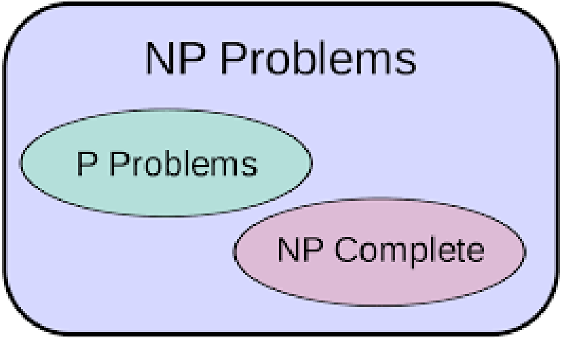
Subexponential algorithms in non sparse classes of graphs

Édouard Bonnet

Department of Computer Science, Middlesex University, London

Lyon, May 15th, 2017



not all the NP-complete problems are as hard when one looks at their optimal running time

Subexponential algorithms

NP-hardness:

> your problem is not solvable in polytime, unless 3-SAT is.
>
> very widely believed but do not give an evidence against algorithms running in, say, $2^{n^{1/100}}$.

Subexponential algorithms

NP-hardness:

your problem is not solvable in polytime, unless 3-SAT is.

very widely believed but do not give an evidence against algorithms running in, say, $2^{n^{1/100}}$.

ETH-hardness:

stronger assumption than P$\neq$NP is ETH asserting that no $2^{o(n)}$ algorithm exists for 3-SAT

Allows to prove stronger conditional lower bounds

linear reduction from 3-SAT: no $2^{o(n)}$ algorithm for your problem, quadratic reduction: no $2^{o(\sqrt{n})}$ algorithm, etc.

Biased viewpoint of this talk

$2^{O(\sqrt{n})}$ (or at least $2^{O(n^c)}$ with $c < 1$) vs no $2^{o(n)}$ under ETH?

For optimisation problems: $n^{O(\sqrt{k})}$ vs no $n^{o(k)}$ (or $n^{o(k/\log k)}$)?

we will not care about the log factors in the exponent.

Biased viewpoint of this talk

$2^{O(\sqrt{n})}$ (or at least $2^{O(n^c)}$ with $c < 1$) vs no $2^{o(n)}$ under ETH?

For optimisation problems: $n^{O(\sqrt{k})}$ vs no $n^{o(k)}$ (or $n^{o(k/\log k)}$)?

we will not care about the log factors in the exponent.

$n^{O(\sqrt{k})} \overset{\text{often}}{\rightsquigarrow} 2^{O(\sqrt{n}\log n)}$ because $k \overset{\text{often}}{\leqslant} n$.

Notable exception: Chordal Completion can even be solved in $2^{O(\sqrt{k}\log k)}$ but not in $2^{o(n)}$ under ETH (k can be $\Omega(n^2)$).

Biased viewpoint of this talk

- $2^{O(\sqrt{n})}$ (or at least $2^{O(n^c)}$ with $c < 1$) vs no $2^{o(n)}$ under ETH?
- For optimisation problems: $n^{O(\sqrt{k})}$ vs no $n^{o(k)}$ (or $n^{o(k/\log k)}$)?
- we will not care about the log factors in the exponent.

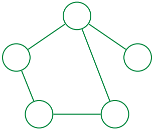
We will focus on geometric non necessarily sparse graphs:

- intersection graphs: disks, balls, segments, strings, etc.
- visibility graphs.

Sparse classes and dense classes

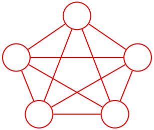
sparse class $\mathcal{C}$: $\exists \alpha_{\mathcal{C}}$, $\forall G \in \mathcal{C}$ such that $|E(G)| \leqslant \alpha_{\mathcal{C}} |V(G)|$.

dense class $\mathcal{C}$: $\exists \alpha_{\mathcal{C}}$ and infinitely many graphs $G \in \mathcal{C}$ with $|E(G)| \geqslant \alpha_{\mathcal{C}} |V(G)|^2$.



sparse

vs



dense

Examples

Sparse classes:

- forests (with $\alpha_{\mathcal{C}} = 1$)
- 3-regular graphs (with $\alpha_{\mathcal{C}} = 3/2$)
- planar graphs (with $\alpha_{\mathcal{C}} = 3$)
- graphs with average degree bounded by d (with $\alpha_{\mathcal{C}} = d/2$)

Dense classes:

- general graphs
- chordal graphs
- intersection graphs of disks
- cliques

Examples

Sparse classes:

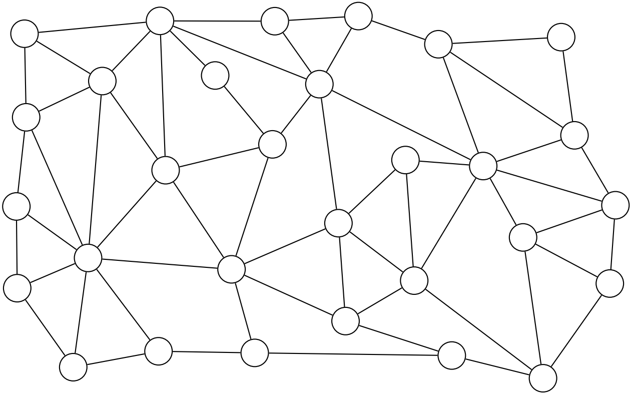
- forests (with $\alpha_{\mathcal{C}} = 1$)
- 3-regular graphs (with $\alpha_{\mathcal{C}} = 3/2$)
- planar graphs (with $\alpha_{\mathcal{C}} = 3$)
- graphs with average degree bounded by d (with $\alpha_{\mathcal{C}} = d/2$)

Dense classes:

- general graphs
- chordal graphs
- intersection graphs of disks
- cliques

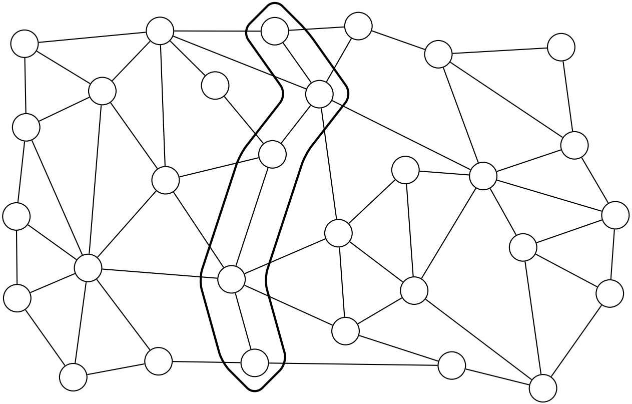
Planar graphs are especially nice sparse graphs since they admit *small* balanced separators...

