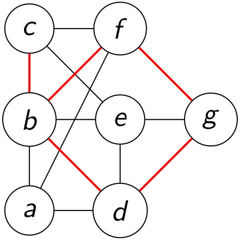
Twin-width and ordered binary structures

Édouard Bonnet
joint work with Ugo Giocanti, Patrice Ossona de Mendez, and Stéphan Thomassé; Pierre Simon and Szymon Toruńczyk
Also Colin Geniet, Eunjung Kim, Jarik Nešetřil, Sebastian Siebertz, and Rémi Watrigant

ENS Lyon, LIP

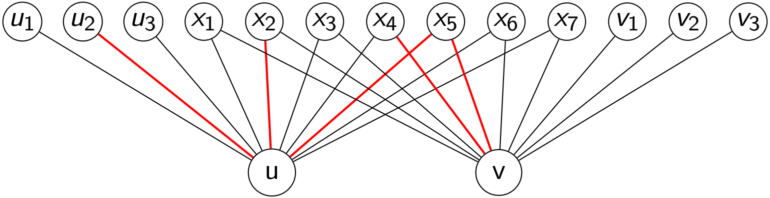
February 19th, 2021, Bordeaux GT G&O

Trigraphs



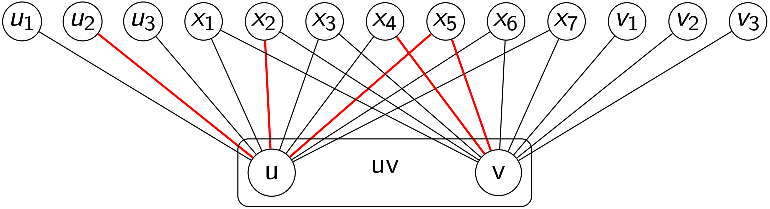
Three outcomes between a pair of vertices:
edge, or non-edge, or red edge (error edge)

Contractions in trigraphs



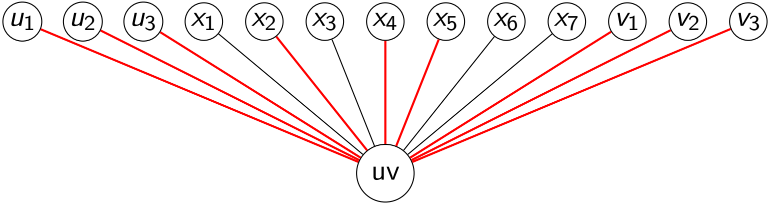
Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



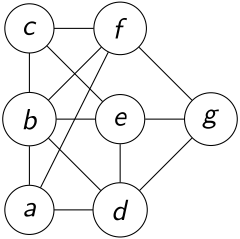
Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



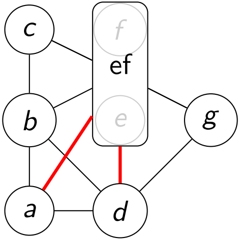
edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Contraction sequence



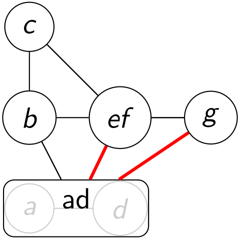
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



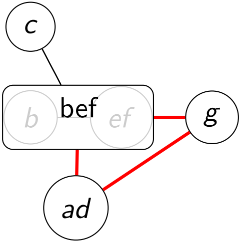
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



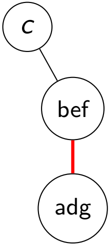
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that
G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



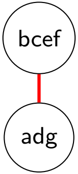
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that
G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



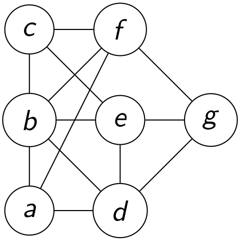
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that
G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence

A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

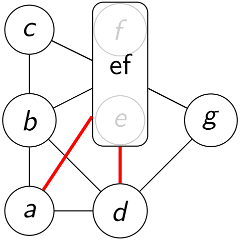


Maximum red degree = 0

overall maximum red degree = 0

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

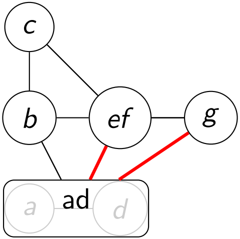


Maximum red degree $= 2$

overall maximum red degree = 2

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

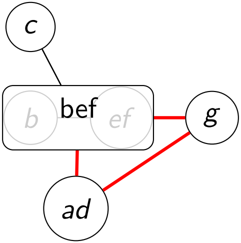


Maximum red degree $= 2$

overall maximum red degree = 2

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

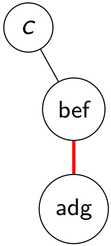


Maximum red degree = 2

overall maximum red degree = 2

Twin-width

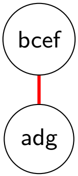
tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.



