Twin-width can be exponential in treewidth

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Abstract

For any small positive real ε and integer $t > \frac{1}{\varepsilon}$, we build a graph with a vertex deletion set of size t to a tree, and twin-width greater than $2^{(1-\varepsilon)t}$. In particular, this shows that the twin-width is sometimes exponential in the treewidth, in the so-called oriented twin-width and grid number, and that adding an apex may multiply the twin-width by at least $2-\varepsilon$. Except for the one in oriented twin-width, these lower bounds are essentially tight.

1 Introduction

Twin-width is a graph parameter introduced by Bonnet, Kim, Thomassé, and Watrigant [12]. It is defined by means of trigraphs. A *trigraph* is a graph with some edges colored black, and some colored red. A (vertex) *contraction* consists of merging two (non-necessarily adjacent) vertices, say, u, v into a vertex w, and keeping every edge wz black if and only if uz and vz were previously black edges. The other edges incident to w become red (if not already), and the rest of the trigraph remains the same. A *contraction sequence* of an n-vertex graph G is a sequence of trigraphs $G = G_n, \dots, G_1 = K_1$ such that G_i is obtained from G_{i+1} by performing one contraction. A *d-sequence* is a contraction sequence in which every vertex of every trigraph has at most d red edges incident to it. The *twin-width* of G, denoted by $\text{tww}(G)$, is then the minimum integer d such that G admits a d-sequence. Figure 1 gives an example of a graph with a 2-sequence, i.e., of twin-width at most 2. Twin-width can be naturally

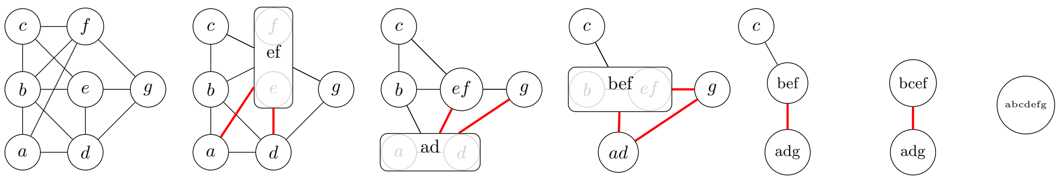

Figure 1 A 2-sequence witnessing that the initial graph has twin-width at most 2.

extended to matrices (unordered [12] or ordered [9]) over a finite alphabet, and hence to any binary structures. Classes of binary structures with bounded twin-width include graphs with bounded treewidth, bounded clique-width, K_t-minor free graphs, posets with antichains of bounded size, strict subclasses of permutation graphs, map graphs, bounded-degree string graphs [12], segment graphs with no $K_{t,t}$ subgraph, visibility graphs of 1.5D terrains without large half-graphs, visibility graphs of simple polygons without large independent sets [6], as well as $\Omega(\log n)$-subdivisions of n-vertex graphs, classes with bounded queue number or bounded stack number, and some classes of cubic expanders [7].

Despite their apparent generality, classes of bounded twin-width are small [7], χ-bounded [8], even quasi-polynomially χ-bounded [17], preserved (albeit with a higher upper bound) by first-order transductions [12], and by the usual graph products when one graph has bounded degree [16, 7], have VC density 1 [11, 19], admit, when $O(1)$-sequences are given, a fixed-parameter tractable first-order model checking [12], an (almost) single-exponential parameterized algorithm for various problems that are W[1]-hard in general [8], as well as a parameterized fully-polynomial linear algorithm for counting triangles [15], an (almost) linear representation [18], a stronger regularity lemma [19], etc.

In all these applications, the upper bound on twin-width, although somewhat hidden in the previous paragraph, plays a role. There is then an incentive to obtain as low as possible upper bounds on particular classes of bounded twin-width. To give one concrete algorithmic example, an independent set of size k can be found in time $O(k^2 d^{2k} n)$ in an n-vertex graph given with a d-sequence [8]. This is relatively practical for moderate values of k, with the guarantee that d is below 10, but not when d is merely upperbounded by 10^{10}. Another motivating example: triangle-free graphs of twin-width at most d are $d+2$-colorable [8], a stronger fact in the former case than in the latter.