Square root phenomenon on planar graphs



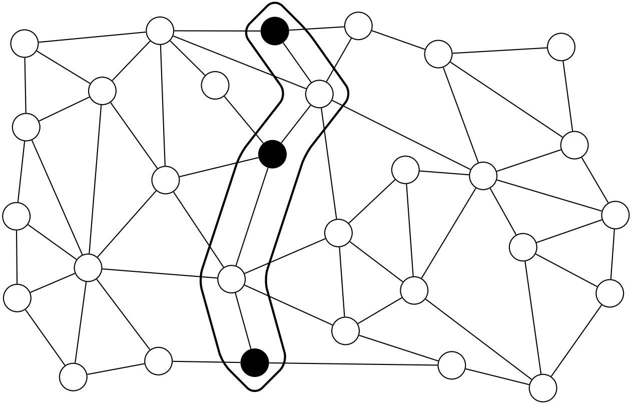
Many problems are solvable in $2^{O(\sqrt{n})}$ in **planar graphs**, and unlikely solvable in $2^{o(n)}$ in general graphs.

Square root phenomenon on planar graphs



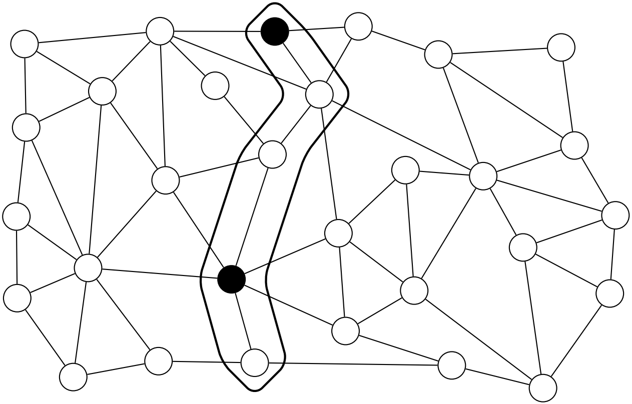
Many problems are solvable in $2^{O(\sqrt{n})}$ in **planar graphs**, and unlikely solvable in $2^{o(n)}$ in general graphs.

Square root phenomenon on planar graphs



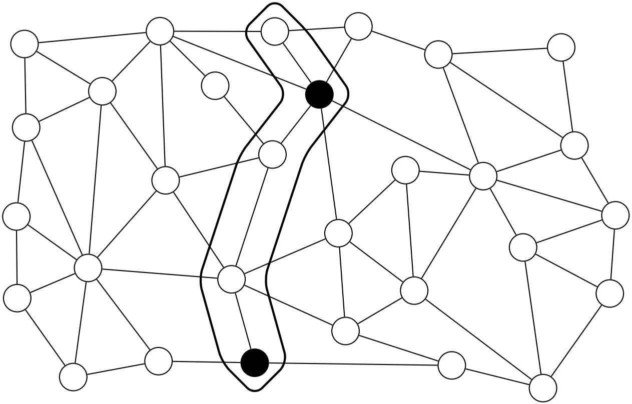
Max Independent Set, 3-Coloring, Hamiltonian Path...

Square root phenomenon on planar graphs



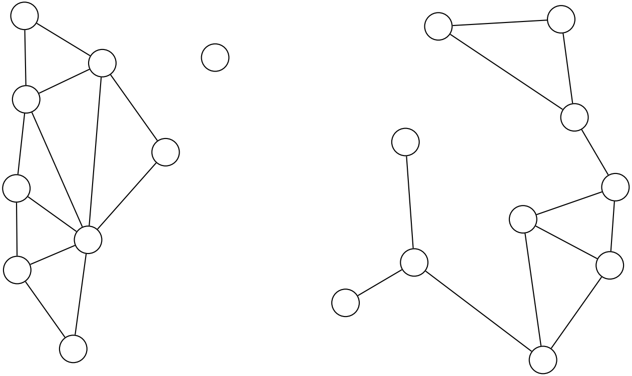
MAX INDEPENDENT SET, 3-COLORING, HAMILTONIAN PATH...

Square root phenomenon on planar graphs



Max Independent Set, 3-Coloring, Hamiltonian Path...

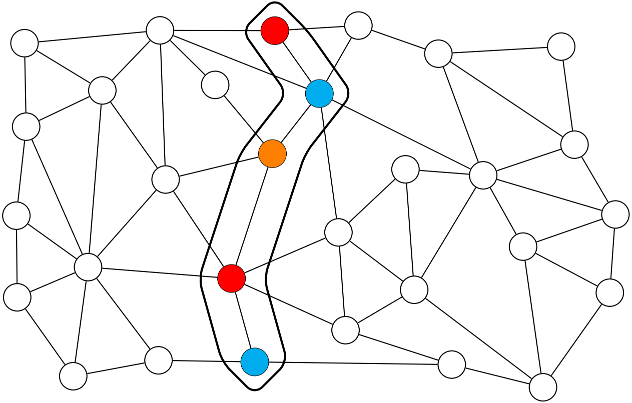
Square root phenomenon on planar graphs



<u>Max Independent Set</u>, 3-Coloring, Hamiltonian Path...

$$T(n) \leqslant 2^{O(\sqrt{n})} T(2n/3) \leqslant 2^{O(\sqrt{n} \log n)}$$

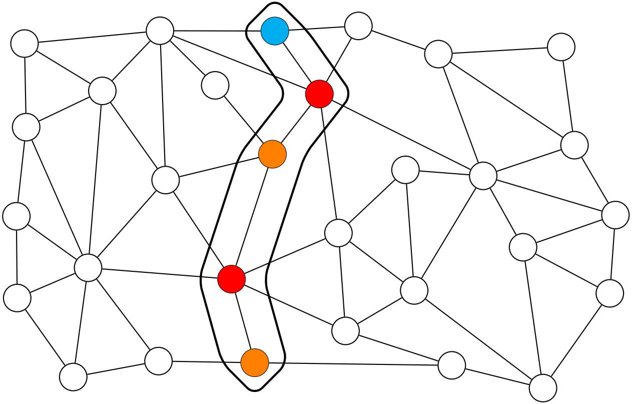
Square root phenomenon on planar graphs



Max Independent Set, 3-Coloring, Hamiltonian Path...