Maximum red degree $= 1$
overall maximum red degree = 2

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.



Maximum red degree $= 1$

overall maximum red degree = 2

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

Maximum red degree = 0

overall maximum red degree = 2

Simple operations preserving small twin-width

- complementation: remains the same
- taking induced subgraphs: may only decrease
- adding one apex: at most "doubles"
- substitution $G(v \leftarrow H)$: max of the twin-width of G and H

Theorem (B., Geniet, Kim, Thomassé, Watrigant '20 & '21)

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size,*
- *unit interval graphs,*
- K_t*-minor free graphs,*
- *map graphs with embedding,*
- d*-dimensional grids,*
- K_t*-free unit* d*-dimensional ball graphs,*
- $\Omega(\log n)$*-subdivisions of all the* n*-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from* K_4*,*
- *flat classes,*
- *subgraphs of every* $K_{t,t}$*-free class above,*
- *first-order transductions of all the above.*

First-order model checking on graphs

GRAPH FO MODEL CHECKING	**Parameter:** $\lvert\varphi\rvert$

Input: A graph G and a first-order sentence $\varphi \in FO(\{E\})$
Question: $G \models \varphi$?

First-order model checking on graphs

GRAPH FO MODEL CHECKING **Parameter:** $|\varphi|$
Input: A graph G and a first-order sentence $\varphi \in FO(\{E\})$
Question: $G \models \varphi$?

Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \forall y \ (E(x,y) \Rightarrow \bigvee_{1 \leqslant i \leqslant k} x = x_i \vee y = x_i)$$

$G \models \varphi$? $\Leftrightarrow$ k-VERTEX COVER

First-order model checking on graphs

GRAPH FO MODEL CHECKING	**Parameter:** $\lvert\varphi\rvert$
Input: A graph G and a first-order sentence $\varphi \in FO(\{E\})$	
Question: $G \models \varphi$?	

Example:

$$\varphi = \exists x_1 \exists y_1 \cdots \exists x_k \exists y_k \bigwedge_{\{x,y\} \in \binom{\{x_1, y_1, \ldots, x_k, y_k\}}{2}} x \neq y$$

$$\wedge\ E(x, y) \Leftrightarrow \bigvee_{1 \leqslant i \leqslant k} (x = x_i \wedge y = y_i) \vee (x = y_i \wedge y = x_i)$$

$G \models \varphi? \Leftrightarrow$

First-order model checking on graphs

GRAPH FO MODEL CHECKING	**Parameter:** $\|\varphi\|$
Input: A graph G and a first-order sentence $\varphi \in FO(\{E\})$	
Question: $G \models \varphi$?	

Example:

$$\varphi = \exists x_1 \exists y_1 \cdots \exists x_k \exists y_k \bigwedge_{\{x,y\} \in \binom{\{x_1, y_1, \dots, x_k, y_k\}}{2}} x \neq y$$

$$\wedge\ E(x,y) \Leftrightarrow \bigvee_{1 \leqslant i \leqslant k} (x = x_i \wedge y = y_i) \vee (x = y_i \wedge y = x_i)$$

$G \models \varphi$? $\Leftrightarrow$ k-INDUCED MATCHING

First-order model checking on graphs

GRAPH FO MODEL CHECKING	**Parameter:** $\vert\varphi\vert$
Input: A graph G and a first-order sentence $\varphi \in FO(\{E\})$	
Question: $G \models \varphi$?	