In that line of work, Balabán and Hlinený show that posets of width k (i.e., with antichains of size at most k) have twin-width at most $9k$ [2]. Unit interval graphs have twin-width at most 2 [8], and proper k-mixed-thin graphs (a recently proposed generalization of unit interval graphs) have twin-width $O(k)$ [3]. Every graph obtained by subdividing at least $2 \log n$ (throughout the paper, all logs are in base 2) times each edge of an n-vertex graph has twin-width at most 4 [4]. Schidler and Szeider report the (exact) twin-width of a collection of graphs [20], obtained via SAT encodings. Jacob and Pilipczuk [14] give the current best upper bound of 183 on the twin-width of planar graphs, while graphs with genus g have twin-width $O(g)$ [13]. Most relevant to our paper, for every graph G, $\text{tww}(G) \leqslant 3 \cdot 2^{\text{tw}(G)-1}$ [14], where $\text{tw}(G)$ denotes the treewidth of G.

Conversely, one may ask the following.

▶ **Question 1.** *What is the largest twin-width a graph of treewidth k can have?*

A lower bound of $\Omega(k)$ comes from the existence of n-vertex graphs with twin-width $\Omega(n)$ (since the treewidth is trivially upperbounded by $n-1$). This is almost surely the case of graphs drawn from $G(n, 1/2)$. Alternatively, the n-vertex Paley graph (for a prime n such that $n \equiv 1 \mod 4$) has precisely twin-width $(n-1)/2$ [1]. Another example to derive the linear lower bound is the power set graph [14]. Improving on this lower bound is not obvious, and $\Theta(k)$ is indeed the answer to Question 1 within the class of planar graphs [14], or when replacing 'treewidth' by 'cliquewidth' or 'pathwidth.'

When switching 'twin-width' and 'treewidth' in Question 1, the gap is basically as large as possible: There are n-vertex graphs with treewidth $\Omega(n)$ and twin-width at most 6, in the iterated 2-lifts of K_4 [7, 5].

An important characterization of bounded twin-width is via the absence of complex divisions of an adjacency matrix. A matrix has a *k-mixed minor* if its row (resp. column) set can be partitioned into k sets of consecutive rows (resp. columns), such that each of the k^2 cells defined by this *k-division* contains at least two distinct rows and at least two distinct columns. The *mixed number of a matrix M* is the largest integer k such that M admits a k-mixed minor. The *mixed number of a graph G*, denoted by $\text{mxn}(G)$, is the minimum, taken among all the adjacency matrices M of G, of the mixed number of M. The following was shown.

▶ **Theorem 1** ([12]). *For every graph G, $(mxn(G)-1)/2 \leqslant tww(G) \leqslant 2^{2^{O(mxn(G))}}$.*

In sparse graphs (here, excluding a fixed $K_{t,t}$ as a subgraph), the previous theorem is both simpler to formulate and has a better dependency. A matrix has a *k-grid minor* if it has a *k-division* with at least one 1-entry in each of its k^2 cells. The *grid number* of a matrix and of a graph G, denoted by $\text{gn}(G)$, are defined analogously to the previous paragraph. We only state the inequality that is useful to bound the twin-width of a sparse class, but is valid in general.

▶ **Theorem 2** (follows from [12]). *For every graph G, $tww(G) \leqslant 2^{O(gn(G))}$.*

Theorems 1 and 2 allow to bound the twin-width of a class $\mathcal{C}$ by exhibiting, for every $G \in \mathcal{C}$, an adjacency matrix of G without large mixed or grid minor. Therefore one merely has to order $V(G)$ (the vertex set of G) in an appropriate way. The double (resp. simple) exponential dependency in mixed number (resp. grid number) implies relatively weak twin-width upper bounds. For several classes whose twin-width was originally upperbounded via Theorem 1, better bounds were later given by avoiding this theorem (see [7, 2, 14, 13, 4]). Still for some geometric graph classes, bypassing Theorem 1 seems complicated (see [6]). And in general (since this theorem is at the basis of several other applications, see for instance [7, 8, 9]) it would help to have an improved upper bound of $\mathrm{tww}(G)$; in particular a negative answer to the following question.

▶ **Question 2.** *Is twin-width sometimes exponential in mixed and grid number?*

A variant of twin-width, called *oriented twin-width*, adds an orientation to the red edges (see [10]). The red edge (arc) is oriented away from the contracted vertex. The *oriented twin-width* d of a graph G, denoted by $\mathrm{otww}(G)$, is then defined similarly as twin-width by tolerating more than d red arcs incident to a vertex, as long as at most d of them are out-going. Rather surprisingly twin-width and oriented twin-width are tied.

▶ **Theorem 3** ([10]). *For every graph G, $otww(G) \leqslant tww(G) \leqslant 2^{2^{O(otww(G))}}$.*

Classic results show that planar graphs have oriented twin-width at most 9 [10]. Thus it would be appreciable to lower the dependency of $\mathrm{tww}(G)$ in $\mathrm{otww}(G)$.