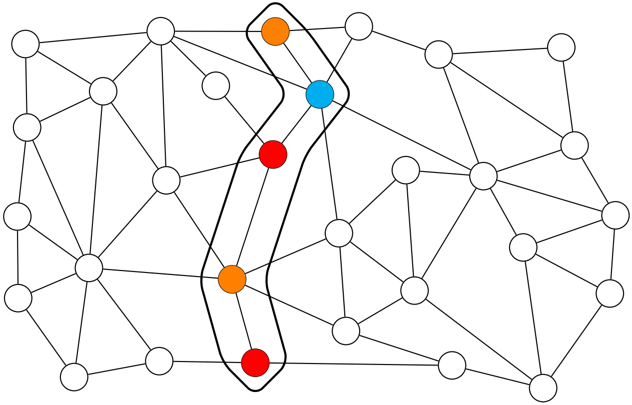
Dynamic programming would spare a log n in the exponent.

Square root phenomenon on planar graphs



MAX INDEPENDENT SET, 3-COLORING, HAMILTONIAN PATH...

Square root phenomenon on planar graphs

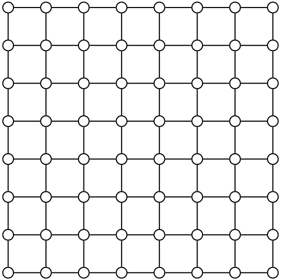


Max Independent Set, 3-Coloring, Hamiltonian Path...

Bidimensionality

Theorem (Robertson & Seymour, Graph Minors)

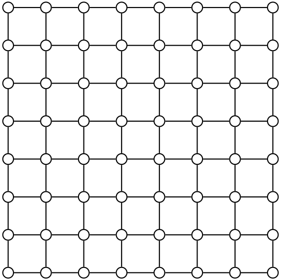
A planar graph with treewidth $\geqslant 5k$ admits a $k \times k$-grid minor.



Bidimensionality

Theorem (Robertson & Seymour, Graph Minors)

A planar graph with treewidth $\geqslant 5k$ admits a $k \times k$-grid minor.

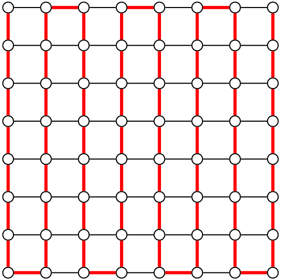


If $\mathrm{tw} < 5\sqrt{k} \rightsquigarrow$ algorithm in $2^{O(\mathrm{tw})} n^{O(1)} = 2^{O(\sqrt{k})} n^{O(1)}$.

Bidimensionality

Theorem (Robertson & Seymour, Graph Minors)

A planar graph with treewidth $\geqslant 5k$ admits a $k \times k$-grid minor.



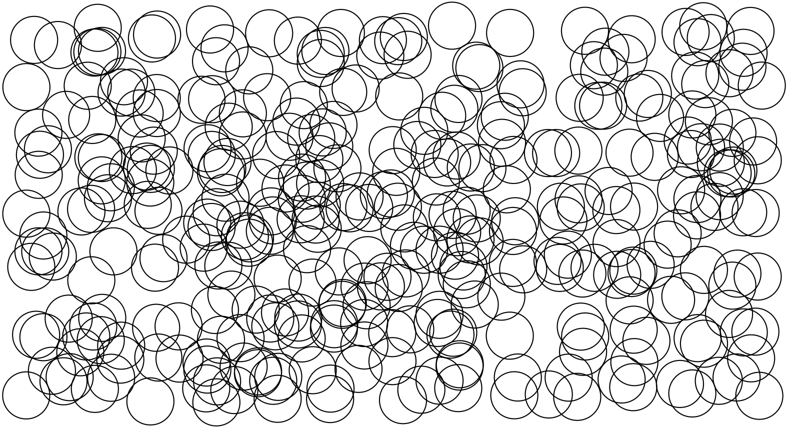
If $\text{tw} < 5\sqrt{k} \rightsquigarrow$ algorithm in $2^{O(\text{tw})} n^{O(1)} = 2^{O(\sqrt{k})} n^{O(1)}$.

If $\text{tw} \geqslant 5\sqrt{k} \rightsquigarrow \exists \sqrt{k} \times \sqrt{k}$-grid minor $\rightsquigarrow$ always yes (always no).

Packing unit disks

Theorem (Alber & Fiala, J. Alg.'04)

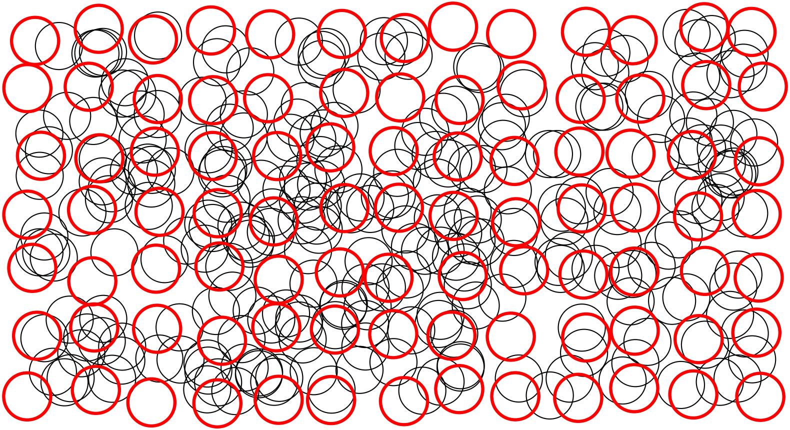
Unit disks can be packed in time $n^{O(\sqrt{k})}$.



Packing unit disks

Theorem (Alber & Fiala, J. Alg.'04)

Unit disks can be packed in time $n^{O(\sqrt{k})}$.

