Small But Unwieldy

Édouard Bonnet, Julien Duron, John Sylvester, Viktor Zamaraev, and Maksim Zhukovskii

ENS Lyon, LIP

September 18th 2023, CoA 2023, Paris

Labeling Schemes

Given a graph G,

an injective *labeling* $\ell : V(G) \to \{0,1\}^*$ and
a *decoder* $A : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ such that:

for every pair $u, v \in V(G)$, $A(\ell(u), \ell(v)) = 1 \Leftrightarrow uv \in E(G)$.

Labeling Schemes

Given a graph G,

an injective *labeling* $\ell : V(G) \to \{0,1\}^*$ and
a *decoder* $A : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ such that:

for every pair $u, v \in V(G)$, $A(\ell(u), \ell(v)) = 1 \Leftrightarrow uv \in E(G)$.

Main goal: make $\max_{u \in V(G)} |\ell(u)|$ as low as possible

Example: Louis's talk

Universal Graphs

An infinite family of graphs $G_1, G_2, \ldots$ such that:
every n-vertex graph of $\mathcal{C}$ ($\mathcal{C}_n$) is an induced subgraph of G_n.

Universal Graphs

An infinite family of graphs $G_1, G_2, \ldots$ such that:
every n-vertex graph of $\mathcal{C}$ ($\mathcal{C}_n$) is an induced subgraph of G_n.

Main goal: make $|V(G_n)|$ as low as possible a function of n

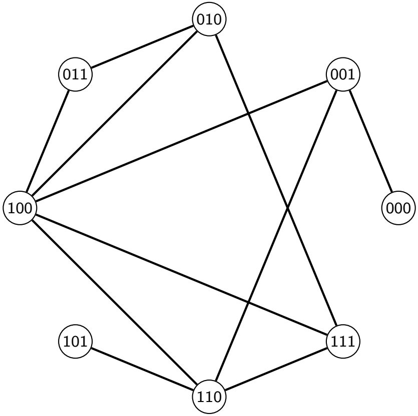
u_n-universal graph for $\mathcal{C}$:
u_n-vertex graph *representing* every graph of $\mathcal{C}_n$

Link between Labeling Schemes and Universal Graphs

$f(n)$-bit labeling schemes $\Leftrightarrow$ $2^{f(n)}$-universal graphs.

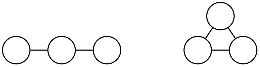
Link between Labeling Schemes and Universal Graphs

$f(n)$-bit labeling schemes $\Leftrightarrow$ $2^{f(n)}$-universal graphs.

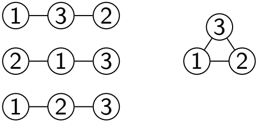


Growth of Graph Classes

Number of unlabeled n-vertex graphs of $\mathcal{C}$ up to isomorphism, or
Number of labeled n-vertex graphs of $\mathcal{C}$

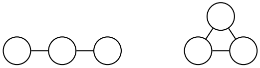


or

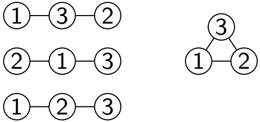


Growth of Graph Classes

Number of unlabeled n-vertex graphs of $\mathcal{C}$ up to isomorphism, or
Number of labeled n-vertex graphs of $\mathcal{C}$



or



Factorial: growth $2^{O(n \log n)}$ (interval graphs)

Small: labeled growth $n!2^{O(n)}$ (graphs of bounded twin-width)

Tiny: unlabeled growth $2^{O(n)}$ (same examples)

Heredity and monotonicity

A graph class is

- *hereditary* if it is closed under taking induced subgraphs
- *monotone* if it is closed under taking subgraphs

A counting observation

A u_n-universal graph represents at most $\binom{u_n}{n}$ graphs on n vertices.

Hence, classes with $n^{O(1)}$-universal graphs (or $O(\log n)$-bit labeling schemes) are factorial.

Implicit Graph Conjecture

Kannan–Naor–Rudich '92, Spinrad '03
Every hereditary factorial graph class has $O(\log n)$*-bit schemes.*

Implicit Graph Conjecture

Kannan–Naor–Rudich '92, Spinrad '03
Every hereditary factorial graph class has $O(\log n)$-bit schemes.

"The Implicit Graph Conjecture is [Completely] False":

There are such classes without $O(n^{0.499})$-bit adjacency schemes

[Hamed and Pooya Hatami, '21].