▶ **Question 3.** *Is twin-width sometimes exponential in oriented twin-width?*

An elementary argument shows that when adding an apex (i.e., an additional vertex with an arbitrary neighborhood) to a graph G, the twin-width of the obtained graph is at most $2 \cdot \mathrm{tww}(G) + 1$. Again it is not clear whether this increase could be made smaller.

▶ **Question 4.** *Does twin-width sometimes essentially double when an apex is added?*

Note that Question 1 is asked by Jacob and Pilipczuk [14], and Question 3 is posed by Bonnet et al. [10], and is closely related to Question 2.

Our contribution.

With a single construction, we answer all these questions. The answer to Questions 2, 3, and 4 is affirmative, while the answer to Question 1 is $2^{\Theta(k)}$, which confirms the intuition of the authors of [14]. More precisely, we show the following.

▶ **Theorem 4.** *For every real $0 < \varepsilon \leqslant 1/2$ and integer $t > 1/\varepsilon$, there is a graph $G_{t,\varepsilon}$ with a feedback vertex set of size t and such that $tww(G_{t,\varepsilon}) > 2^{(1-\varepsilon)t}$.*

The graph $G_{t,\varepsilon}$ has in particular treewidth at most $t+1$, grid number at most $t+2$, and oriented twin-width at most $t+1$. Thus

- $\mathrm{tww}(G_{t,\varepsilon}) > 2^{(1-\varepsilon)(\mathrm{tw}(G_{t,\varepsilon})-1)}$,
- $\mathrm{tww}(G_{t,\varepsilon}) > 2^{(1-\varepsilon)(\mathrm{gn}(G_{t,\varepsilon})-2)}$, and
- $\mathrm{tww}(G_{t,\varepsilon}) > 2^{(1-\varepsilon)(\mathrm{otww}(G_{t,\varepsilon})-1)}$.

Hence Theorem 4 has the following consequences.

▶ **Corollary 5.** *For every small $\varepsilon > 0$, there is a family $\mathcal{F}$ of graphs with unbounded twin-width such that for every $G \in \mathcal{F}$: $tww(G) > 2^{(1-\varepsilon)(tw(G)-1)}$.*

Up to multiplicative factors, it matches the known upper bound [14, 12], and essentially settles Question 1. The following answers Question 2.

▶ **Corollary 6.** *For every small $\varepsilon > 0$, there is a family $\mathcal{F}$ of graphs with unbounded twin-width such that for every $G \in \mathcal{F}$: $tww(G) > 2^{(1-\varepsilon)(gn(G)-2)}$.*

The following answers Question 3.

▶ **Corollary 7.** *For every small $\varepsilon > 0$, there is a family $\mathcal{F}$ of graphs with unbounded twin-width such that for every $G \in \mathcal{F}$: $tww(G) > 2^{(1-\varepsilon)(otww(G)-1)}$.*

The following answers Question 4.

▶ **Corollary 8.** *For every small $\varepsilon > 0$, there is a family $\mathcal{F}$ of graphs with unbounded twin-width such that for every $G \in \mathcal{F}$: $tww(G) > (2-\varepsilon)tww(G - \{v\})$, where v is a single vertex of G.*

We leave as an open question if the twin-width upper bound in oriented twin-width and mixed number can be made single-exponential.

2 Preliminaries

For i and j two integers, we denote by $[i, j]$ the set of integers that are at least i and at most j. For every integer i, $[i]$ is a shorthand for $[1, i]$. We use the standard graph-theoretic notations: $V(G)$ denotes the vertex set of a graph G, $E(G)$ denotes its edge set, $G[S]$ denotes the subgraph of G induced by S, etc.

We give an alternative approach to contraction sequences. The *twin-width* of a graph, introduced in [12], can be defined in the following way (complementary to the one given in introduction). A *partition sequence* of an n-vertex graph G, is a sequence $\mathcal{P}_n, \ldots, \mathcal{P}_1$ of partitions of its vertex set $V(G)$, such that $\mathcal{P}_n$ is the set of singletons $\{\{v\} : v \in V(G)\}$, $\mathcal{P}_1$ is the singleton set $\{V(G)\}$, and for every $2 \leqslant i \leqslant n$, $\mathcal{P}_{i-1}$ is obtained from $\mathcal{P}_i$ by merging two of its parts into one. Two parts P, P' of a same partition $\mathcal{P}$ of $V(G)$ are said *homogeneous* if either every pair of vertices $u \in P, v \in P'$ are non-adjacent, or every pair of vertices $u \in P, v \in P'$ are adjacent. Two non-homogeneous parts are also said *red-adjacent*. The *red degree* of a part $P \in \mathcal{P}$ is the number of other parts of $\mathcal{P}$ which are red-adjacent to P. Finally the twin-width of G, denoted by $\text{tww}(G)$, is the least integer d such that there is a partition sequence $\mathcal{P}_n, \ldots, \mathcal{P}_1$ of G with every part of every $\mathcal{P}_i$ ($1 \leqslant i \leqslant n$) having red degree at most d.