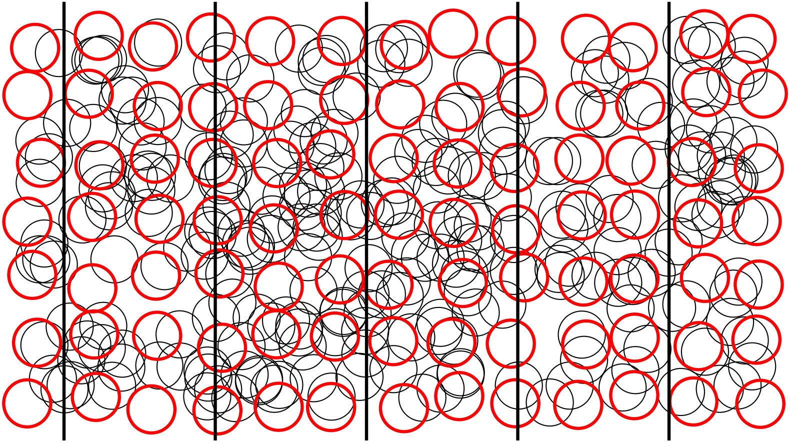


Imagine a solution.

Packing unit disks

Theorem (Alber & Fiala, J. Alg.'04)

Unit disks can be packed in time $n^{O(\sqrt{k})}$.

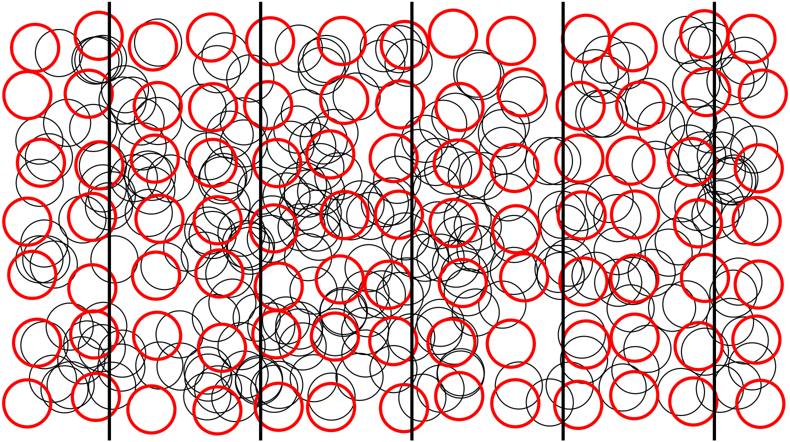


Translate by one unit $\sqrt{k}$ vertical lines distant by $\sqrt{k}$.

Packing unit disks

Theorem (Alber & Fiala, J. Alg.'04)

Unit disks can be packed in time $n^{O(\sqrt{k})}$.

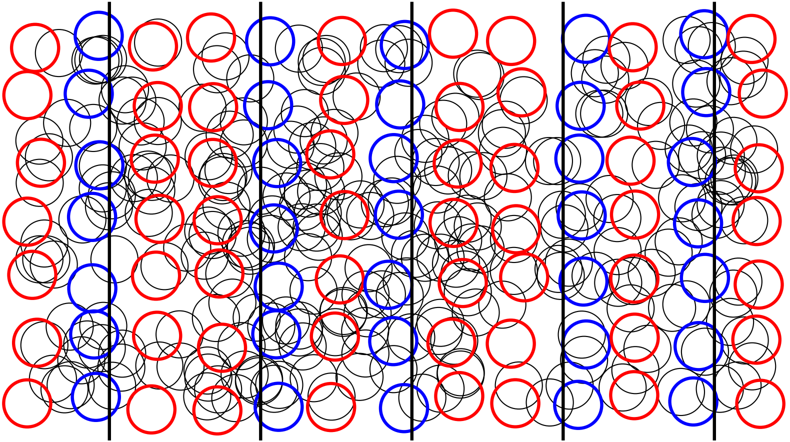


At some point, the lines intersect it on at most $\sqrt{k}$ disks.

Packing unit disks

Theorem (Alber & Fiala, J. Alg.'04)

Unit disks can be packed in time $n^{O(\sqrt{k})}$.

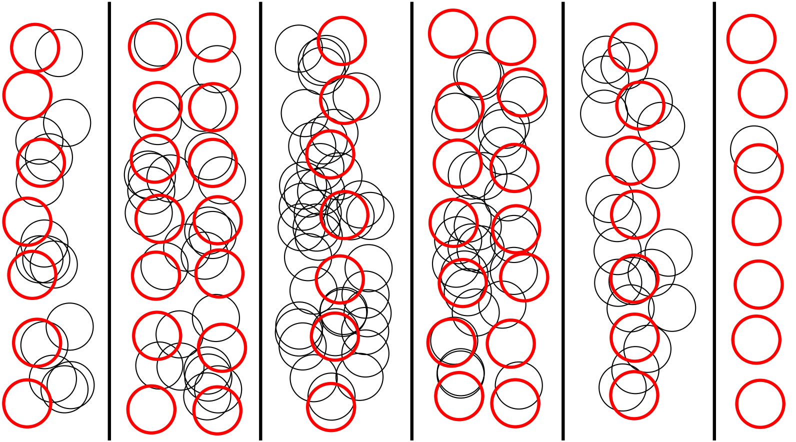


Guess that intersection in $n^{\sqrt{k}}$.

Packing unit disks

Theorem (Alber & Fiala, J. Alg.'04)

Unit disks can be packed in time $n^{O(\sqrt{k})}$.

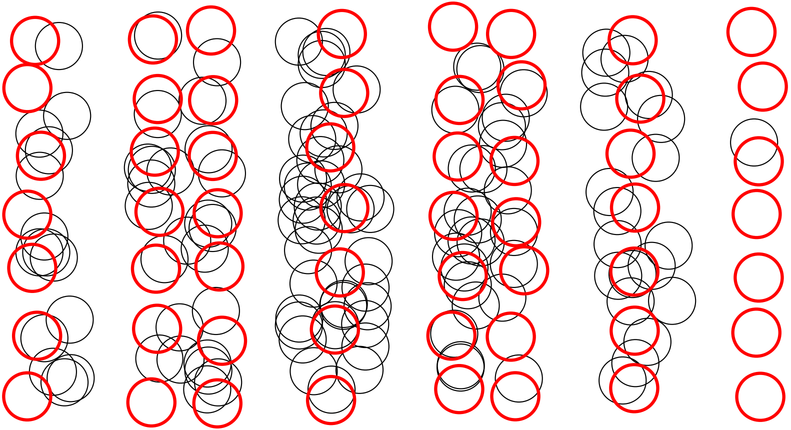


Remove the disks touched by the lines or this intersection.

