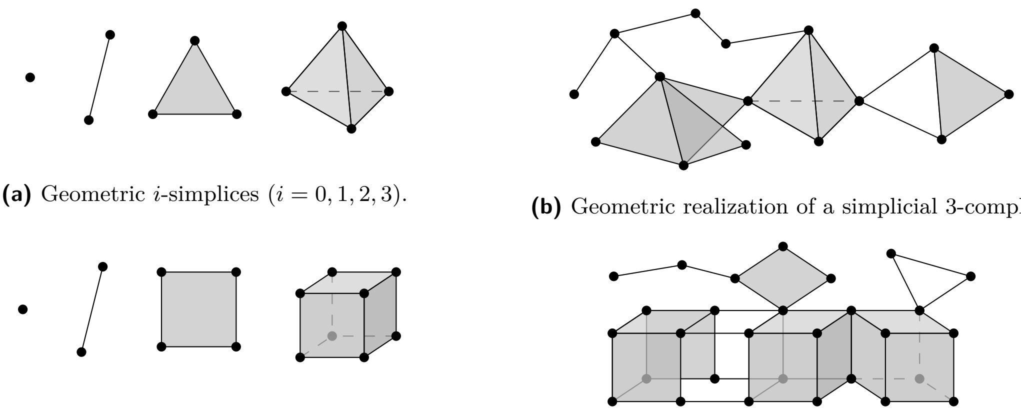
Geometric realization of simplicial complexes. Every abstract simplicial complex $\mathcal{X}$ may be realized geometrically as follows. To each abstract i-simplex $\sigma = \{v_0, v_1, \ldots, v_i\} \in \mathcal{X}$ we associate a *geometric i-simplex* $\|\sigma\| = [v_0, v_1, \ldots, v_i] \subset \mathbb{R}^i$ defined as the convex hull of $i+1$ affinely independent points in $\mathbb{R}^i$. We equip $\|\sigma\|$ with the subspace topology inherited from $\mathbb{R}^i$. Next, we consider the disjoint union $\bigsqcup_{\sigma\in\mathcal{X}} \|\sigma\|$ of these geometric simplices, and perform identifications along their faces that reflect their relationship in $\mathcal{X}$. The resulting space $\|\mathcal{X}\|$, equipped with the quotient topology, is called the *geometric realization of* $\mathcal{X}$, see Figure 1. The geometric realization of a simplicial complex is unique up to homeomorphism.

▶ **Theorem 10** (folklore; cf. [41, Theorem 3.15]). *Let $\|\mathcal{X}\|$ be the geometric realization of a d-dimensional simplicial complex $\mathcal{X}$. Then there exists an embedding (i.e., a continuous, injective map) $f\colon \|\mathcal{X}\| \to \mathbb{R}^{2d+1}$. Furthermore, f can be chosen to be simplex-wise linear.*

▶ Remark 11. Theorem 10 generalizes the well-known fact that every graph admits a straight-line embedding in $\mathbb{R}^3$.

Cubical complexes. Analogous to simplicial complexes, a *cubical complex* $\mathcal{X}$ over a ground set $\mathcal{S}$ is a set system $\mathcal{X} \subset 2^{\mathcal{S}}$ that consists of "cubes" instead of simplices. The terminology is the same as in the simplicial case. The only difference is that a *geometric i-cube* is a topological space homeomorphic to $[0,1]^i$, where $[0,1]$ denotes the closed unit interval.

Cubical or simplicial complexes in this paper will typically be defined geometrically, and as such they will naturally come with a geometric realization.



(a) Geometric i-simplices ($i = 0, 1, 2, 3$).

(b) Geometric realization of a simplicial 3-complex.

(c) Geometric i-cubes ($i = 0, 1, 2, 3$).

(d) Geometric realization of a cubical 3-complex.

Figure 1 The geometric perspective on simplicial and cubical complexes.

The hypercubic honeycomb. Let n and d be positive integers and consider the d-dimensional cube $[1,n]^d \subset \mathbb{R}^d$. The *$d$-dimensional hypercubic honeycomb* $\mathbf{H}^{d,n}$ is a cubical d-complex that decomposes $[1,n]^d$ into $(n-1)^d$ geometric cubes. The properties of this familiar object play an important role in this work, so we describe it for completeness. We define $\mathbf{H}^{d,n}$ geometrically, in a bottom-up fashion. First, the vertex set $\mathbf{H}^{d,n}(0)$ consists of precisely those points in $[1,n]^d$, which have only integral coordinates. Next, a 1-cube (i.e., an edge) is attached along its endpoints to vertices $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ in $\mathbf{H}^{d,n}(0)$ if and only if $\sum_{i=1}^{d} |u_i - v_i| = 1$. Thus the 1-skeleton $\mathbf{H}^{d,n}_1$ is just the d-dimensional grid graph P^d_n encountered in the end of Section 2.1. Finally, the higher dimensional skeleta of $\mathbf{H}^{d,n}$ are *induced* by its 1-skeleton: for each subcomplex $\mathcal{Y} \subset \mathbf{H}^{d,n}_i$ isomorphic to the boundary of a $(i+1)$-cube, we attach an $(i+1)$-cube to $\mathbf{H}_i^{d,n}$ along $\mathcal{Y}$. In words, starting from the 1-skeleton $\mathbf{H}_1^{d,n}$, whenever we have the possibility to attach a cube of dimension at least two (because its boundary is already present), we attach it. See, e.g., Figure 2.

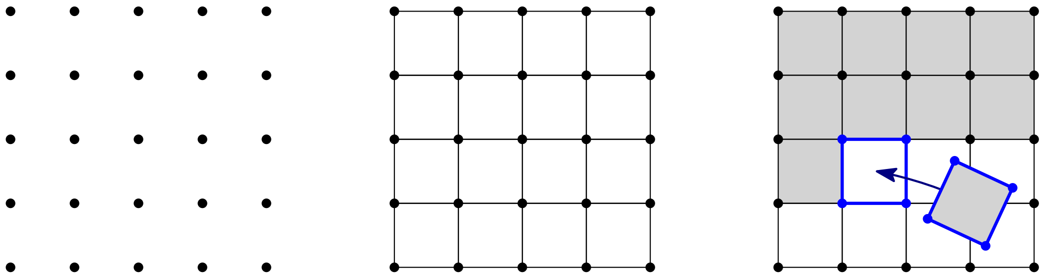
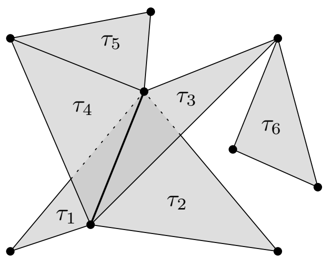
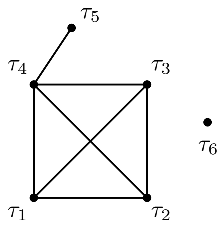


Figure 2 Constructing the complex $\mathbf{H}^{2,5}$. Its 1-skeleton $\mathbf{H}_1^{2,5}$ is isomorphic to the grid P_5^2.