The definition of the previous paragraph is equivalent to the one given in introduction, via contraction sequences. Indeed the trigraph G_i is obtained from partition $\mathcal{P}_i$, by having one vertex per part of $\mathcal{P}_i$, a black edge between any fully adjacent pair of parts, and a red edge between red-adjacent parts. A *partial contraction sequence* is a sequence of trigraphs $G_n, \ldots, G_i$, for some $i \in [n]$. A (full) *contraction sequence* is one such that $i = 1$. We naturally consider the trigraph G_j to come *after* (resp. *before*) $G_{j'}$ if $j < j'$ (resp. $j > j'$). Thus when we write *the first trigraph of the sequence $\mathcal{S}$ to satisfy X* (or *the first time a trigraph of $\mathcal{S}$ satisfies X*) we mean the trigraph G_j with largest index j among those satisfying X. The same goes for partition sequences.

If u is a vertex of a trigraph H, then $u(G)$ denotes the set of vertices of G eventually contracted into u in H. We denote by $\mathcal{P}_G(H)$ (and $\mathcal{P}(H)$ when G is clear from the context) the partition $\{u(G) : u \in V(H)\}$ of $V(G)$. We may refer to a *part* of H as any set in $\{u(G) : u \in V(H)\}$. We may also refer to a *part* of a contraction/partition sequence as any part of one its trigraphs/partitions. A contraction *involves* a vertex v if it produces a new part (of size at least 2) containing v. In general, we use trigraphs and partitioned graphs somewhat interchangeably, when one notion appears more convenient than the other.

3 Proof of Theorem 4

We fix once and for all, $0 < \varepsilon \leqslant 1/2$, a possibly arbitrarily small positive real. We build for every integer $t > 1/\varepsilon$, a graph $G_{t,\varepsilon}$, that we shorten to G_t. We set

$$f(t) = \left\lceil 2 + C_t 2^{(1-\varepsilon)t(2+C_t(2^{(1-\varepsilon)t}+1))} \right\rceil$$

where $C_t = 2^{(1-\varepsilon)t}/\varepsilon$.

Construction of G_t. Let T be the full 2^t-ary tree of depth $f(t)$, i.e., with root-to-leaf paths on $f(t)$ edges. Let X be a set of t vertices, that we may identify to $[t]$. The vertex set of G_t is $X \uplus V(T)$. The edges of G_t are such that $G[X]$ is an independent set, and $G[V(T)] = T$. The edges between $V(T)$ are X are such that

- the root of T has no neighbor in X, and
- the 2^t children (in T) of every internal node of T each have a distinct neighborhood in X.

Note that this defines a single graph up to isomorphism. By a slight abuse of language, we may utilize the usual vocabulary on trees directly on G_t. By *root*, *internal node*, *child*, *parent*, *leaf* of G_t, we mean the equivalent in T.

We start with this straightforward observation.

▶ **Lemma 9.** *G_t has treewidth at most $t + 1$.*

Proof. The set X is a feedback vertex set of G_t of size t, thus $\text{tw}(G_t) \leqslant \text{fvs}(G_t)+1 \leqslant t+1$. ◀

The following is the core lemma, which occupies us for the remainder of the section.

▶ **Lemma 10.** *G_t has twin-width greater than $2^{(1-\varepsilon)t}$.*

Proof. We assume, by way of contradiction, that G_t admits a d-sequence with $d \leqslant 2^{(1-\varepsilon)t}$. We consider the partial d-sequence $\mathcal{S}$, starting at G_t, and ending right before the first contraction involving a child of the root. We first show that no vertex of X can be involved in a contraction of $\mathcal{S}$. Note that it implies, in particular, that the root cannot be involved in a contraction of $\mathcal{S}$.

▷ Claim 11. No part of $\mathcal{S}$ contains more than one vertex of X.