Packing unit disks

Theorem (Alber & Fiala, J. Alg.'04)

Unit disks can be packed in time $n^{O(\sqrt{k})}$.



Those instances can be solved by DP in $n^{O(\sqrt{k})}$ due to their width.

Packing disks

Idea of Sariel Har-Peled to get geometric QPTAS:
use the Voronoi diagram of an assumptive solution.

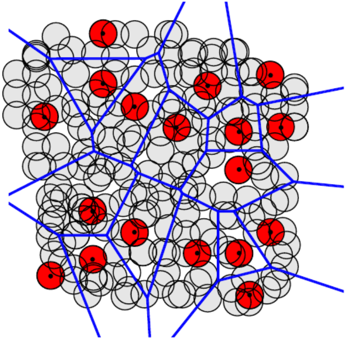
Theorem (Marx & Pilipczuk, ESA '15)

Disks can be packed in time $n^{O(\sqrt{k})}$.

Packing disks

Theorem (Marx & Pilipczuk, ESA '15)

Disks can be packed in time $n^{O(\sqrt{k})}$.

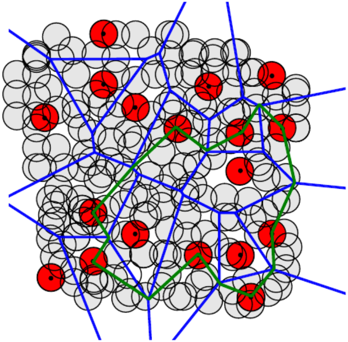


The Voronoi diagram of a solution is a planar graph with k faces.

Packing disks

Theorem (Marx & Pilipczuk, ESA '15)

Disks can be packed in time $n^{O(\sqrt{k})}$.

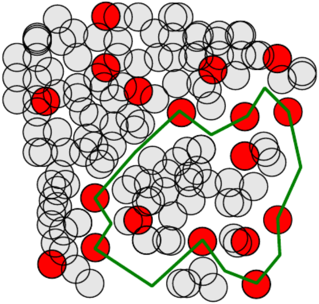


It has a $O(\sqrt{k})$ face-balanced noose. Guess it in $n^{O(\sqrt{k})}$.

Packing disks

Theorem (Marx & Pilipczuk, ESA '15)

Disks can be packed in time $n^{O(\sqrt{k})}$.

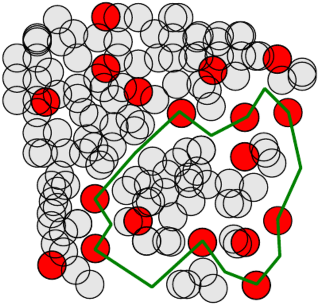


Remove the disks touched by it or by the intersected solution.

Packing disks

Theorem (Marx & Pilipczuk, ESA '15)

Disks can be packed in time $n^{O(\sqrt{k})}$.

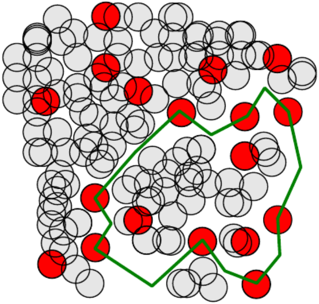


Recurse: $T(n,k) \leqslant n^{O(\sqrt{k})}\, T(n, 2k/3) \leqslant n^{O(\sqrt{k})}$.

Packing disks

Theorem (Marx & Pilipczuk, ESA '15)

Disks can be packed in time $n^{O(\sqrt{k})}$.

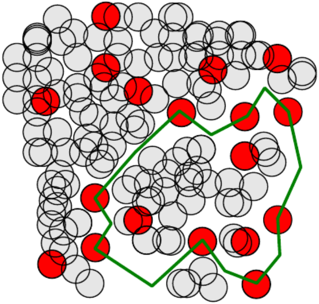


For non unit disks, use *distance* $d(c, p) := \|c - p\|_2 - r(p)$.

Covering points with disks

Theorem (Marx & Pilipczuk, ESA '15)

Selecting k objects among a set of disks covering a set of points can be solved in time $n^{O(\sqrt{k})}$.



With more subtle rules to make the inside and outside independent.

Essentially best algorithms: GRID TILING

Theorem (Chen et al., CCC '04)

k-CLIQUE cannot be solved in time $f(k)n^{o(k)}$ under ETH.

GRID TILING: *Embed k-CLIQUE* in a k-by-k grid.
In each cell select one pair among a prescribed subset of $[n] \times [n]$.

Two horizontally (vertically) adjacent cells should agree on their first (second) coordinate.

Essentially best algorithms: Grid Tiling

Theorem (Marx, ESA '05)

Grid Tiling *cannot be solved in time* $f(k)n^{o(k)}$ *under ETH.*

Grid Tiling: *Embed* k-Clique in a k-by-k grid.
In each cell select one pair among a prescribed subset of $[n] \times [n]$.

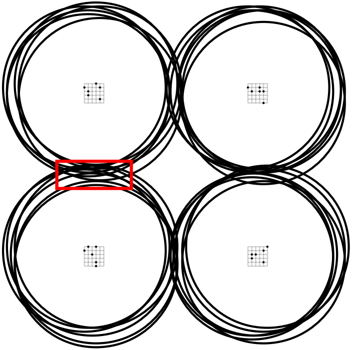
Two horizontally (vertically) adjacent cells should agree on their first (second) coordinate.

The choice of vertices are made along the diagonal: (u_i, u_i).
Checking the edge $u_i u_j$ is done in the cell (i, j).