Scaled-Down Implicit Graph Conjecture

Every hereditary small graph class has $(1 + o(1)) \log n$*-bit schemes.*

Question attributed to [Gavoille and Labourel, '07]

Scaled-Down Implicit Graph Conjecture

Every hereditary small graph class has $(1 + o(1)) \log n$*-bit schemes.*

Question attributed to [Gavoille and Labourel, '07]

Equivalent to $\boldsymbol{n^{1+o(1)}}$**-universal graphs**

Shown for planar graphs [DujmovicEGJMM '21]

Tiny Classes with Polynomially-Large Universal Graphs

Theorem (B., Duron, Sylvester, Zamaraev, Zhukovskii '23)

For every s, there is a monotone tiny class without universal graphs of size n^s.

Or equivalently, without $s \log n$-bit labeling scheme.

Subset counting trick [Hatami & Hatami]

Counting observation:
A single u_n-vertex graph represents only so many graphs

Subset counting trick: *All u_n-vertex graphs combined represent only so many κ_n-subsets of graphs*

$$2^{u_n^2} \cdot \binom{\binom{u_n}{n}}{\kappa_n}$$

Subset counting trick [Hatami & Hatami]

Counting observation:
A single u_n-vertex graph represents only so many graphs

Subset counting trick: *All u_n-vertex graphs combined represent only so many κ_n-subsets of graphs*

$$2^{u_n^2} \cdot \binom{\binom{u_n}{n}}{\kappa_n} \leqslant 2^{u_n^2 + n\kappa_n \log u_n}$$

Subset counting trick [Hatami & Hatami]

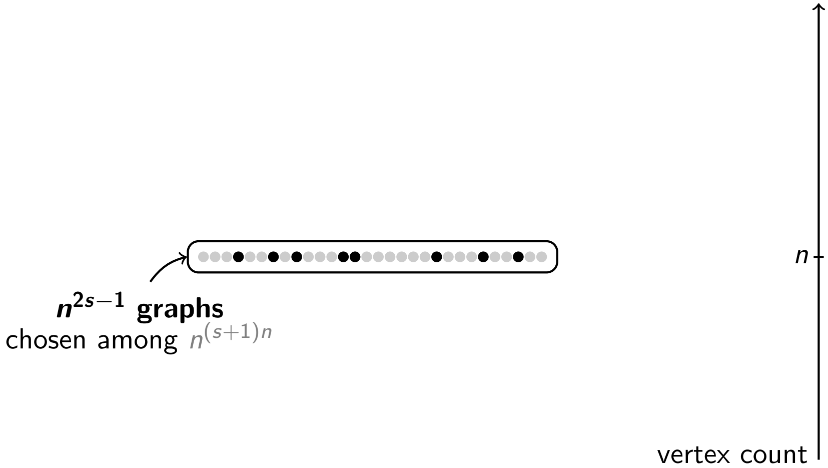
Counting observation:
A single u_n-vertex graph represents only so many graphs

Subset counting trick: *All u_n-vertex graphs combined represent only so many κ_n-subsets of graphs*

$$2^{u_n^2} \cdot \binom{\binom{u_n}{n}}{\kappa_n} \leqslant 2^{u_n^2 + n\kappa_n \log u_n} = 2^{n^{2s}(1+s \log n)}$$

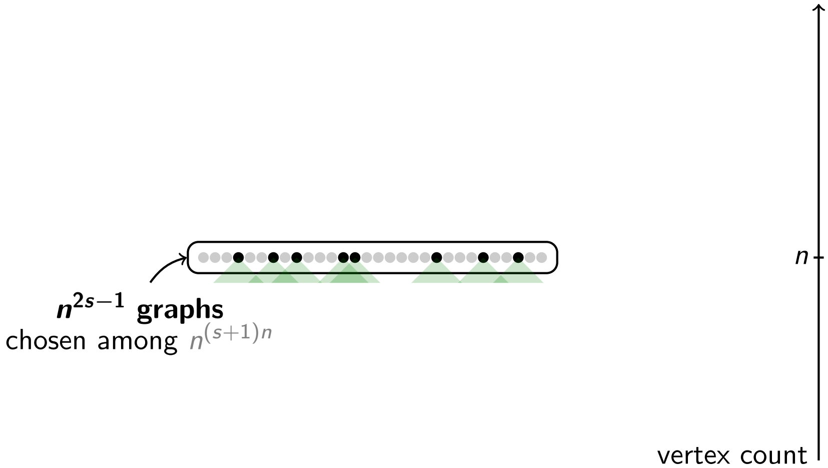
With $u_n = n^s$ and $\kappa_n = n^{2s-1}$,
$\to$ too many κ_n-subsets of n-vertex graphs among tiny classes