Proof of the Claim: Observe that, for every $i \neq j \in [t]$, there are 2^{t-1} sets of $2^{[t]}$ containing exactly one of i, j: 2^{t-2} only contain i, and 2^{t-2} only contain j. Recall now that by assumption, in every trigraph of $\mathcal{S}$, every child of the root is alone in its part. Thus a part P of $\mathcal{S}$ such that $|P \cap X| \geqslant 2$ would have red degree at least $2^{t-1} > 2^{(1-\varepsilon)t} \geqslant d$. ◊

▷ Claim 12. No part of $\mathcal{S}$ intersects both X and $V(T)$.

Proof of the Claim: For the sake of contradiction, consider the first occurrence of a part $P \supseteq \{x, v\}$ with $x \in X$ and $v \in V(T)$. Vertex x is adjacent to half of the children of the root, whereas v is adjacent to at most one of them, or all of them (if v is itself the root). In both cases, this entails at least $2^{t-1} - 1$ red edges for P towards children of the root. If v is not a grandchild of the root, the red degree of P is at least 2^{t-1}. We thus assume that v is a grandchild of the root.

As $t \geqslant 2$, there is a $y \in X \setminus \{x\}$. Let v' be the child of v whose neighborhood in X is exactly $\{y\}$. This vertex exists since $f(t) \geqslant 3$. If P contains v', P is also red-adjacent to $\{y\}$ (indeed a part, by Claim 11). If instead, P does not contain v', then P is also red-adjacent to the part containing v'.

Thus, in any case, the red degree of P is at least $2^{t-1} > 2^{(1-\varepsilon)t} \geqslant d$. ◇

From Claims 11 and 12, we immediately obtain:

▷ Claim 13. Every part of $\mathcal{S}$ intersecting X is a singleton.

Crucial to the proof, we introduce two properties $\mathscr{P}$, and later $\mathscr{Q}$, on internal nodes $v \in V(T)$ in trigraphs $H \in \mathcal{S}$. Property $\mathscr{P}$ is defined by

$$\mathscr{P}(v, H) = \text{“At least } 2^{\varepsilon t} \text{ children of } v \text{ are in the same part of } \mathcal{P}(H)\text{.”}$$

We first remark that any internal node in a non-singleton part verifies $\mathscr{P}$.

▷ Claim 14. Let H be any trigraph of $\mathcal{S}$ and v be any internal node of T whose part in $\mathcal{P}(H)$ is not a singleton. Then $\mathscr{P}(v, H)$ holds.

Proof of the Claim: Let P be the part of v (i.e., the one containing v) in $\mathcal{P}(H)$, and $u \in P \setminus \{v\}$. At least $2^t - 1$ children of v are not adjacent to u. Thus these $2^t - 1$ vertices have to be in at most $d + 1 \leqslant 2^{(1-\varepsilon)t} + 1$ parts. These parts are part P, plus at most d parts linked to P by a red edge. Since $(2^{\varepsilon t} - 1)(2^{(1-\varepsilon)t} + 1) < 2^t - 1$ (recall that $\varepsilon < 1/2$), one of these parts (possibly P) contains at least $2^{\varepsilon t}$ children of v. ◇

As the merge of a singleton part $\{v\}$ with any other part does not change the intersections of parts with the set of children of v, we get a slightly stronger claim.

▷ Claim 15. Let v be an internal node of T, and H be the last trigraph of $\mathcal{S}$ for which v is in a singleton part of $\mathcal{P}(H)$. Then $\mathscr{P}(v, H)$ holds.

A *preleaf* is an internal node of T adjacent to a leaf, i.e., the parent of some leaves. We obtain the following as a direct consequence of Claim 14.

▷ Claim 16. In any trigraph $H \in \mathcal{S}$, any non-preleaf internal node $v \in V(T)$ that verifies $\mathscr{P}(v, H)$ has at least $2^{\varepsilon t}$ children u verifying $\mathscr{P}(u, H)$.

We define the property $\mathscr{Q}$ on internal nodes v of T and trigraphs $H \in \mathcal{S}$ by induction:

$$\mathscr{Q}(v, H) = \begin{cases} \mathscr{P}(v, H) & \textit{if } v \textit{ is a preleaf, and otherwise} \\ \mathscr{Q}(u_1, H) \wedge \mathscr{Q}(u_2, H) & \textit{for some pair } u_1 \neq u_2 \textit{ of children of } v. \end{cases}$$

That is, $\mathscr{Q}$ is defined as $\mathscr{P}$ for preleaves, and otherwise, $\mathscr{Q}$ holds when it holds for at least two of its children. Observe that $\mathscr{P}$ and $\mathscr{Q}$ are monotone in the following sense: If $\mathscr{P}(v, H)$ (resp. $\mathscr{Q}(v, H)$) holds, then $\mathscr{P}(v, H')$ (resp. $\mathscr{Q}(v, H')$) holds for every subsequent trigraph H' of the partial d-sequence $\mathcal{S}$. We may write that v *satisfies* $\mathscr{P}$ *(resp.* $\mathscr{Q}$*) in* H when $\mathscr{P}(v, H)$ (resp. $\mathscr{Q}(v, H)$) holds, and may add *for the first time* if no trigraph $H' \in \mathcal{S}$ before H is such that $\mathscr{P}(v, H')$ (resp. $\mathscr{Q}(v, H')$) holds.