Essentially best algorithms: GRID TILING

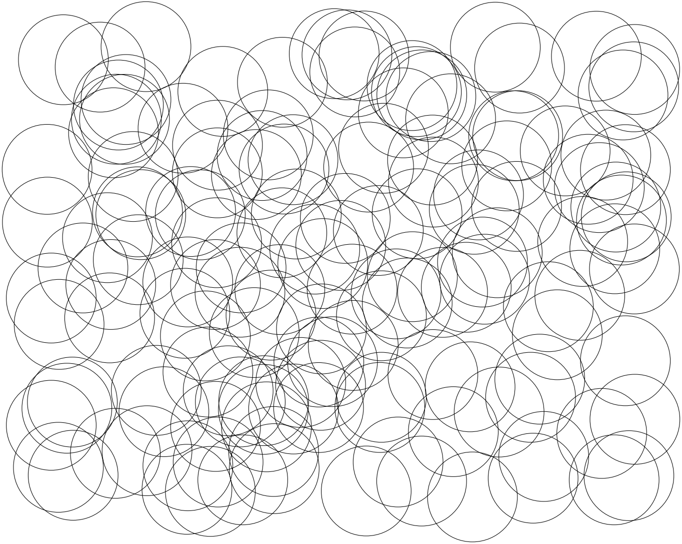
Theorem (Marx, ESA '05)

MIS *on UDG cannot be solved in time* $f(k)n^{o(\sqrt{k})}$ *under ETH.*

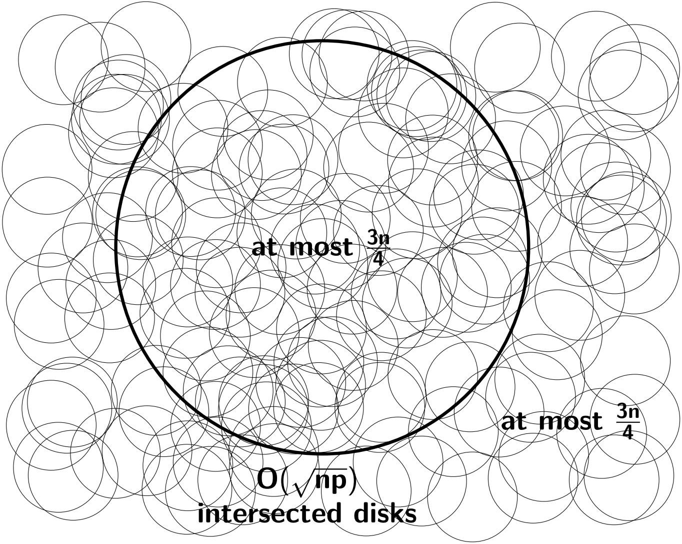


Syntactical reduction to MIS on UDG.

Smith and Wormald '98: $\forall n$ disks with ply p,



Smith and Wormald '98: $\forall n$ disks with ply p, $\exists$ **O**



Actually more general

Theorem (Smith & Wormald, FOCS '98)

For every $d \geq 1$ and $B \geq 0$, there exists a constant $c = c(d, B)$, such that for every B-fat collection $\mathcal{S}$ of n d-dimensional convex sets with ply at most ℓ, there exists a d-dimensional sphere Q, such that:

at most $\frac{d+1}{d+2}n$ elements of $\mathcal{S}$ are entirely inside Q,

at most $\frac{d+1}{d+2}n$ elements of $\mathcal{S}$ are entirely outside Q,

at most $cn^{1-1/d}\ell^{1/d}$ elements of $\mathcal{S}$ intersect Q.

Simple algorithm for ℓ-coloring disks

Win-win based on the value of the ply:

$p > \ell \rightsquigarrow$ answer NO.

Simple algorithm for ℓ-coloring disks

Win-win based on the value of the ply:

$p > \ell \rightsquigarrow$ answer NO.
$p \leqslant \ell \rightsquigarrow$ balanced separator of size $O(\sqrt{n\ell}) \rightsquigarrow$ treewidth $\tilde{O}(\sqrt{n\ell}) \rightsquigarrow$ coloring in time $2^{\tilde{O}(\sqrt{n\ell})}$

Simple algorithm for ℓ-coloring disks

Win-win based on the value of the ply:

$p > \ell \rightsquigarrow$ answer NO.
$p \leqslant \ell \rightsquigarrow$ balanced separator of size $O(\sqrt{n\ell}) \rightsquigarrow$
treewidth $\tilde{O}(\sqrt{n\ell}) \rightsquigarrow$ coloring in time $2^{\tilde{O}(\sqrt{n\ell})}$

For ℓ-coloring d-dimensional balls, the same argument gives running time $2^{\tilde{O}(n^{1-1/d}\ell^{1/d})}$.

Essentially best algorithms: GRID COLORING

Theorem (Biro et al., SoCG '17)

For any $\alpha \in [0,1]$, coloring n unit disks with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{o(n^{\frac{1+\alpha}{2}})} = 2^{o(\sqrt{n\ell})}$, under the ETH.

Essentially best algorithms: Grid Coloring

Theorem (Biro et al., SoCG '17)

For any $\alpha \in [0,1]$, coloring n unit disks with $\ell = \Theta(n^{\alpha})$ colors cannot be solved in time $2^{o(n^{\frac{1+\alpha}{2}})} = 2^{o(\sqrt{n\ell})}$, under the ETH.

Constant number of colors $\rightsquigarrow$ square root phenomenon.
Linear number of colors $\rightsquigarrow$ no subexponential-time algorithm.

Essentially best algorithms: GRID COLORING

Theorem (Biro et al., SoCG '17)

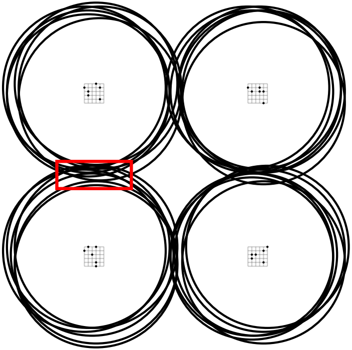
For any $\alpha \in [0,1]$, coloring n unit disks with $\ell = \Theta(n^{\alpha})$ colors cannot be solved in time $2^{o(n^{\frac{1+\alpha}{2}})} = 2^{o(\sqrt{n\ell})}$, under the ETH.

Constant number of colors $\rightsquigarrow$ square root phenomenon.
Linear number of colors $\rightsquigarrow$ no subexponential-time algorithm.

And everything in between (hard part).
For instance, $\sqrt{n}$-coloring cannot be done in $2^{o(n^{3/4})}$.

