An algorithmic weakening of the Erdős-Hajnal conjecture

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Joint work with Stéphan Thomassé, Xuan Thang Tran, and Rémi Watrigant

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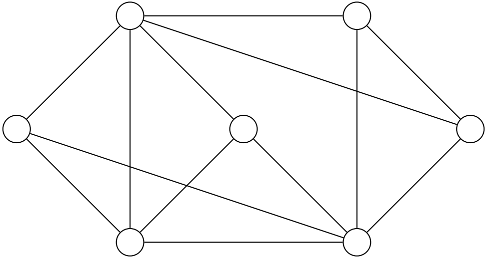
Max Independent Set

Problem: Given a graph



Find a largest subset of vertices that are pairwise non-adjacent

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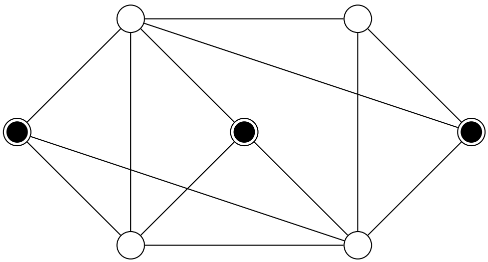

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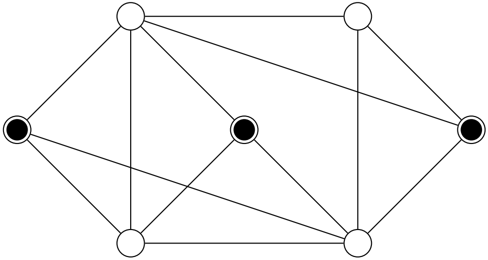


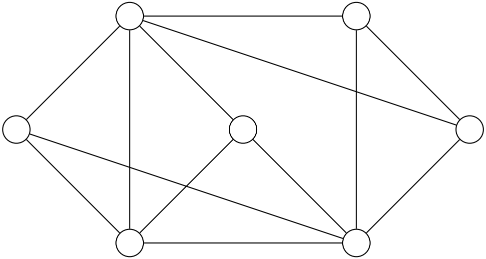
Find a largest subset of vertices that are pairwise non-adjacent

NP-complete even to approximate $\alpha(G)$

$\forall \varepsilon > 0$, an $n^{1-\varepsilon}$-approximation algorithm would imply P=NP

Max Independent Set in H-free graphs

Excluding a fixed graph H as an induced subgraph



Example: this graph is P_4-free

MAX INDEPENDENT SET in H-free graphs

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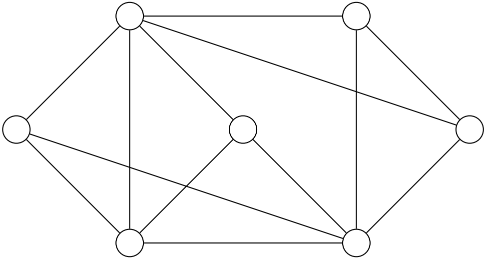

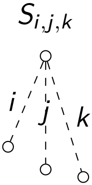

Example: this graph is P_4-free

- P_6-free: polynomial algorithm [Grzesik et al. '19]
- P_t-free: quasipolynomial algorithm [Gartland, Lokshtanov '20]
- $S_{i,j,k}$-free: QPTAS [Chudnovsky et al. '20]
- other H-free: APX-hard [Poljak '73, Alekseev '82]

Dichotomies of MIS in H-free graphs

P vs NP-complete: all the open cases ($S_{i,j,k}$) believed to be in P

QPTAS vs APX-hard: same dichotomy but proven

FPT vs W[1]-hard: even the dichotomy statement is unclear

Improved approximation property

Conjecture

The only hereditary class where $\forall \varepsilon > 0$*, it is NP-hard to* $n^{1-\varepsilon}$*-approximate* MIS *is the class of all graphs.*

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Definition

A graph H has the *improved approximation property* if $\exists \delta > 0$ such that MIS in H-free graphs can be $n^{1-\delta}$-approximated in randomized polynomial time.