Pure complexes and their dual graphs. A (cubical or simplicial) complex $\mathcal{X}$ is *pure* if every face of $\mathcal{X}$ is contained in a face of dimension $\dim \mathcal{X}$. It follows that, for every i with $0 \leqslant i \leqslant \dim \mathcal{X}$, the i-skeleton $\mathcal{X}_i$ of a pure complex $\mathcal{X}$ is also pure. Examples of pure complexes include nonempty graphs without isolated vertices, or triangulations of manifolds (see Section 2.2.3). Given a pure complex $\mathcal{X}$, the *dual graph* $\Gamma(\mathcal{X}_i) = (V, E)$ of its i-skeleton is defined as the graph, where the vertex set V corresponds to the set $\mathcal{X}(i)$ of i-faces, and $\{\sigma, \tau\} \in E$ if and only if σ and τ share an $(i-1)$-dimensional face in $\mathcal{X}$, see Figure 3.



(a) A pure simplicial 2-complex $\mathcal{X}$ formed by six triangles τ_1, τ_2, τ_3, τ_4, τ_5, and τ_6, four of which meet along a single edge.



(b) The dual graph $\Gamma(\mathcal{X}_2)$ of $\mathcal{X}_2$. As τ_6 shares no edge with any other triangle in $\mathcal{X}$, its corresponding vertex is isolated.

Figure 3 Example of a pure simplicial 2-complex and its dual graph.

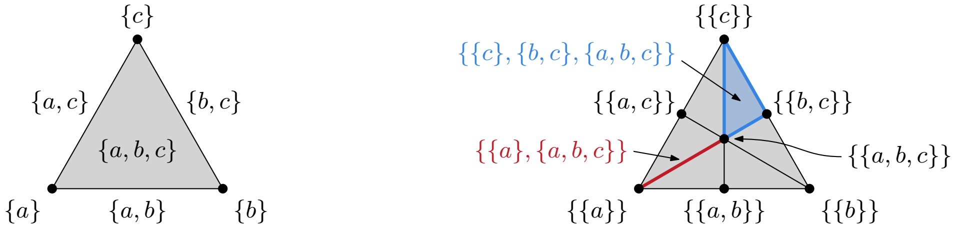
2.2.2 Barycentric subdivisions

Given a (cubical or simplicial) complex $\mathcal{X}$, its *barycentric subdivision* is a **simplicial** complex $\mathcal{X}'$ defined abstractly as follows. For the ground set $\mathcal{S}'$ of $\mathcal{X}'$ we have $\mathcal{S}' = \mathcal{X}$. A $(k+1)$-tuple $\{\sigma_0, \ldots, \sigma_k\} \subset \mathcal{S}'$ forms a k-simplex of $\mathcal{X}'$ if and only if $\sigma_i \subset \sigma_j$ for every $0 \leqslant i < j \leqslant k$. We denote the 2^{nd} (resp. ℓ^{th}) iterated barycentric subdivision of a complex $\mathcal{X}$ by $\mathcal{X}''$ (resp. $\mathcal{X}^{(\ell)}$). See Figures 4 and 5 for examples. The following are simple consequences of the definitions.

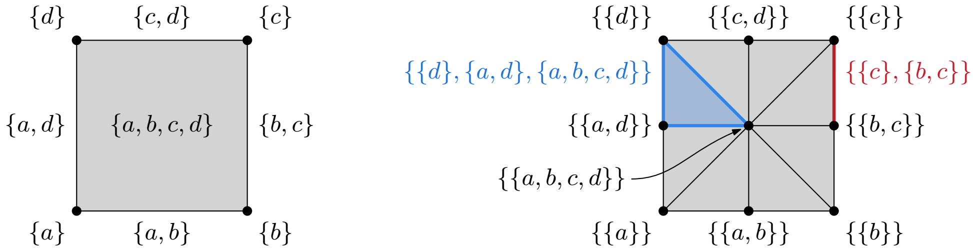
▶ **Observation 12.** *For any $0 \leqslant l \leqslant k$, the k-simplex has $\binom{k}{l}$ l-faces.*

▶ **Observation 13.** *The barycentric subdivision of the k-simplex contains $k!$ k-simplices.*

▶ **Observation 14.** *The barycentric subdivision of the k-cube contains $2^k k!$ k-simplices.*

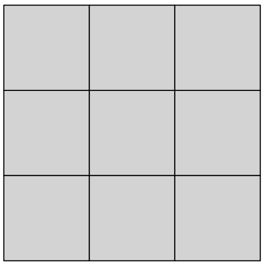


(a) A triangle presented as a simplicial complex (left) and its barycentric subdivision (right).

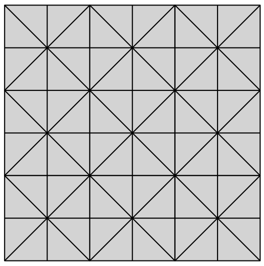


(b) A square presented as a cubical complex (left) and its barycentric subdivision (right).

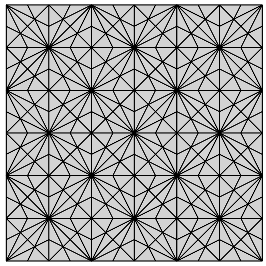
Figure 4 The effect of barycentric subdivision on a triangle and on a square.



(a) $\mathbf{H}^{2,4}$



(b) $(\mathbf{H}^{2,4})'$



(c) $(\mathbf{H}^{2,4})''$

Figure 5 The effect of two barycentric subdivisions on the 2-dimensional hypercubic honeycomb.