Example:

$$\varphi = \bigvee_{1 \leqslant q \leqslant k,\ q \text{ is odd}} \exists x_1 \notin \{s\}\ E(s, x_1) \wedge (\forall x_2 \notin \{s, x_1\}\ \neg E(x_1, x_2) \vee$$

$$(\exists x_3 \notin \{s, x_1, x_2\}\ E(x_2, x_3) \wedge (\forall x_4 \cdots (\exists x_q \notin \{s, x_1, \ldots, x_{q-1}\} E(x_{q-1}, x_q)$$

$$\wedge\ (\forall x_{q+1}\ \neg E(x_q, x_{q+1}) \vee x_{q+1} \in \{s, x_1, \ldots, x_q\})) \cdots)))$$

$G \models \varphi? \Leftrightarrow$

First-order model checking on graphs

Graph FO Model Checking	**Parameter:** $\lvert\varphi\rvert$
Input: A graph G and a first-order sentence $\varphi \in FO(\{E\})$	
Question: $G \models \varphi$?	

Example:

$$\varphi = \bigvee_{1 \leqslant q \leqslant k,\ q \text{ is odd}} \exists x_1 \notin \{s\}\ E(s, x_1) \wedge (\forall x_2 \notin \{s, x_1\}\ \neg E(x_1, x_2) \vee$$

$$(\exists x_3 \notin \{s, x_1, x_2\}\ E(x_2, x_3) \wedge (\forall x_4 \cdots (\exists x_q \notin \{s, x_1, \ldots, x_{q-1}\} E(x_{q-1}, x_q)$$

$$\wedge\ (\forall x_{q+1}\ \neg E(x_q, x_{q+1}) \vee x_{q+1} \in \{s, x_1, \ldots, x_q\})) \cdots)))$$

$G \models \varphi$? $\Leftrightarrow$ Short Generalized Geography

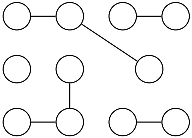
FO interpretations and transductions

FO interpretation: redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

FO interpretations and transductions

FO interpretation: redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

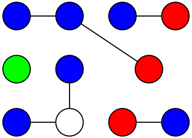
FO transduction: color by $O(1)$ unary relations, interpret, delete



FO interpretations and transductions

FO interpretation: redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

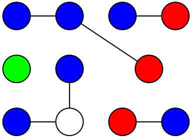
FO transduction: color by $O(1)$ unary relations, interpret, delete



FO interpretations and transductions

FO interpretation: redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

FO transduction: color by $O(1)$ unary relations, interpret, delete

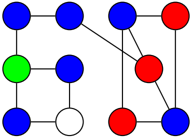


$$\varphi(x, y) = E(x, y) \vee (G(x) \wedge B(y) \wedge \neg\exists z R(z) \wedge E(y, z))$$
$$\vee (R(x) \wedge B(y) \wedge \exists z R(z) \wedge E(y, z) \wedge \neg\exists z B(z) \wedge E(y, z))$$

FO interpretations and transductions

FO interpretation: redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

FO transduction: color by $O(1)$ unary relations, interpret, delete

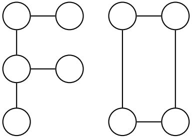


$$\varphi(x, y) = E(x, y) \vee (G(x) \wedge B(y) \wedge \neg\exists z R(z) \wedge E(y, z))$$
$$\vee(R(x) \wedge B(y) \wedge \exists z R(z) \wedge E(y, z) \wedge \neg\exists z B(z) \wedge E(y, z))$$

FO interpretations and transductions

FO interpretation: redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

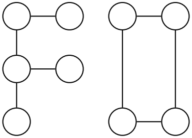
FO transduction: color by $O(1)$ unary relations, interpret, delete



FO interpretations and transductions

FO interpretation: redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

FO transduction: color by $O(1)$ unary relations, interpret, delete



Theorem (B., Kim, Thomassé, Watrigant '20)

Transductions of bounded twin-width classes have bounded twin-width.

Dependence and monadic dependence

A class $\mathscr{C}$ is
dependent, if the class of all graphs is not an interpretation of $\mathscr{C}$
monadically dependent, if the class of all graphs is not a transduction of $\mathscr{C}$ [Baldwin, Shelah '85]

Dependence and monadic dependence

A class $\mathscr{C}$ is
dependent, if the class of all graphs is not an interpretation of $\mathscr{C}$
monadically dependent, if the class of all graphs is not a transduction of $\mathscr{C}$ [Baldwin, Shelah '85]