With $\kappa_n = n^{2s-1}$,
$\rightarrow \#\kappa_n$-subsets of n-vertex graphs in tiny classes $> 2^{n^{2s}(1+s \log n)}$



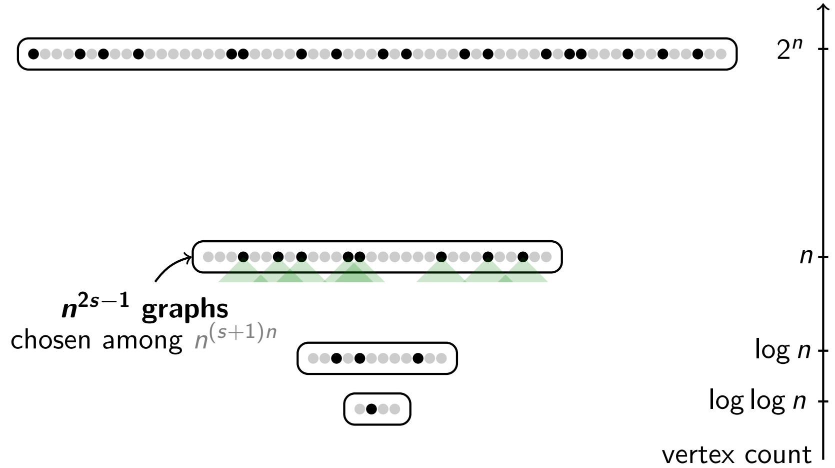
$2^{(s+1)n^{2s} \log n}$ choices $> 2^{n^{2s}(1+s \log n)}$

With $\kappa_n = n^{2s-1}$,
$\rightarrow \#\kappa_n$-subsets of n-vertex graphs in tiny classes $> 2^{n^{2s}(1+s\log n)}$



$n^{(s+1)n}$ graphs on n vertices with a *tiny cone*

With $\kappa_n = n^{2s-1}$,
$\rightarrow \#\kappa_n$-subsets of n-vertex graphs in tiny classes $> 2^{n^{2s}(1+s\log n)}$



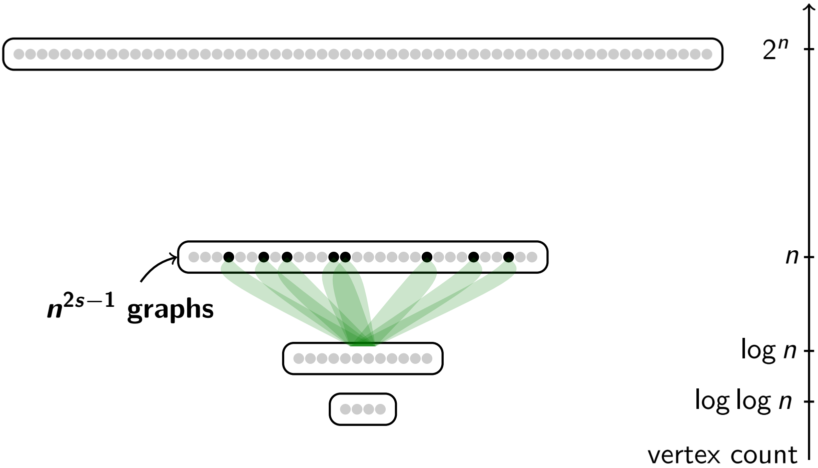
But we have to stack (even sparsely) infinitely many cones

Monotone tiny cones

G is monotone c-tiny: $\forall k$, G has $\leqslant c^k$ unlabeled k-subgraphs

Monotone tiny cones

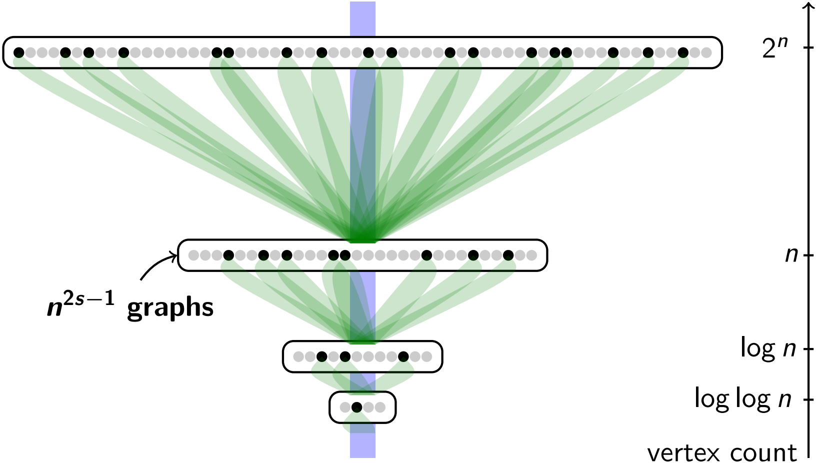
G is monotone c-tiny: $\forall k$, G has $\leqslant c^k$ unlabeled k-subgraphs