2.2.3 Manifolds and their triangulations

Manifolds—the main objects of interest in this paper—can be regarded as higher dimensional analogs of surfaces. A *d-dimensional topological manifold with boundary* (or *d-manifold*, for short) is a topological space[15] $\mathcal{M}$, where every point has an open neighborhood homeomorphic to $\mathbb{R}^d$, or to the closed upper half-space $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 \geqslant 0\}$. The points of $\mathcal{M}$ that do **not** have a neighborhood homeomorphic to $\mathbb{R}^d$ constitute the *boundary* $\partial\mathcal{M}$ of $\mathcal{M}$. If a manifold $\mathcal{M}$ satisfies $\partial\mathcal{M} = \emptyset$, then $\mathcal{M}$ is called a *closed* manifold.

In this paper $\mathcal{M}$ always denotes a compact manifold.

Smooth manifolds. The main result of this paper (Theorem 1) applies for manifolds that have an additional property, namely *smoothness*. As we will not need to work with the actual definition of smoothness, but merely rely on it, we only give a brief definition here. For more background on smooth manifolds, we refer to [41, Chapter 5] and [32].

[15] As a topological space a manifold is required to be *second countable* [41, p. 2] and *Hausdorff* [41, p. 87].

Given a connected and open subset $U \subset \mathcal{M}$ and a homeomorphism $\varphi\colon U \to \varphi(U)$ onto an open subset of $\mathbb{R}^d$, the pair (U, φ) is called a *chart*. Given two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) with $U_\alpha \cap U_\beta \neq \emptyset$, the map $\tau_{\alpha,\beta}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ defined via $\tau_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is called a *transition map*. A *smooth structure* on a topological manifold $\mathcal{M}$ with $\partial\mathcal{M} = \emptyset$ is a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of charts that satisfies the following three properties.

1. The sets U_α cover $\mathcal{M}$, that is $\bigcup_{\alpha \in A} U_\alpha = \mathcal{M}$.
2. For any $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, the transition map $\tau_{\alpha,\beta}$ is smooth.[16]
3. The collection $\mathcal{A}$ is maximal in the sense that if (U, φ) is a chart and for every $\alpha \in A$ with $U \cap U_\alpha \neq \emptyset$ the transition maps $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are smooth, then $(U, \varphi) \in \mathcal{A}$.

A topological manifold together with a smooth structure is called a *smooth manifold*. By an appropriate modification of property **2** above, the definition extends to manifolds with non-empty boundary as well. We refer to [41, Section 5.1.1] for details.

Triangulations. Let $\mathcal{M}$ be a compact topological d-manifold. A simplicial complex $\mathcal{T}$ whose geometric realization $\|\mathcal{T}\|$ is homeomorphic to $\mathcal{M}$ is called a *triangulation* of $\mathcal{M}$. It follows that $\mathcal{T}$ is a pure simplicial complex of dimension d (see Section 2.2.1). For $d \leqslant 3$, every topological d-manifold admits a triangulation [39, 42], however, for $d > 3$ this is generally not true (see [36] for an overview). Smooth manifolds can nevertheless always be triangulated, irrespective of their dimension, e.g., by work of Whitney [44, Chapter IV.B] (cf. Section 3).

Given a d-dimensional triangulation $\mathcal{T}$, recall that its *dual graph* $\Gamma(\mathcal{T})$ is the graph with vertices corresponding to the d-simplices of $\mathcal{T}$, and edges to the face gluings, i.e, those $(d-1)$-simplices of $\mathcal{T}$ that are contained in precisely two d-simplices. Note that $\deg(v) \leqslant d+1$ for any vertex v of $\Gamma(\mathcal{T})$. The proof of the next proposition is left to the reader.

▶ **Proposition 15.** *Let $\mathcal{T}$ be a triangulation of a d-manifold $\mathcal{M}$ and $\mathcal{T}'$ be a collection of d-simplices of $\mathcal{T}$ that define a submanifold of $\mathcal{M}$. Then $\Gamma(\mathcal{T}')$ is an induced subgraph of $\Gamma(\mathcal{T})$.*

3 Whitney's method

A seminal result of Whitney states that a smooth d-dimensional manifold $\mathcal{M}$ always admits a smooth embedding into a $2d$-dimensional Euclidean space.

▶ **Theorem 16** (strong Whitney embedding theorem [43, Theorem 5], cf. [32, Theorem 6.19])**.** *For $d > 0$, every smooth d-manifold admits a smooth embedding into $\mathbb{R}^{2d}$.*

An important consequence of Theorem 16 is that smooth manifolds can be triangulated.

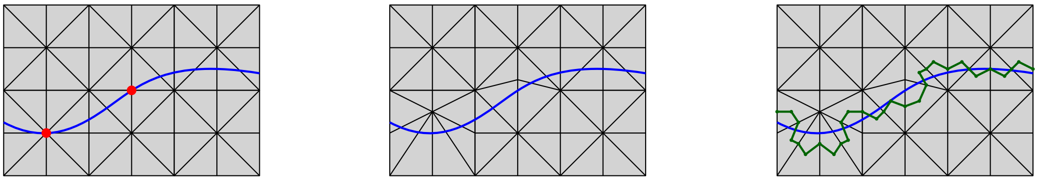
▶ **Theorem 17** (triangulation theorem [44, Chapter IV.B], cf. [5, Theorem 1.1])**.** *Every compact, smooth d-manifold $\mathcal{M}$ embedded in some Euclidean space $\mathbb{R}^m$ admits a triangulation.*

Next, we give a very brief and high-level overview of Whitney's method of triangulating smooth manifolds based on [44, Chapter IV.B] sufficient for our purposes. However, we highly recommend [5], where Whitney's method is recast in a modern, computational setting.

[16] See [41, p. 185] for a discussion of (smooth) maps of manifolds.

Triangulating smooth manifolds. Let $\mathcal{M}$ be a compact smooth d-manifold. By Theorem 16 there exists a smooth embedding $\iota\colon \mathcal{M} \to \mathbb{R}^{2d}$. Given such an embedding, we choose a sufficiently fine (with respect to ι) hypercubic honeycomb decomposition of $\mathbb{R}^{2d}$, denoted by L_0. Next, we pass to the first barycentric subdivision L of L_0. (Geometrically, L is obtained from L_0 by subdividing each k-dimensional hypercube of L_0 into $(2k)!!$ simplices.) By slightly perturbing the vertices of L we obtain a triangulation L^* of $\mathbb{R}^{2d}$, which is combinatorially isomorphic to L, but is in *general position* with respect to $\iota(\mathcal{M}) \subset \mathbb{R}^{2d}$. Now, by the work of Whitney, L^* induces a triangulation $\mathcal{T}$ of $\mathcal{M}$, where, importantly, $\mathcal{T}$ is contained in the d-skeleton of the barycentric subdivision $(L^*)'$ of L^*.