Conjecture: Every H has the improved approximation property

The Erdős-Hajnal conjecture

Conjecture (Erdős-Hajnal '89)

For every H, there is a $\delta > 0$ such that every H-free graph G has an independent set or a clique of size at least $|V(G)|^{\delta}$.

The Erdős-Hajnal conjecture

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For every H, there is a $\delta > 0$ such that every H-free graph G has an independent set or a clique of size at least $|V(G)|^{\delta}$.

- proven for every 5-vertex graph but P_5 and its complement
- closure under substitution [Alon et al. '01]
- line of work excluding a forest and a complement of a forest
- C_5, $\{C_6, \overline{C_6}\}$, $\{C_7, \overline{C_7}\}$, etc. [Chudnovsky et al. '21]
- P_5? What to do in the dense case?

Effective Erdős-Hajnal

For every H, there is a $\delta > 0$ and a polynomial-time algorithm $\mathcal{A}$ that inputs H-free graphs G, and outputs an independent set or a clique of G of size at least $|V(G)|^{\delta}$.

Effective Erdős-Hajnal $\Rightarrow$ improved approximation

For every H, there is a $\delta > 0$ and a polynomial-time algorithm $\mathcal{A}$ that inputs H-free graphs G, and outputs an independent set or a clique of G of size at least $|V(G)|^{\delta}$.

Theorem

If H has the effective Erdős-Hajnal property, then H has the improved approximation property.

Effective Erdős-Hajnal ⇒ improved approximation

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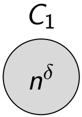

$\mathcal{A}$ yields a large clique C_1, otherwise we get an $n^{1-\delta}$-approximation

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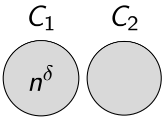

We run $\mathcal{A}$ on $G - C_1$ and get clique C_2

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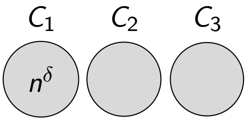

We run $\mathcal{A}$ on $G - \bigcup_{1 \leqslant i \leqslant k-1} C_i$ and get clique C_k

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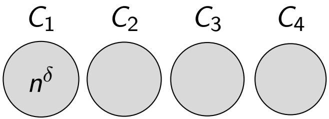

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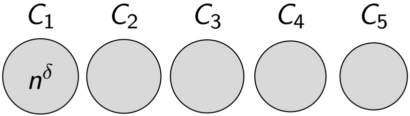

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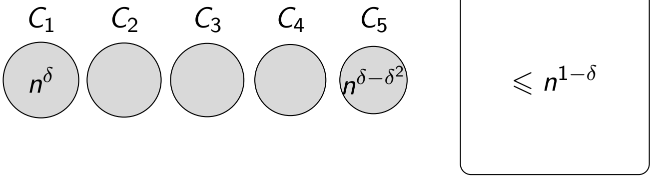

Stop when the remaining graph has at most $n^{1-\delta}$ vertices

Effective Erdős-Hajnal $\Rightarrow$ improved approximation

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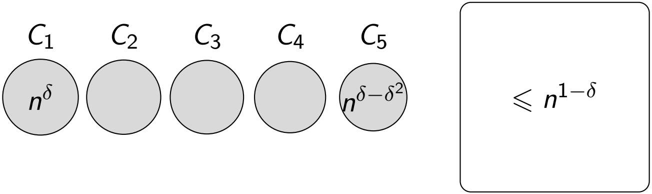


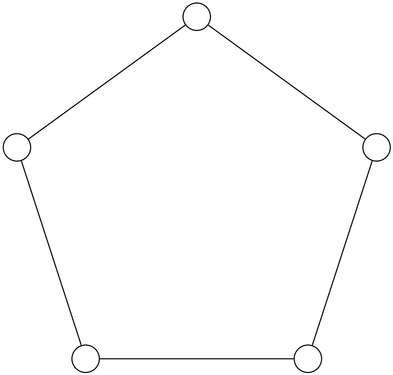
$\leqslant n^{1-\delta+\delta^2}$ cliques C_i's, so $\alpha(G) \leqslant n^{1-\delta+\delta^2} + n^{1-\delta} \leqslant n^{1-\delta+\delta^2+\frac{1}{\log n}}$

Particular approximation strategy

For some $\delta > 0$:

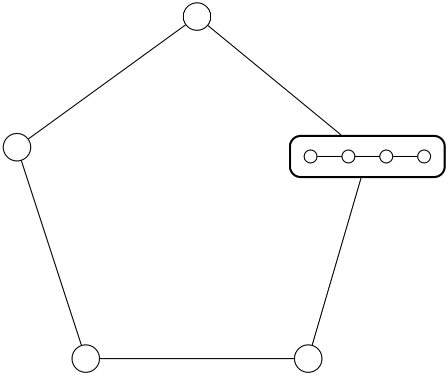
(A) Find an independent set of size n^δ, and output it or
(B) establish that $\alpha(G) \leqslant n^{1-\delta}$, and output a single vertex

Substitution



$G = C_5$

Substitution



$G = C_5$, $H = P_4$, substitution $G[v \leftarrow H]$

Closure under substitution

Theorem

If H_1 and H_2 have the improved approximation property, then so does $H_1[v \leftarrow H_2]$ for every $v \in V(H_1)$.

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$h_i = |V(H_i)|$, and $\mathcal{A}_i$ $n^{1-\varepsilon_i}$-approximates MIS in H_i-free graphs

$(\varepsilon = \min(\varepsilon_1, \varepsilon_2, 1/2), \quad \gamma = \frac{\varepsilon}{2h_1}, \quad \delta = \frac{\varepsilon\gamma}{2+\varepsilon\gamma})$

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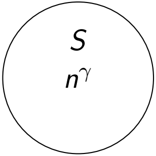

H_1-free w.h.p (at least $(1 - \frac{1}{n^{h_1-\varepsilon/2}})^{n^{h_1-\varepsilon}}$)

If G has less than $n^{h_1-\varepsilon}$ induced copies of H_1, sample n^γ vertices S

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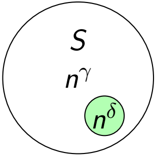

H_1-free

Run $\mathcal{A}_1$ on $G[S]$: either we get a large enough independent set

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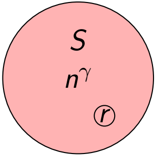

H_1-free

Run $\mathcal{A}_1$ on $G[S]$: or $\alpha(G[S]) \leqslant r|S|^{1-\varepsilon} < n^{\delta}|S|^{1-\varepsilon} < |S|^{1-\varepsilon/2}$

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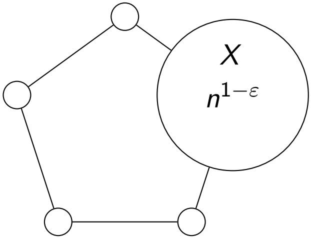

If G has $> n^{h_1-\varepsilon}$ copies of H_1 $\Rightarrow$ copy of $H_1 - v$ with large X

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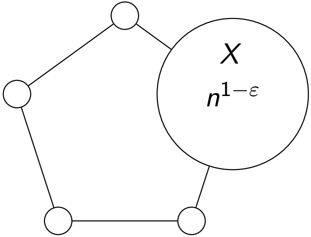

$G[X]$ is H_2-free, so we run $\mathcal{A}_2$ on $G[X]$

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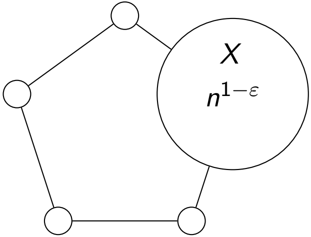

If we never output a large independent set $\Rightarrow \alpha(G) \leqslant n^{1-\delta}$

H with improved approximation but unknown Erdős-Hajnal

P_5, P_6, and all the new graphs obtained by substitution

Can the recent progress on solving MIS on P_t-free graphs help?

- potential maximal cliques have been around long enough
- quasipolynomial-time algorithm succeeds exactly in the hard regime of Erdős-Hajnal (linear degree)

Approximation algorithm in $\triangle$-free graphs

MIS has an easy $\sqrt{n}$-approximation in $\triangle$-free graphs

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Take any vertex v,
if $|N(v)| \geqslant \sqrt{n}$ then output $N(v)$,
else add v to the solution.