▷ Claim 17. For any trigraph $H \in \mathcal{S}$ and internal node v of T, $\mathscr{P}(v, H)$ implies $\mathscr{Q}(v, H)$.

Proof of the Claim: This is a tautology if v is a preleaf. The induction step is ensured by Claim 16, since $2^{\varepsilon t} \geqslant 2$. ◇

At the end of the partial d-sequence $\mathcal{S}$, we know, by Claim 15, that at least one child of the root satisfies $\mathscr{P}$, hence satisfies $\mathscr{Q}$, by Claim 17. Thus the first time in the partial d-sequence $\mathcal{S}$ that $\mathscr{Q}(v, H)$ holds, for a trigraph $H \in \mathcal{S}$ and a child v of the root, is well-defined. We call F this trigraph, and v_0 a child of the root such that $\mathscr{Q}(v_0, F)$ holds.

We now find many nodes satisfying $\mathscr{Q}$ in F, whose parents form a vertical path of singleton parts.

▷ Claim 18. There is a set $Q \subset V(T)$ of at least $f(t) - 2$ internal nodes such that

- for every $v \in Q$, $\mathscr{Q}(v, F)$ holds,
- the parent of any $v \in Q$ is in a singleton part of $\mathcal{P}(F)$, and
- and *no* two distinct nodes of Q are in an ancestor-descendant relationship.

Proof of the Claim: We construct by recurrence two sequences $(v_i)_{i \in [f(t)-2]}$, $(q_i)_{i \in [0, f(t)-3]}$ of internal nodes of T such that for all $i \in [f(t)-2]$, v_i is a child of v_{i-1}, v_{i-1} is in a singleton part of $\mathcal{P}(F)$, and v_{i-1} has a child $q_{i-1} \neq v_i$ for which $\mathscr{Q}(q_{i-1}, F)$ holds.

Assume that the sequence is defined up to v_i, for some $i < f(t) - 2$. We will maintain the additional invariant that v_i satisfies $\mathscr{Q}$ for the first time in F. This is the case for $i = 0$.

As v_i is not a preleaf, it satisfies $\mathscr{Q}$ for the first time when a second child of v_i satisfies $\mathscr{Q}$. Let v_{i+1} be this second child, and q_i be the first child to satisfy $\mathscr{Q}$ (breaking ties arbitrarily if both children satisfy $\mathscr{Q}$ for the first time in F). The vertex v_{i+1} satisfies $\mathscr{Q}$ for the first time in F. Thus our invariant is preserved.

For every $i \in [f(t) - 2]$, v_i is in a singleton part of $\mathcal{P}(F)$. Indeed, by Claim 15, if v_i was not in a singleton part of $\mathcal{P}(F)$, v_i would satisfy $\mathscr{P}$, hence $\mathscr{Q}$, in the trigraph preceding F; a contradiction.

The set Q can thus be defined as $\{q_i \; : \; i \in [0, f(t) - 3]\}$. We already checked that the first two requirements of the lemma are fulfilled. No pair in Q is in an ancestor-descendant relationship since the nodes of Q are all children of a root-to-leaf path made by the v_is (see Figure 2). ◇

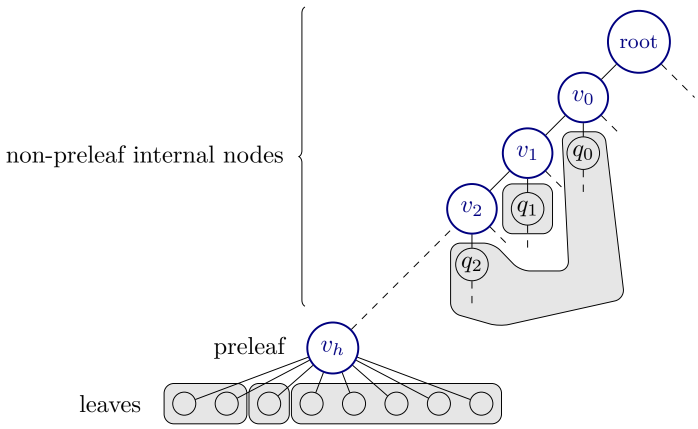

Figure 2 The nodes $(v_i)_{i \in [0,h]}$ and $(q_i)_{[0,h-1]}$ ($h = f(t) - 2$) satisfy $\mathscr{P}$ and $\mathscr{Q}$ in F. The v_is and the root (nodes circled in blue) are in singleton parts of F. The other represented nodes can be in larger parts (shaded areas).