See Figure 6 for an illustration of this procedure for $d = 1$.



(a) **(b)** **(c)**

Figure 6 Illustration of Whitney's method of triangulating ambient submanifolds ($d = 1$).

▶ **Proposition 18.** *For any* Whitney triangulation[17] $\mathcal{T}$ *of a compact smooth d-manifold* $\mathcal{M}$*, we have that the dual graph* $\Gamma(\mathcal{T})$ *is an induced subgraph of* $\Gamma(((\mathbf{H}^{2d,n})'')_d)$.

4 The proof of Theorem 1

In this section we prove our main result, i.e., every compact smooth d-manifold $\mathcal{M}$ has twin-width $\operatorname{tww}(\mathcal{M}) \leqslant d^{O(d)}$. To streamline the notation, we let $G_{d,n} = \Gamma(((\mathbf{H}^{2d,n})'')_d)$, i.e., $G_{d,n}$ denotes the dual graph of the d-skeleton[18] of the second barycentric subdivision of the hybercubic honeycomb $\mathbf{H}^{2d,n}$. The result is based on the following property of $G_{d,n}$.

▶ **Theorem 19.** *For the twin-width of the dual graph* $G_{d,n} = \Gamma(((\mathbf{H}^{2d,n})'')_d)$ *of the d-skeleton of the second barycentric subdivision of the hypercubic honeycomb* $\mathbf{H}^{2d,n}$ *we have*

$$\operatorname{tww}(G_{d,n}) \leqslant d^{O(d)}.$$

Before proving Theorem 19, we show how it implies Theorem 1.

Proof of Theorem 1. Let $\mathcal{M}$ be a compact, smooth d-dimensional manifold. Consider a Whitney triangulation $\mathcal{T}$ of $\mathcal{M}$. By Proposition 18, $\Gamma(\mathcal{T})$ is an induced subgraph of $G_{d,n}$ and by Theorem 19, $\operatorname{tww}(G_{d,n}) \leqslant d^{O(d)}$. Hence, since twin-width is monotone under taking induced subtrigraphs (Proposition 5), we obtain $\operatorname{tww}(\Gamma(\mathcal{T})) \leqslant \operatorname{tww}(G_{d,n}) \leqslant d^{O(d)}$. ◀

To complete the proof of Theorem 1, it remains to show Theorem 19.

[17] Recall that a triangulation $\mathcal{T}$ of a compact smooth manifold $\mathcal{M}$ is called a *Whitney triangulation*, if $\mathcal{T}$ is obtained via Whitney's method discussed in Section 3.

[18] Recall that the dual graph $\Gamma(\mathcal{X})$ of a pure k-dimensional complex $\mathcal{X}$ has vertices corresponding to the k-faces of $\mathcal{X}$ and two vertices are connected if and only if their corresponding k-faces share a $(k-1)$-face.

Proof of Theorem 19. We establish the theorem by exhibiting a $d^{O(d)}$-contraction sequence $\mathfrak{S}\colon G_{d,n} \rightsquigarrow \bullet$. We will obtain $\mathfrak{S}$ by concatenating two contraction sequences $\mathfrak{S}_1 : G_{d,n} \rightsquigarrow G^*_{d,n}$ and $\mathfrak{S}_2 : G^*_{d,n} \rightsquigarrow \bullet$, referred to as the *first* and the *second epoch*, where $G^*_{d,n}$ is an appropriate subtrigraph of $\mathrm{red}(D_{n,2d})$, the $2d$-dimensional red n-grid with diagonals. In the following we regard $\mathbf{H}^{2d,n}$ and its subdivisions mainly as abstract complexes, but we will also take advantage of their geometric nature.

Preparations. Consider the $(2d+1)$-coloring $\mathcal{C}\colon \mathbf{H}^{2d,n} \to \{0,\dots,2d\}$, where we color the cubes of $\mathbf{H}^{2d,n}$ by their dimension, that is, for $c \in \mathbf{H}^{2d,n}$ we set $\mathcal{C}(c) = \dim(c)$ (Figure 7a).

The coloring $\mathcal{C}$ induces a $(2d+1)$-coloring $\mathcal{C}''\colon (\mathbf{H}^{2d,n})'' \to \{0,\dots,2d\}$ of the simplices of the second barycentric subdivision $(\mathbf{H}^{2d,n})''$ as follows. The vertices of the first barycentric subdivision $(\mathbf{H}^{2d,n})'$ are in one-to-one correspondence with the cubes in $\mathbf{H}^{2d,n}$, hence $\mathcal{C}$ induces a coloring $\mathcal{C}'_0\colon (\mathbf{H}^{2d,n})'(0) \to \{0,\dots,2d\}$ via $\mathcal{C}'_0(v_c) = \mathcal{C}(c)$, where v_c denotes the vertex of $(\mathbf{H}^{2d,n})'$ corresponding to the cube $c \in \mathbf{H}^{2d,n}$ (Figure 7b). Geometrically, v_c is in the barycenter of the cube c. When we pass to the second barycentric subdivision, the vertices of $(\mathbf{H}^{2d,n})'$ become vertices of $(\mathbf{H}^{2d,n})''$, thus there is a natural inclusion $\iota\colon (\mathbf{H}^{2d,n})'(0) \to (\mathbf{H}^{2d,n})''(0)$. Let $\mathcal{V} = \mathrm{im}(\iota) \subset (\mathbf{H}^{2d,n})''(0)$ be the image of $(\mathbf{H}^{2d,n})'(0)$ under this inclusion ι. We color the elements of $\mathcal{V}$ identically to $\mathcal{C}'_0$, that is, we consider the coloring $\mathcal{C}''_{\mathcal{V}}\colon \mathcal{V} \to \{0,\dots,2d\}$ defined as $\mathcal{C}''_{\mathcal{V}}(v) = \mathcal{C}'_0(\iota^{-1}(v))$, see Figure 7c. Now, pick a simplex $\sigma \in (\mathbf{H}^{2d,n})''$ and note that $\bigcup_{v\in\mathcal{V}} \overline{\mathrm{st}}(v) = (\mathbf{H}^{2d,n})''$, i.e., the closed stars of the vertices $v \in \mathcal{V}$ cover $(\mathbf{H}^{2d,n})''$. Let $\mathcal{V}_\sigma = \{v \in \mathcal{V} : \sigma \in \overline{\mathrm{st}}(v)\}$. We now define $\mathcal{C}''(\sigma)$ as