Essentially best algorithms: Grid Coloring

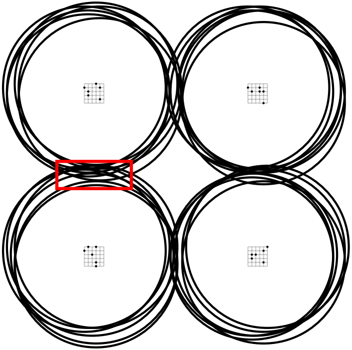
k-by-k Grid Tiling instance with t legal pairs in each cell.



Instead of selecting one center in each small grid...

Essentially best algorithms: Grid Coloring

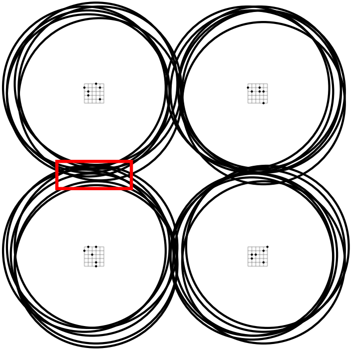
k-by-k Grid Tiling instance with t legal pairs in each cell.



...we color them with a different color each...

Essentially best algorithms: Grid Coloring

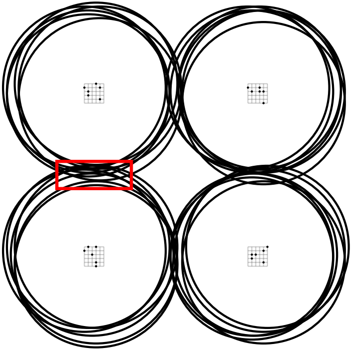
k-by-k Grid Tiling instance with t legal pairs in each cell.



...such that each color class corresponds to a clique

Essentially best algorithms: Grid Coloring

k-by-k Grid Tiling instance with t legal pairs in each cell.



...such that each color class corresponds to a clique

Grid Coloring cannot be solved in time $2^{o(tk)}$ under ETH.

Segment intersection graphs

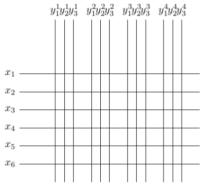
The subexponential algorithm generalizes to other **fat** objects.

Segment intersection graphs

The subexponential algorithm generalizes to other **fat** objects.

Theorem (Rzążewski)

4-Coloring *on 2-DIR cannot be solved in time* $2^{o(n)}$.



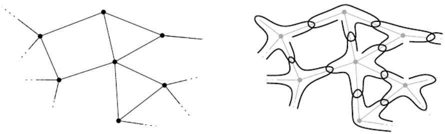
No subexponential algorithms even for a constant number of colors.

3-COLORING on segments?

3-COLORING on segments?

Subexponential algorithm even on string graphs!

String graphs: intersection graphs of curves in the plane.



Theorem

(LIST) 3-COLORING *on strings can be solved in time* $2^{O(n^{2/3})}$.

Subexponential algorithms on string graphs

Important fact: string graphs have separators of size $O(\sqrt{m})$.

Subexponential algorithms on string graphs

Important fact: string graphs have separators of size $O(\sqrt{m})$.

Win-win:

high maximum degree: good branching

low maximum degree: small separator

Theorem (Fox & Pach, SODA '11)

Max Independent Set *has a subexponential algorithm on string graphs.*

Subexponential algorithms on string graphs

Important fact: string graphs have separators of size $O(\sqrt{m})$.

Win-win:

high maximum degree: good branching

low maximum degree: small separator

Theorem (Fox & Pach, SODA '11)

MAX INDEPENDENT SET *has a subexponential algorithm on string graphs.*

What about MIN DOMINATING SET? MIN INDEPENDENT DOMINATING SET? MAX CLIQUE?

Subexponential algorithms on string graphs

Important fact: string graphs have separators of size $O(\sqrt{m})$.

Win-win:

high maximum degree: good branching

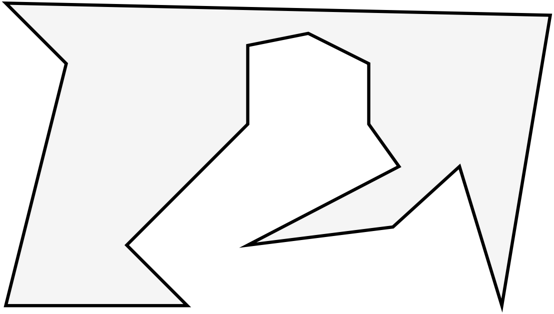
low maximum degree: small separator

Theorem (Fox & Pach, SODA '11)

MAX INDEPENDENT SET *has a subexponential algorithm on string graphs.*

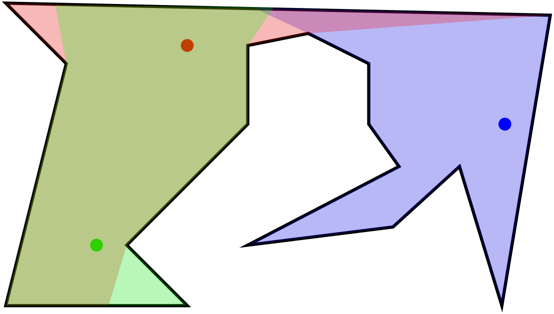
What about MIN DOMINATING SET? MIN INDEPENDENT DOMINATING SET? MAX CLIQUE? No, No, No

Art Gallery Problem



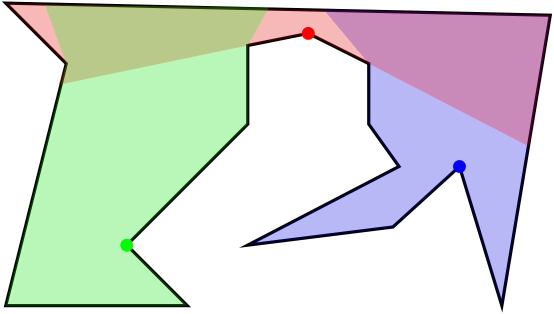
Simple polygon with n vertices.

Art Gallery Problem



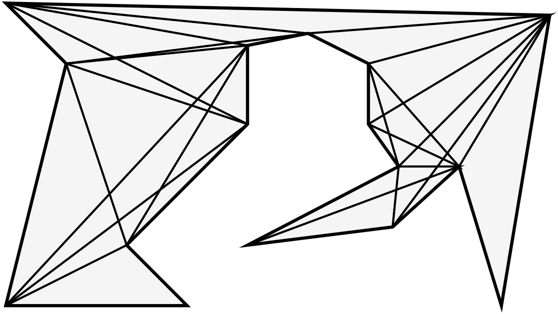
Simple polygon with n vertices.
Guard the gallery with k **points**.