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Theorem (direct generalization)

MIS *has an* $n^{\frac{t-2}{t-1}}$*-approximation algorithm in* K_t*-free graphs.*

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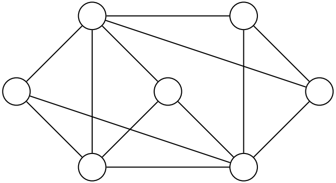
MIS *has an* $n^{\frac{t-2}{t-1}}$*-approximation algorithm in* K_t*-free graphs.*

$R(s,t) = \tilde{O}(t^{s-1})$, so $\alpha(G) = \tilde{\Omega}(n^{\frac{1}{t-1}})$ for an n-vertex K_t-free G

Hardness of approximation in △-free graphs

$\alpha(G) \leqslant N^{\varepsilon}$ or $\alpha(G) \geqslant N^{1-\varepsilon}$? is NP-h



N vertices

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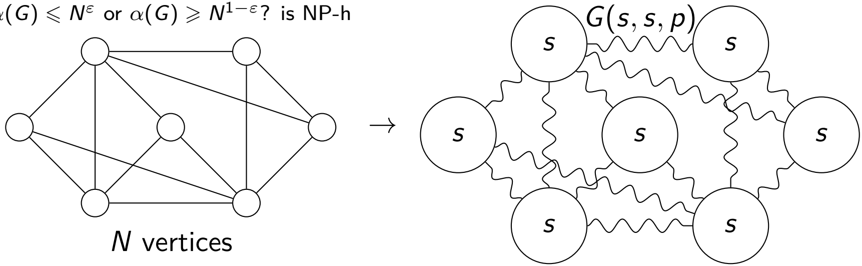
$s = N^5$, $p = N^{-4-\frac{2\varepsilon}{3}}$



N^6 vertices

Vertex $v \to$ independent set I_v of size s

edge $\to G(s, s, p)$

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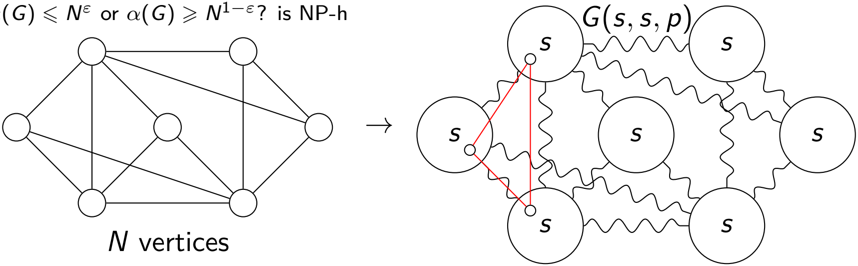

N vertices

N^6 vertices

$$\mathbb{E}(\#\triangle) \leqslant N^3 (N^5)^3 p^3 = N^{6-2\varepsilon}$$

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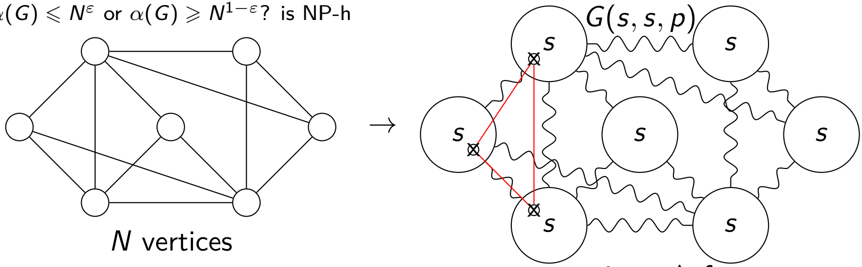

N vertices

n vertices, $\triangle$-free

Removing every triangle, we keep $n \geqslant N^6/2$ vertices

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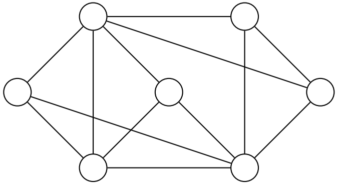