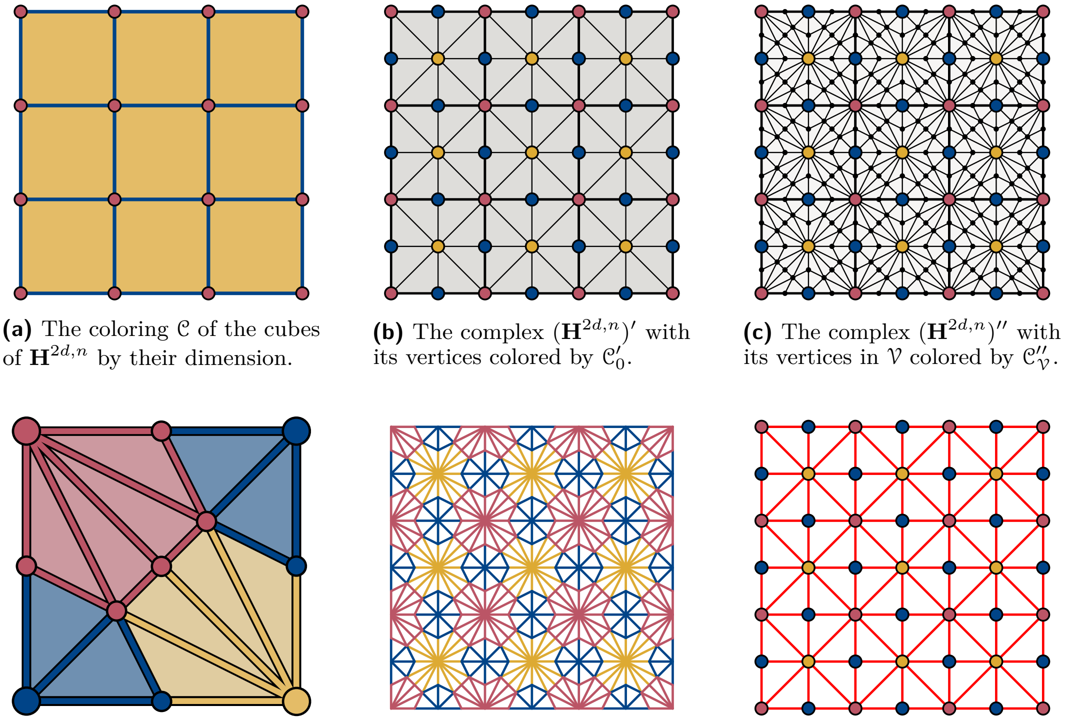
$$\mathcal{C}''(\sigma) = \min\{\mathcal{C}''_{\mathcal{V}}(v) : v \in \mathcal{V}_\sigma\}. \tag{4}$$

In words, $\mathcal{C}''(\sigma)$ is defined as the smallest dimension of any cube $c \in \mathbf{H}^{2d,n}$ such that σ belongs to the closed star of the vertex $\iota(v_c)$ in $(\mathbf{H}^{2d,n})''$, where v_c is the vertex of $(\mathbf{H}^{2d,n})'$ corresponding to c. We refer to Figure 7d for an example.

The first epoch. We start the description of the *first epoch* $\mathfrak{S}_1 : G_{d,n} \rightsquigarrow G^*_{d,n}$ by considering the restriction $\mathcal{C}''_d\colon (\mathbf{H}^{2d,n})''(d) \to \{0,\dots,2d\}$ of the coloring $\mathcal{C}''$ to the d-simplices of $(\mathbf{H}^{2d,n})''$, see Figure 7e. Note that for each $i \in \{0,\dots,2d\}$, the i-colored d-simplices of $(\mathbf{H}^{2d,n})''$ form a family $\mathcal{F}_i = \{F_{i,c} : c \in \mathbf{H}^{2d,n}(i)\}$ of pairwise-disjoint connected subcomplexes of the d-skeleton of $(\mathbf{H}^{2d,n})''$, where $F_{i,c}$ denotes the subcomplex induced by the i-colored d-simplices of $(\mathbf{H}^{2d,n})''$ belonging to the closed star of the vertex $\iota(v_c)$, where v_c is the vertex of $(\mathbf{H}^{2d,n})'$ corresponding to the i-cube c in $\mathbf{H}^{2d,n}$ (Figure 7e). We also let $\mathcal{F} = \bigcup_{i=0}^{2d} \mathcal{F}_i$.

Since $G_{d,n}$ is defined as the dual graph of the d-skeleton of $(\mathbf{H}^{2d,n})''$, the nodes of $G_{d,n}$ are in one-to-one correspondence with the d-simplices of $(\mathbf{H}^{2d,n})''$. Let $\gamma\colon (\mathbf{H}^{2d,n})''(d) \to V(G_{d,n})$ denote the bijection realizing this correspondence. Henceforth, by a slight abuse of notation, we also consider $\mathcal{C}''_d$ to be a $(2d+1)$-coloring of $V(G_{d,n})$ via $\mathcal{C}''_d(v) = \mathcal{C}''_d(\gamma^{-1}(v))$. Let $C_{i,c}$ denote the subtrigraph of $G_{d,n}$ induced by the nodes $\{\gamma(\sigma) : \sigma \in F_{i,c}\}$. Note that $\mathcal{C}''_d$ assigns the color i to all nodes of $C_{i,c}$. The first epoch $\mathfrak{S}_1 : G_{d,n} \rightsquigarrow G^*_{d,n}$ is merely the contraction sequence, where we contract each $C_{i,c}$ (where $0 \leqslant i \leqslant 2d$ and c is running over $\mathbf{H}^{2d,n}$) to a single node, in any order, obtaining a trigraph $G^*_{d,n}$ (Figure 7f).

The second epoch. $\mathfrak{S}_2 : G^*_{d,n} \rightsquigarrow \bullet$ is defined as an optimal contraction sequence of the trigraph $G^*_{d,n}$ to a single vertex. Since $G^*_{d,n}$ is a subtrigraph of $\mathrm{red}(D_{n,2d})$, the $2d$*-dimensional red* n*-grid with diagonals*, by Theorem 9 the width of $\mathfrak{S}_2$ is at most $2(3^{2d}-1)$.