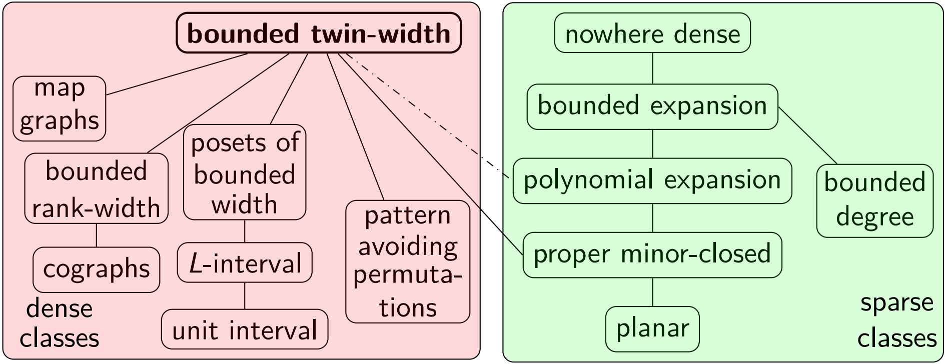
Theorem (Downey, Fellows, Taylor '96)

FO model checking is $AW[*]$*-complete on general graphs,*

thus on independent classes

Could it be that on every dependent class, it is FPT?

Classes with known tractable FO model checking



Theorem (B., Kim, Thomassé, Watrigant '20)

FO MODEL CHECKING *solvable in* $f(|\varphi|, d)n$ *on graphs with a* d*-sequence.*

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

Bounded twin-width classes are small.

Unifies and extends the same result for:
σ-free permutations [Marcus, Tardos '04]
K_t-minor free graphs [Norine, Seymour, Thomas, Wollan '06]

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

Bounded twin-width classes are small.

Subcubic graphs, interval graphs, triangle-free unit segment graphs have *unbounded* twin-width

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

Bounded twin-width classes are small.

Is the converse true for hereditary classes?

Conjecture (small conjecture)

A hereditary class has bounded twin-width if and only if it is small.

Recap of the main questions

- Can we efficiently approximate twin-width?
- Can we solve FO model checking on every dependent class?
- Is every hereditary small class of bounded twin-width?

Recap of the main questions

- Can we efficiently approximate twin-width?
- Can we solve FO model checking on every dependent class?
- Is every hereditary small class of bounded twin-width?

We answer all these questions positively in the case of ordered binary structures $\equiv$ matrices on a finite alphabet

Twin-width for unordered matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Encode a bipartite graph (or, if symmetric, a graph)

Twin-width for unordered matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Contraction of two columns (similar with two rows)

Twin-width for unordered matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & r & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & r & 0 & 1 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The red degree is now the max number of r per row/column

Twin-width for unordered matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & r & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & r & 0 & 1 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

In the non-bipartite case, we force symmetric pairs of contractions

Twin-width for matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

That was *not* the twin-width of **ordered** matrices

Twin-width for matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & r & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & r & 0 & 1 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Let's also record the columns disagreeing with the contration

Twin-width for matrices

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & r & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & r & 0 & 1 & 1 & 0 \\
1 & 0 & r & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}
$$

$$\max_{\mathit{row},\mathit{column}} (\text{number of red entries} + \text{red degree})$$

Twin-width for matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

If you find it too clumsy, encode the linear order

Twin-width for matrices

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 3 & 3 & 1 \\ 2 & 3 & 3 & 2 & 2 & 0 & 1 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and we're back to the unordered definition

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Maximum number of non-constant zones per column or row part
$=$ error value

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Maximum number of non-constant zones per column or row part
. . . until there are a single row part and column part

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Twin-width as maximum error value
of a contraction sequence

Matrix FO model checking

Signature for 0,1-matrices $\sigma = \{R^{(1)}, <^{(2)}, E^{(2)}\}$
($E^{(2)}$ becomes $E_1^{(2)}, \ldots, E_t^{(2)}$ for $[0, t]$-matrices)

Matrix FO model checking

Signature for 0,1-matrices $\sigma = \{R^{(1)}, <^{(2)}, E^{(2)}\}$
($E^{(2)}$ becomes $E_1^{(2)}, \ldots, E_t^{(2)}$ for $[0, t]$-matrices)

- $M \models R(x)$ iff x is a row index
- $M \models x < y$ iff x is a smaller index than y
- $M \models E(x, y)$ iff $M_{x,y} = 1$

Matrix FO model checking

Signature for 0,1-matrices $\sigma = \{R^{(1)}, <^{(2)}, E^{(2)}\}$
($E^{(2)}$ becomes $E_1^{(2)}, \ldots, E_t^{(2)}$ for $[0, t]$-matrices)