Let B the vertices $w \in V(F)$ such that $w(G)$ contains at least $2^{\varepsilon t}$ children of the same node of T. Each vertex of B is red-adjacent to at least $\log(2^{\varepsilon t}) = \varepsilon t$ (singleton) parts of X. Therefore, since the red degree of (singleton) parts of X is at most $2^{(1-\varepsilon)t}$:

$$|B| \leqslant \frac{2^{(1-\varepsilon)t}}{\varepsilon}.$$

Next we show that there is relatively large set of vertices of F each corresponding to a non-singleton part that contains an internal node of T.

▷ Claim 19. There is a set $B' \subseteq V(F)$ of size at least

$$\frac{1}{(1-\varepsilon)t} \log\left(\frac{f(t)-2}{|B|}\right) - 1$$

such that for every $b \in B'$ there is an internal node v of T with $v \in b(G_t)$ and $|b(G_t)| \geqslant 2$.

Proof of the Claim: Let $s := \frac{1}{(1-\varepsilon)t}\log(\frac{f(t)-2}{|B|}) - 1$. Our goal is to construct a sequence $(b_i)_{i \in [0,s]}$ of distinct vertices of F such that for every $i \in [s]$,

part $b_i(G_t)$ is *not* a singleton and contains an internal node of T. (1)

We first focus on finding b_0. Note that b_0 need not satisfy Invariant (1), but will be chosen to force the existence of b_1 itself satisfying (1) and starting the induction.

Let $Q := \{q_j \ : \ 0 \leqslant j \leqslant f(t)-3\} \subset V(T)$ be as described in Claim 18. Every $q_j \in Q$ has (at least) one descendant q'_j that is a preleaf and satisfies $\mathcal{Q}$, hence $\mathcal{P}$, in F. The q'_js are pairwise distinct because no two nodes of Q are in an ancestor-descendant relationship. We set $Q' := \{q'_j \ : \ 0 \leqslant j \leqslant f(t)-3\}$.

Now for every q'_j, at least $2^{\varepsilon t}$ of its children are in the same part of $\mathcal{P}(F)$; hence, this part corresponds to a vertex in B. By the pigeonhole principle, there is a $b_0 \in B$ that contains at least $2^{\varepsilon t}$ children of at least $(f(t)-2)/|B|$ nodes of Q'.

For each b_i, we define $Q_i \subset Q$ as the set of vertices q_j such that
- $b_i(G_t)$ contains a (not necessarily strict) descendant z of q_j, and
- no part $b_{i'}(G_t)$ with $i' < i$ contains a node on the path between q_j and z in T.

Thus $|Q_0| \geqslant (f(t)-2)/|B|$.

We now assume that $b_i \in V(F)$, for some $0 \leqslant i < s$, has been found with

$$|Q_i| \geqslant \frac{f(t)-2}{|B| \cdot 2^{i(1-\varepsilon)t}}. \tag{2}$$

Observe that Q_0 satisfies (2). We construct b_{i+1}, Q_{i+1} satisfying the invariants (1) and (2).

For each $q_j \in Q_i$, consider the highest descendant z_j of q_j in $b_i(G_t)$, and z'_j the parent of z_j in T. By construction, the part P_j of $\mathcal{P}(F)$ containing z'_j is not a $b_k(G_t)$ for any $k \leqslant i$. Part P_j is linked to $b_i(G_t)$ by a red edge. Therefore there are at most $2^{(1-\varepsilon)t}$ such parts P_j. In particular, there is a $b_{i+1} \in V(F)$ such that $b_{i+1}(G_t)$ contains at least

$$\frac{|Q_i|}{d} \geqslant \frac{f(t)-2}{|B| \cdot 2^{i(1-\varepsilon)t}} \cdot \frac{1}{2^{(1-\varepsilon)t}} = \frac{f(t)-2}{|B| \cdot 2^{(i+1)(1-\varepsilon)t}}$$

parents z'_j of highest descendants z_j.