(a) The coloring $\mathcal{C}$ of the cubes of $\mathbf{H}^{2d,n}$ by their dimension.

(b) The complex $(\mathbf{H}^{2d,n})'$ with its vertices colored by $\mathcal{C}'_0$.

(c) The complex $(\mathbf{H}^{2d,n})''$ with its vertices in $\mathcal{V}$ colored by $\mathcal{C}''_{\mathcal{V}}$.

(d) The coloring $\mathcal{C}''$ near the upper left corner of $(\mathbf{H}^{2d,n})''$. The four larger disks represent vertices that belong to $\mathcal{V}$.

(e) The coloring $\mathcal{C}''$ restricted to the d-simplices of $(\mathbf{H}^{2d,n})''$. The color classes correspond to the families $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$.

(f) The trigraph $G^*_{d,n}$ obtained from $G_{d,n}$ by contracting each subtrigraph $C_{i,c}$ to a single node. All edges in $G^*_{d,n}$ are red.

Figure 7 (a)–(c) Illustrations of the cubical complex $\mathbf{H}^{2d,n}$ for $d = 1$ and $n = 4$, and of its first and second barycentric subdivisions (which are simplicial complexes) displaying the colorings described in the proof of Theorem 19. **(d)–(f)** Three essential steps in the proof of Theorem 19.

Bounding the width of the first epoch. We now give a rough estimate to show that the first epoch $\mathfrak{S}_1 \colon G_{d,n} \rightsquigarrow G^*_{d,n}$ is a $d^{O(d)}$-contraction sequence. The estimate is based on Claim 20 below, which follows from elementary properties of the hypercubic honeycomb and barycentric subdivisions.

▷ Claim 20. For any color $i \in \{0,\dots,2d\}$ and cube $c \in \mathbf{H}^{2d,n}$, the subcomplex $F_{i,c}$ of $\mathbf{H}^{2d,n}$ (resp. the subtrigraph $C_{i,c}$ of $G_{d,n}$) defined above
1. has less than $h_1(d) = 4^d((2d)!)^2\binom{2d}{d}$ d-simplices (resp. nodes), and
2. less than $h_2(d) = 9^d$ incident subcomplexes $F_{i',c'} \in \mathcal{F}$ (resp. adjacent subtrigraphs $C_{i',c'}$).

Indeed, these two facts imply that by sequentially contracting each $C_{i,c}$ into a single node, the maximum red degree remains bounded by $O(h_1(d)^3 h_2(d))$ throughout the first epoch.

Proof of Claim 20. To bound the number of d-simplices in $F_{i,c}$, note that
- $F_{i,c}$ is covered by an appropriate translate of a $2d$-dimensional cube of $\mathbf{H}^{2d}$,
- the barycentric subdivision of a $2d$-cube contains $2^{2d}(2d)!$ $2d$-simplices (Observation 14),
- the barycentric subdivision of a $2d$-simplex contains $(2d)!$ $2d$-simplices (Observation 13),
- a $2d$-simplex has $\binom{2d}{d}$ faces of dimension d (Observation 12).

Multiplying these numbers, we obtain the first part of the claim.

To bound the number of subcomplexes $F_{i',c'} \in \mathcal{F}$ incident to $F_{i,c}$, observe that the incidences between these subcomplexes reflect those between the handles in the canonical handle decomposition of $\mathbf{H}^{2d,n}$ induced by its cubical structure. Thus, the number of subcomplexes in $\mathcal{F}$ incident to $F_{i,c}$ equals $3^{2d}-1$ if $i \in \{0,2d\}$ and $3^i + 3^{2d-i} - 2$ otherwise. Both of these numbers are less than 9^d, so the second part of the claim also holds. ◁

The width of $\mathfrak{S}$ is the maximum of the widths of $\mathfrak{S}_1$ and $\mathfrak{S}_2$, hence $\mathrm{tww}(G_{d,n}) \leqslant d^{O(d)}$. ◀

5 Triangulations with dual graph of arbitrary large twin-width

In Section 4 we showed that every compact, smooth d-manifold admits a triangulation with dual graph of twin-width at most $d^{O(d)}$. In this section we prove complementary results, showing that triangulations with dual graphs of large twin width are abundant. We shed light on this fact in two ways. First, we show that for any fixed dimension $d \geqslant 3$, the class of $(d+1)$-regular graphs that can be dual graphs of triangulated d-manifolds is *not* small. Second, we show that the d-dimensional ball admits triangulations with a dual graph of arbitrarily large twin-width, which extends to every piecewise-linear (hence smooth) d-manifold. Both of these results rely on counting arguments, and thus are not constructive.

5.1 The class of dual graphs of triangulations is not small

Let us fix an integer $d \geqslant 3$. Following [20], we let $M_d(n)$ denote the number of colored[19] triangulations of closed orientable d-dimensional manifolds consisting of n d-simplices labeled from 1 to n. We assume for convenience that n is even. By [20, Theorem 1.1] we have[20]

$$n! \cdot n^{n/(2d)} \preceq M_d(n). \tag{5}$$

[19] We refer to [20, Section 2.1] for the precise definitions. Since colored triangulations form a subfamily of uncolored triangulations (those considered in this paper), any lower bound on $M_d(n)$ is automatically a lower bound on the number of uncolored d-dimensional labeled triangulations with n simplices.

[20] Here "$\preceq$" denotes a comparison where exponential factors are ignored: more precisely, $f(n) \preceq g(n)$ means that there exists some constant $K > 0$, such that for n large enough, we have $f(n) \leqslant K^n g(n)$.

Now, any $(d+1)$-regular graph G with n vertices can be the dual graph of at most $d!^{n(d+1)/2}$ different d-dimensional triangulations. Indeed, if G is the dual graph of some d-dimensional triangulation, then each of the $n(d+1)/2$ edges of G corresponds to a face gluing, i.e., an identification of two $(d-1)$-dimensional simplices via a simplicial isomorphism, of which there are $d!$ many.[21] Hence by (5), for the number $\Gamma_d(n)$ of $(d+1)$-regular graphs on n labeled vertices that are dual graphs of some d-dimensional triangulations, we have

$$\frac{n! \cdot n^{n/(2d)}}{d!^{n(d+1)/2}} \preceq \Gamma_d(n). \tag{6}$$

As the left-hand side of (6) grows super-exponentially in n, the next theorem directly follows.