Art Gallery Problem



Simple polygon with n vertices.
Guard the gallery with k **vertices**.

Art Gallery Problem

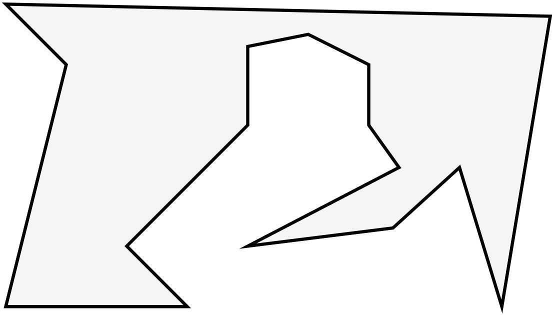


Simple polygon with n vertices.

DOMINATING SET in the visibility graph of a simple polygon.

Those three problems are NP-hard.

Art Gallery Problem



Simple polygon with n vertices.
DOMINATING SET in the visibility graph of a simple polygon.

Those three problems are NP-hard.

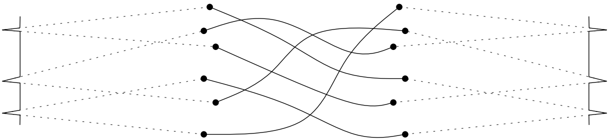
Algorithm in time $f(k)n^{O(\sqrt{k})}$? Algorithm in time $f(k)n^c$?

Intermediate problem and linker

Theorem (B. & Miltzow, ESA '16)

No algorithm in time $f(k)n^{o(k/\log k)}$ *unless the ETH fails.*

two instances of min hitting set of *intervals*
taking a point in instance A forces a specific point in instance B

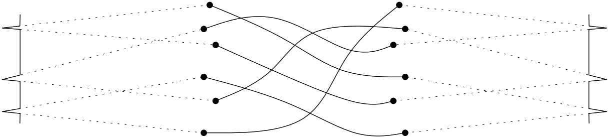


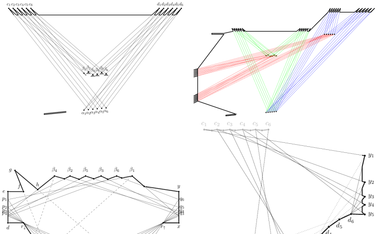
Intermediate problem and linker

Theorem (B. & Miltzow, ESA '16)

No algorithm in time $f(k)n^{o(k/\log k)}$ unless the ETH fails.

two instances of min hitting set of *intervals*
taking a point in instance A forces a specific point in instance B



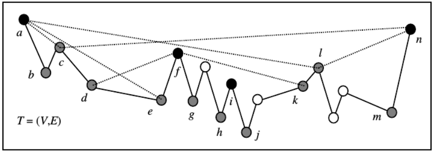


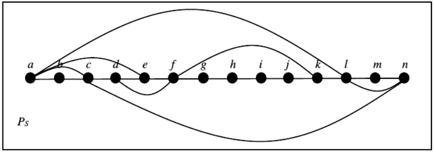
Terrain Guarding

Guarding an x-monotone polygonal curve with k vertices.

Theorem (Ashok et al. SoCG'17)

Terrain Guarding *is solvable in* $n^{O(\sqrt{k})}$*, hence in* $2^{O(\sqrt{n}\log n)}$.





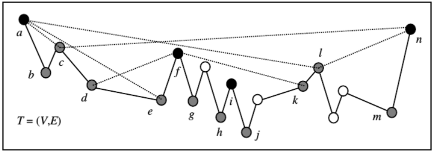
A planar graph with domination number k has treewidth $O(\sqrt{k})$.

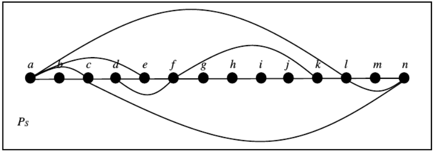
Terrain Guarding

Guarding an x-monotone polygonal curve with k vertices.

Theorem (Ashok et al. SoCG'17)

TERRAIN GUARDING *is solvable in* $n^{O(\sqrt{k})}$*, hence in* $2^{O(\sqrt{n}\log n)}$.





A planar graph with domination number k has treewidth $O(\sqrt{k})$.
Non-trivial divide-and-conquer based on this separator.

Subexponential algorithms of geometric graphs

- Algorithmic techniques: guessing a *small* separator relative to a hypothetical solution (Voronoi diagram, planar graph, etc.), separator theorems (for disk graph, string graphs; generalizing the planar separator theorem), win-win approach.
- ETH-based lowerbounds: reductions from Grid Tiling, Grid Coloring, 2-Track Hitting Set.

Separator-based techniques also lead to approximation algorithms: Instead of brute-forcing on the separator, ignore it.

Subexponential algorithms of geometric graphs

Algorithmic techniques: guessing a *small* separator relative to a hypothetical solution (Voronoi diagram, planar graph, etc.), separator theorems (for disk graph, string graphs; generalizing the planar separator theorem), win-win approach.

ETH-based lowerbounds: reductions from Grid Tiling, Grid Coloring, 2-Track Hitting Set.

Separator-based techniques also lead to approximation algorithms: Instead of brute-forcing on the separator, ignore it.

Some open questions:

Optimal complexity of MIS, 3-coloring, on string graphs?

Lowerbound or better algorithm for Terrain Guarding?

Subexponential algorithms of geometric graphs

Algorithmic techniques: guessing a *small* separator relative to a hypothetical solution (Voronoi diagram, planar graph, etc.), separator theorems (for disk graph, string graphs; generalizing the planar separator theorem), win-win approach.

ETH-based lowerbounds: reductions from GRID TILING, GRID COLORING, 2-TRACK HITTING SET.

Separator-based techniques also lead to approximation algorithms: Instead of brute-forcing on the separator, ignore it.

Some open questions:

Optimal complexity of MIS, 3-COLORING, on string graphs?

Lowerbound or better algorithm for TERRAIN GUARDING?

Thanks for your attention!