N vertices

$\rightarrow$

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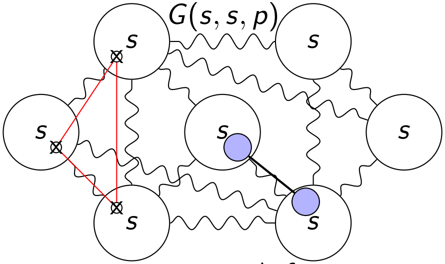
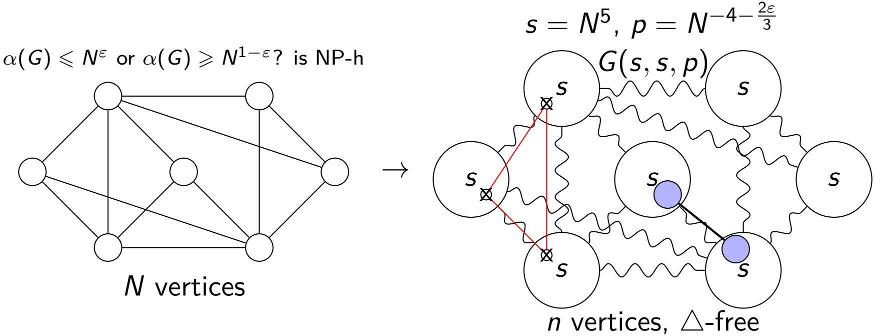


n vertices, $\triangle$-free

If I_v, I_w are adjacent, $X \subseteq I_v$, $Y \subseteq I_w$ with $|X|, |Y| \geqslant N^{4+\frac{4\varepsilon}{3}}$
there is an edge between X and Y w.h.p

Hardness of approximation in △-free graphs

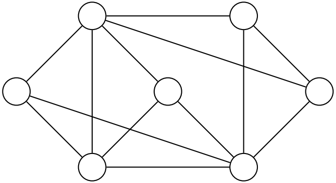


$$(1-p)^{|X|\cdot|Y|} \leqslant \left(1 - \frac{1}{N^{4+\frac{2\varepsilon}{3}}}\right)^{N^{2(4+\frac{4\varepsilon}{3})}} \leqslant e^{-N^4+2\varepsilon}$$

Stomach the $\binom{N^5}{N^{4+\frac{4\varepsilon}{3}}}^2$ pairs of X, Y by union bound

Hardness of approximation in △-free graphs

$\alpha(G) \leqslant N^{\varepsilon}$ or $\alpha(G) \geqslant N^{1-\varepsilon}$? is NP-h



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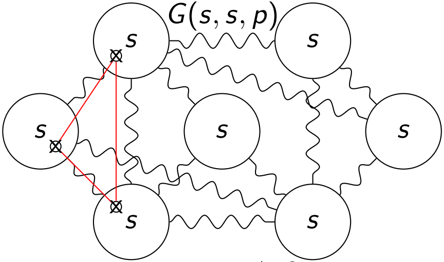
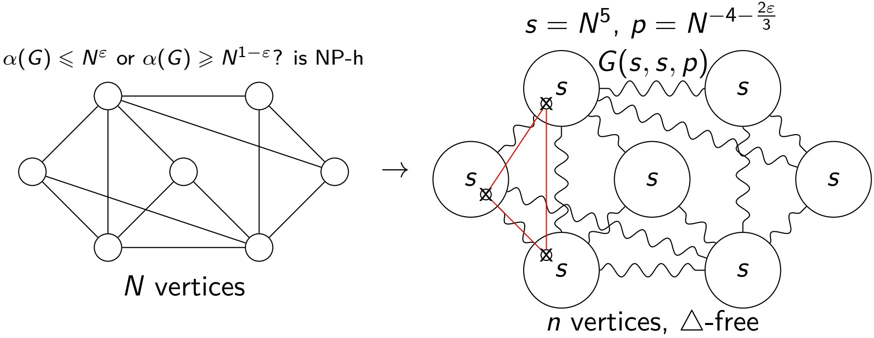


n vertices, △-free

YES-instance $\rightarrow$ independent set $\approx N^5 \cdot N^{1-\varepsilon} = N^{6-\varepsilon}$

Hardness of approximation in △-free graphs



$$\text{NO-instance} \rightarrow \text{independent set} \leqslant N^5 \cdot N^{\varepsilon} + N^{4+\frac{2\varepsilon}{3}} \cdot N \leqslant N^{5+\varepsilon'}$$