▶ **Theorem 21.** *For every $d \geqslant 3$, the class of $(d+1)$-regular graphs that are dual graphs of triangulations of d-manifolds is not small. In particular, there are graphs with arbitrarily large twin-width in this class.*

5.2 Complicated triangulations of the d-dimensional ball

In this section we show Theorem 3, according to which every piecewise-linear (PL) d-manifold ($d \geqslant 3$) admits triangulations with a dual graph of arbitrary large twin-width. To show the existence of such triangulations for every PL-manifold, we rely on the monotonicity of twin-width with respect to taking induced subtrigraphs (Proposition 5), the fact that the class of d-subdivisions of $(d+1)$-regular graphs is *not* small (Proposition 7) together with the following classical result from the theory of PL-manifolds.

▶ **Theorem 22** ([1, Corollary 1])**.** *Any triangulation of the boundary of a compact piecewise-linear (PL) manifold can be extended to a triangulation of the whole manifold.*

We first show that already the d-dimensional ball $\mathcal{B}^d = \{x \in \mathbb{R}^d : \|x\| \leqslant 1\}$ admits triangulations with dual graph of arbitrary large twin-width. More precisely, we prove:

▶ **Theorem 23.** *For every $m \in \mathbb{N}$ there is a triangulation $\mathcal{T}_m$ of the d-dimensional ball $\mathcal{B}^d$, such that* $\operatorname{tww}(\Gamma(\mathcal{T}_m)) \geqslant m$ *and $\partial\mathcal{T}_m = \partial\Delta^d$, the boundary of the standard d-simplex Δ^d.*

Proof. Let G_m be a d-subdivision of a $(d+1)$-regular graph G such that $\operatorname{tww}(G_m) \geqslant m$. The existence of such a graph is guaranteed by Proposition 7. Let $\mathcal{N}$ be a d-manifold homeomorphic to a closed regular neighborhood of a straight-line embedding of G in $\mathbb{R}^d$. Informally, $\mathcal{N}$ can be seen as a d-dimensional thickening of the graph G.

Construct an abstract triangulation of $\mathcal{N}$ as follows. Take a d-simplex σ_v for each node v of G, and fix a one-to-one correspondence between the $d+1$ facets of σ_v and the $d+1$ arcs incident to v in G. For every arc $\{u,v\} \in E(G)$, take a simplicial d-prism $P_{\{u,v\}} = \sigma_{\{u,v\}} \times [0,1]$, where $\sigma_{\{u,v\}}$ is a $(d-1)$-simplex, and attach $P_{\{u,v\}}$ to the simplices σ_u and σ_v by identifying $\sigma_{\{u,v\}} \times \{0\}$ (resp. $\sigma_{\{u,v\}} \times \{1\}$) with the facet of σ_u (resp. σ_v) that corresponds to the arc $\{u,v\}$. Now triangulate each prism $P_{\{u,v\}}$ with a minimal triangulation consisting of d d-simplices stacked onto each other, see Example 25 below.

Let $\mathcal{T}_\mathcal{N}$ denote the resulting triangulation of $\mathcal{N}$.

▷ Claim 24. For the dual graph of the triangulation $\mathcal{T}_\mathcal{N}$ we have $\Gamma(\mathcal{T}_\mathcal{N}) = G_m$.

[21] This is because a simplicial isomorphism between two $(d-1)$-simplices σ and τ is determined by a perfect matching between the d vertices of σ and the d vertices of τ.

Proof. The claim follows immediately from the facts that the dual graph of the constructed triangulation of the simplicial d-prism is merely a path of length d, and G_m is the d-subdivision of the $(d+1)$-regular graph G on which the triangulation $\mathcal{T}_\mathcal{N}$ is modeled. $\lhd$

Pick a sufficiently large $\ell \in \mathbb{N}$ such that the ℓ^{th} iterated barycentric subdivision $\mathcal{T}_\mathcal{N}^{(\ell)}$ of $\mathcal{T}_\mathcal{N}$ embeds linearly in $\mathbb{R}^d$ and fix a simplex-wise linear embedding $\mathcal{E}\colon \|\mathcal{T}_\mathcal{N}^{(\ell)}\| \to \mathbb{R}^d$. Consider a large geometric d-simplex $\Sigma \subset \mathbb{R}^d$ that contains the image $\mathrm{im}(\mathcal{E})$ of $\mathcal{E}$ in its interior. Now $\Sigma_\circ = \Sigma \setminus \mathrm{int}(\mathrm{im}(\mathcal{E}))$ is a PL manifold with triangulated boundary, so by Theorem 22 this boundary triangulation can be extended to a triangulation $\mathcal{T}_{\Sigma_\circ}$ of the entire manifold $\Sigma_\circ$. The boundary of $\mathcal{T}_{\Sigma_\circ}$ has two connected components: $\partial_1\mathcal{T}_{\Sigma_\circ} \cong \partial(\mathcal{T}_\mathcal{N}^{(\ell)})$ and $\partial_2\mathcal{T}_{\Sigma_\circ} \cong \partial\Delta^d$.