At most $n^{2s-1}c^k$ unlabeled k-subgraphs from one layer

Monotone tiny cones

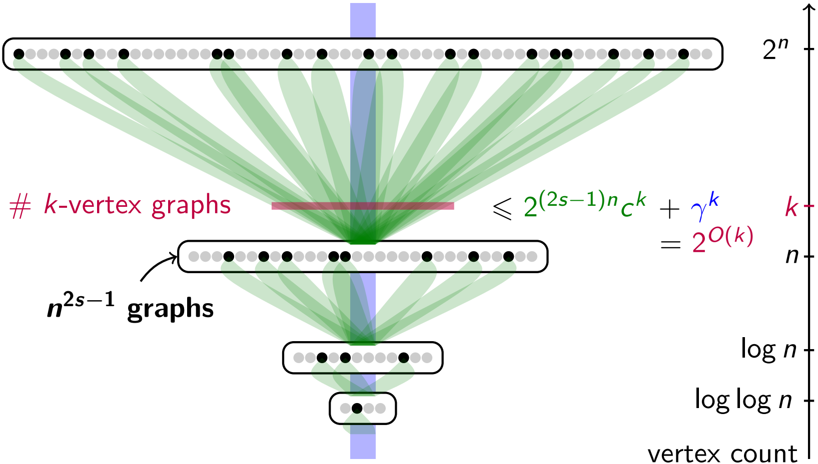
G is monotone c-tiny: $\forall k$, G has $\leqslant c^k$ unlabeled k-subgraphs



Below "$\log n$" the subgraphs should all fall in a same tiny class

Monotone tiny cones

G is monotone c-tiny: $\forall k$, G has $\leqslant c^k$ unlabeled k-subgraphs



We need monotone c-tiny graphs with "$\log n$" threshold property

Bounded average degree graphs

Key: A polynomial fraction of graphs of average degree $d := 2s + 3$ are monotone c-tiny with the "$\log n$" threshold property

...and numerous enough

Bounded average degree graphs

Key: A polynomial fraction of graphs of average degree $d := 2s + 3$ are monotone c-tiny with the "$\log n$" threshold property

...and numerous enough

Analysis on $G(n, p)$ with $p = \frac{d}{n}$

$G(n, \frac{d}{n})$ has the "$\log n$" threshold property

Connected k-subgraphs of $G(n, O(1/n))$ with $k \leqslant \log n$ have only at most $k - 1 + \frac{k}{\log k}$ edges

making a monotone and tiny class (central blue strip)

$G(n, \frac{d}{n})$ are monotone c-tiny

$$\mathbb{E}[\#k\text{-subgraphs of } G(n, d/n) \text{ with } m \text{ edges}] \leqslant \binom{n}{k}\binom{\binom{k}{2}}{m}\left(\frac{d}{n}\right)^m$$

$G(n, \frac{d}{n})$ are monotone c-tiny

$$\mathbb{E}[\#k\text{-subgraphs of } G(n, d/n) \text{ with } m \text{ edges}] \leqslant \binom{n}{k}\binom{\binom{k}{2}}{m}\left(\frac{d}{n}\right)^m$$

Single-exponential in k when $m \geqslant k - 1 + \frac{k}{\log k}$

Open questions

- Refuting the Implicit Graph Conjecture with a small class?
- $(1 + o(1)) \log n$-bit labeling schemes for classes of bounded clique-width? bounded twin-width?
- Is small and tiny equivalent for hereditary classes?

Open questions

- Refuting the Implicit Graph Conjecture with a small class?
- $(1 + o(1)) \log n$-bit labeling schemes for classes of bounded clique-width? bounded twin-width?
- Is small and tiny equivalent for hereditary classes?

Thank you for your attention!

Spring school "Graphs and Algorithms: Conjectures" in May 13–17, 2024, in Aussois

`https://perso.ens-lyon.fr/edouard.bonnet/springSchool.htm`