Hardness of approximation in △-free graphs

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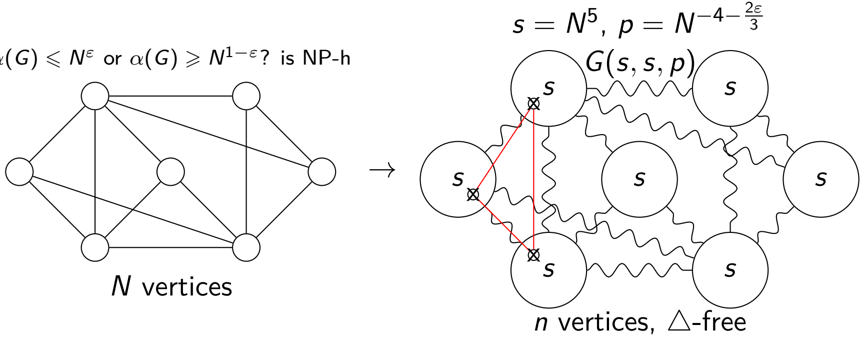
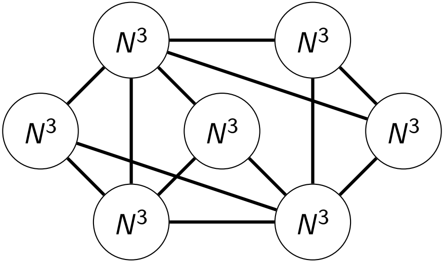


$\alpha(G) \geqslant n^{1-\varepsilon}$ or $\alpha(G) \leqslant n^{\frac{5}{6}-\varepsilon}$? in △-free graphs is NP-h

Sharper inapproximability in △-free graphs

- now blow-up vertices into independent sets of size $N^3 \to G$



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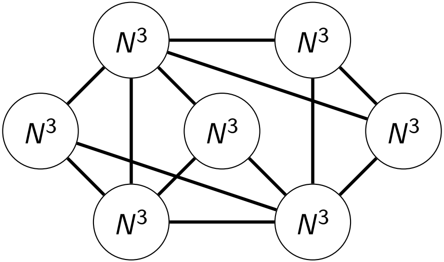
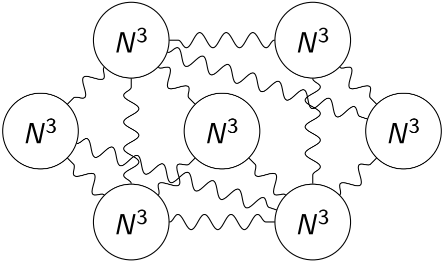
- now blow-up vertices into independent sets of size $N^3 \to G$
- build a △-free graph G' on N^4 vertices with $\alpha(G') \approx N^2$



Sharper inapproximability in $\triangle$-free graphs

- now blow-up vertices into independent sets of size $N^3 \to G$
- build a $\triangle$-free graph G' on N^4 vertices with $\overline{\beta}(G') \approx N^2$
- identify arbitrarily $V(G)$ and $V(G')$ and AND the edge sets



Sharper inapproximability in $\triangle$-free graphs

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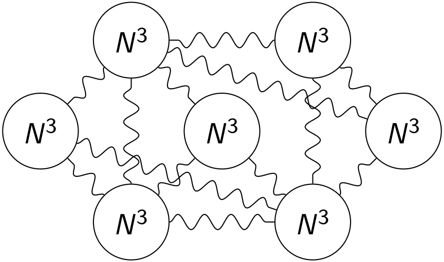

Theorem

There is no $n^{\frac{1}{4}-\varepsilon}$-approximation algorithm for MIS in $\triangle$-free graphs, unless BPP=NP.

Summary and open questions

- new conjecture with two possible approaches
- same closure by substitution
- (Effective) Erdős-Hajnal implies improved approximation
- direct improved approximation in C_5-free graphs?
- off-diagonal Ramsey $\leftrightarrow$ best possible ratio in K_t-free graphs?
- what is the smallest δ such that MIS has a n^δ-approximation in $\triangle$-graphs? $(\frac{1}{4} \leqslant \delta \leqslant \frac{1}{2})$

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Thank you for your attention!