Let $\mathcal{Q}$ be a d-manifold homeomorphic to $\partial\mathcal{T}_\mathcal{N} \times [0,1]$. Take a triangulation $\mathcal{T}_\mathcal{Q}$ of $\mathcal{Q}$, such that for the two boundary components we have $\partial_1\mathcal{T}_\mathcal{Q} \cong \partial\mathcal{T}_\mathcal{N}$ and $\partial_2\mathcal{T}_\mathcal{Q} \cong \partial(\mathcal{T}_\mathcal{N}^{(\ell)})$. One way to construct such a triangulation is as follows. First, consider the decomposition $\mathcal{P}_\mathcal{Q}$ of $\mathcal{Q}$ into simplicial d-prisms induced by the product structure $\partial\mathcal{T}_\mathcal{N} \times [0,1]$. That is, $\mathcal{P}_\mathcal{Q}$ consists of d-prisms $P_\sigma \cong \sigma \times [0,1]$, one for each $(d-1)$-simplex σ of $\partial\mathcal{T}_\mathcal{N}$, glued together along their vertical boundary prisms the same way as the simplices of $\partial\mathcal{T}_\mathcal{N}$. The boundary $\partial\mathcal{P}_\mathcal{Q}$ of $\mathcal{P}_\mathcal{Q}$ has two connected components: $\partial_1\mathcal{P}_\mathcal{Q}$ and $\partial_2\mathcal{P}_\mathcal{Q}$, each combinatorially isomorphic to $\partial\mathcal{T}_\mathcal{N}$. Next, pass to the ℓ^{th} iterated barycentric subdivision of $\partial_2\mathcal{P}_\mathcal{Q}$. This operation turns each prism P_σ of $\mathcal{P}_\mathcal{Q}$ into a polyhedral cell R_σ. These cells form a polyhedral decomposition $\mathcal{R}_\mathcal{Q}$ of $\mathcal{Q}$, where $\partial_1\mathcal{R}_\mathcal{Q} \cong \partial\mathcal{T}_\mathcal{N}$ and $\partial_2\mathcal{R}_\mathcal{Q} \cong (\partial\mathcal{T}_\mathcal{N})^{(\ell)} = \partial(\mathcal{T}_\mathcal{N}^{(\ell)})$. Triangulate R_σ as follows. Consider an order $c_1 \prec c_2 \prec \cdots$ of the vertical cells[22] of R_σ, where $c_i \prec c_j$ implies $\dim(c_i) \leqslant \dim(c_j)$. Place a new vertex v_i in the barycenter of c_i and, iterating over the vertical cells in the above order, triangulate c_i by coning from v_i over its (already triangulated) boundary ∂c_i. It is clear that the resulting triangulation of R_σ is symmetric with respect to the symmetries of its base simplex σ. Applying this procedure for each polyhedral cell R_σ of $\mathcal{R}_\mathcal{Q}$ yields a triangulation $\mathcal{T}_\mathcal{Q}$ of $\mathcal{Q}$ with the desired properties.

Now the triangulation $\mathcal{T}_m$ of $\mathcal{B}^d$ is obtained by gluing together $\mathcal{T}_\mathcal{N}$, $\mathcal{T}_\mathcal{Q}$ and $\mathcal{T}_{\Sigma_\circ}$ via the identity maps along the isomorphic boundary-pairs $\partial\mathcal{T}_\mathcal{N} \cong \partial_1\mathcal{T}_\mathcal{Q}$ and $\partial_2\mathcal{T}_\mathcal{Q} \cong \partial_1\mathcal{T}_{\Sigma_\circ}$.

To conclude, note that the d-simplicies of $\mathcal{T}_\mathcal{N}$ triangulate a submanifold of $\|\mathcal{T}_m\|$, hence by Proposition 15 the graph $\Gamma(\mathcal{T}_\mathcal{N})$ is an induced subgraph of $\Gamma(\mathcal{T}_m)$. By Claim 24, $\Gamma(\mathcal{T}_\mathcal{N}) = G_m$ and by the initial assumption $\mathrm{tww}(G_m) \geqslant m$, hence by Proposition 5, $\Gamma(\mathcal{T}_m) \geqslant m$ as well. ◀

Proof of Theorem 3. Let $\mathcal{M}$ be an arbitrary PL-manifold possibly with non-empty boundary and $\mathcal{T}_\circ$ be a simplicial triangulation of $\mathcal{M}$. Consider a triangulation $\mathcal{T}_m$ of the d-ball with $\mathrm{tww}(\Gamma(\mathcal{T}_m)) \geqslant m$ as in Theorem 23. Let Δ be a d-simplex of $\mathcal{T}_\circ$ that is disjoint from $\partial\mathcal{M}$ (if $\mathcal{T}_\circ$ does not contain such a simplex, just replace $\mathcal{T}_\circ$ with its second barycentric subdivision). Since $\mathcal{T}_\circ$ is simplicial, Δ is embedded in $\mathcal{T}_\circ$ and is topologically a d-ball. Now replace Δ with $\mathcal{T}_m$ by first removing Δ from $\mathcal{T}_\circ$, thereby creating a boundary component isomorphic to $\partial\Delta$, then gluing $\partial\mathcal{T}_m$ to this new boundary component via a simplicial isomorphism. (Note that this is possible since $\partial\mathcal{T}_m \cong \partial\Delta$.) Let $\mathcal{T}$ denote the resulting triangulation of $\mathcal{M}$. By Propositions 5 and 15, it follows that $\mathrm{tww}(\Gamma(\mathcal{T})) \geqslant \mathrm{tww}(\Gamma(\mathcal{T}_m)) \geqslant m$. ◀

▶ **Example 25** (triangulating simplicial d-prisms)**.** Let σ be a $(d-1)$-simplex with vertices $\{v_1, \ldots, v_d\}$, and let $P_\sigma = \sigma \times [0,1]$ be its associated simplicial d-prism. Note that for the vertex set of P_σ we have $P_\sigma(0) = \{(v_1,0), \ldots, (v_d,0), (v_1,1), \ldots, (v_d,1)\}$. A triangulation of P_σ with d-simplices $\{\overline{\sigma}_1, \ldots, \overline{\sigma}_d\}$ can be obtained as follows. We define $\overline{\sigma}_i$ iteratively,

[22] These are precisely those cells that are not contained in the boundary of $\mathcal{R}_\mathcal{Q}$.

through their vertex sets. First, set $\overline{\sigma}_1 = \{(v_1, 0), \dots, (v_d, 0), (v_1, 1)\}$. Next, for $2 \leqslant i \leqslant d$ the vertex set of $\overline{\sigma}_i$ is simply obtained from that of $\overline{\sigma}_{i-1}$ by replacing $(v_i, 0)$ with $(v_i, 1)$.

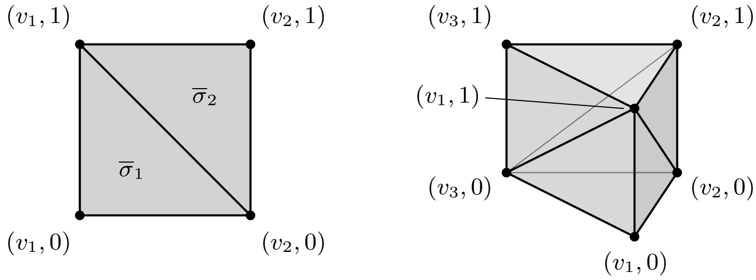


Figure 8 The considered triangulations of the simplicial d-prism for $d = 2$ and $d = 3$.

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