Remark that $b_{i+1}(G_t)$ has size at least two while $(f(t)-2)/(|B| \cdot 2^{(i+1)(1-\varepsilon)t}) > 1$, which holds since $i < s$. Thus $b_{i+1}(G_t)$ does not contain any parent v_j of a q_j (since the v_js are in singleton parts). In particular, $|Q_{i+1}| \geqslant (f(t)-2)/(|B| \cdot 2^{(i+1)(1-\varepsilon)t})$, and b_{i+1}, Q_{i+1} satisfy (1) and (2).

Finally, the set $B' := \{b_i \ : \ 1 \leqslant i \leqslant s\}$ has the required properties. ◊

We can now finish the proof of the lemma.

For every $b_i \in B'$, let $u_i \in b_i(G_t)$ be an internal node of T. As $b_i(G_t) \geqslant 2$, u_i satisfies $\mathscr{P}$ in F. This implies that b_i or a red neighbor of b_i is in B. Therefore, the total number of red edges incident to a vertex of B is at least $|B'| - |B|$. Thus there is a vertex in B with red degree at least $(|B'| - |B|)/|B|$. This is a contradiction since

$$\frac{|B'| - |B|}{|B|} = \frac{|B'|}{|B|} - 1 \geqslant \left(\frac{1}{(1-\varepsilon)t} \log\left(\frac{f(t)-2}{|B|}\right) - 1\right) \cdot \frac{1}{|B|} - 1$$
$$\geqslant \left(\frac{1}{(1-\varepsilon)t} \log\left(2^{(1-\varepsilon)t(2+C_t\cdot(2^{(1-\varepsilon)t}+1))}\right) - 1\right) \cdot \frac{1}{|B|} - 1$$
$$= \left((2 + C_t \cdot (2^{(1-\varepsilon)t} + 1)) - 1\right) \cdot \frac{1}{|B|} - 1 > 2^{(1-\varepsilon)t} + 1 - 1 = 2^{(1-\varepsilon)t} \geqslant d.$$

since, we recall, $f(t) = \left\lceil 2 + C_t \cdot 2^{(1-\varepsilon)t(2+C_t\cdot(2^{(1-\varepsilon)t}+1))} \right\rceil$ and $C_t = \dfrac{2^{(1-\varepsilon)t}}{\varepsilon} \geqslant |B|$. ◀

Since X is a feedback vertex set of size t of G_t, Lemma 10 implies Theorem 4, and hence Corollary 5.

As the twin-width of T is 2, adding the t apices in X, multiplies the twin-width by at least $2^{t(1-\varepsilon-\frac{1}{t})}$. Thus one apex in X multiplies the twin-width by at least $2^{1-\varepsilon-\frac{1}{t}}$, which can be made arbitrarily close to 2. This establishes Corollary 8.

4 Oriented twin-width and grid number

In this section, we check that G_t has oriented twin-width at most $t+1$, and grid number at most $t+2$.

A *(partial) oriented contraction sequence* is defined similarly as a *(partial) contraction sequence* with every red edge replaced by a red arc leaving the newly contracted vertex. Then a *(partial) oriented d-sequence* is such that all the vertices of all its *ditrigraphs* have at most d out-going red arcs. The *oriented twin-width* of a graph G, denoted by $\text{otww}(G)$, is the minimum integer d such that G admits an oriented d-sequence.

▶ **Lemma 20.** *The oriented twin-width of G_t is at most $t+1$.*

Proof. We observe that the 2-sequence for trees [12] is an oriented 1-sequence. We contract T to a single vertex (without touching X) in that manner. This yields a partial oriented $t+1$-sequence for G_t ending on a $t+1$-vertex ditrigraph, which can be contracted in any way. This contraction sequence witnesses that $\text{otww}(G_t) \leqslant t+1$. ◀

Thus Corollary 7 holds.

We finish by establishing Corollary 6.

▶ **Lemma 21.** *The grid number of G_t is at most $t+2$.*

Proof. Recall that $V(G_t) = X \uplus V(T)$. Let $\prec$ be the total order on $V(G_t)$ that puts first all the vertices of X in any order, then from left to right, all the leaves of T, followed by the preleaves, the nodes at depth $f(t)-2$, the nodes at depth $f(t)-3$, and so on, up to the root. We denote by M the adjacency matrix of G_t ordered by $\prec$.

Let M_T be the submatrix of M obtained by deleting the t rows and t columns corresponding to X. Note that the grid number of M is at most $\mathrm{gn}(M_T) + t$. We claim that there is no 3-grid minor in M_T.

Indeed, in the order $\prec$, above the diagonal of M_T there is no pair of 1-entries in strictly decreasing positions. Thus overall there is no triple of 1-entries in strictly decreasing positions. Thus no 3-grid minor is possible in M_T. ◀

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