- $M \models R(x)$ iff x is a row index
- $M \models x < y$ iff x is a smaller index than y
- $M \models E(x, y)$ iff $M_{x,y} = 1$

tractable class: FO model checking solvable in time $f(\varphi)|M|^{O(1)}$

Growth of classes

Our matrix *classes* are closed under taking submatrices

- Small class: $\# n \times n$ matrices is $2^{O(n)}$
- Subfactorial: ultimately, $\# n \times n$ matrices $< n!$

No non-trivial automorphism in totally ordered structures, so no need for labels

Equivalences in the matrix language

Theorem

For every matrix class $\mathcal{M}$, the following are equivalent.

(i) $\mathcal{M}$ *has bounded twin-width.*

(ii) $\mathcal{M}$ *has bounded grid rank. (division property)*

(iii) $\mathcal{M}$ *is pattern-avoiding.*
(not including any of 16 "permutation-universal" classes)

(iv) $\mathcal{M}$ *is dependent.*

(v) $\mathcal{M}$ *is monadically dependent.*

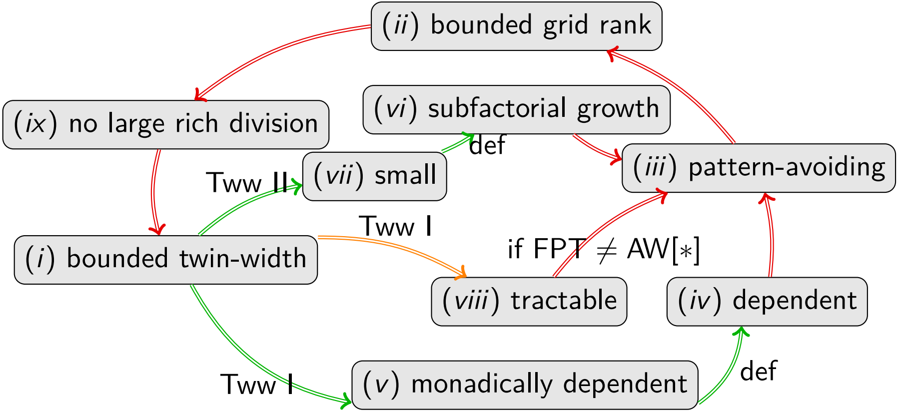
(vi) $\mathcal{M}$ *has subfactorial growth.*

(vii) $\mathcal{M}$ *is small.*

(viii) $\mathcal{M}$ *is tractable. (only if* $\mathsf{FPT} \neq \mathsf{AW}[*]$*.)*

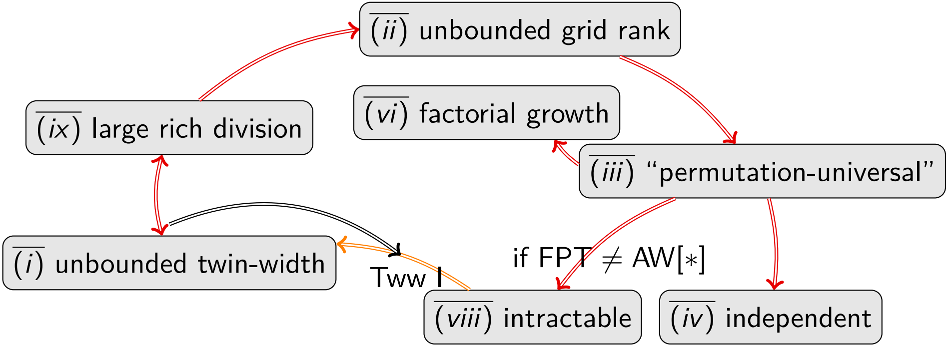
(ix) $\mathcal{M}$ *has no large rich division. (division property)*

Roadmap



(ii) bounded grid rank
(ix) no large rich division
(vi) subfactorial growth
def
(vii) small
(iii) pattern-avoiding
Tww II
Tww I
(i) bounded twin-width
if FPT ≠ AW[∗]
(viii) tractable
(iv) dependent
def
Tww I
(v) monadically dependent

Roadmap



$\overline{(ii)}$ unbounded grid rank
$\overline{(vi)}$ factorial growth
$\overline{(ix)}$ large rich division
$\overline{(iii)}$ "permutation-universal"
$\overline{(i)}$ unbounded twin-width
Tww I
if FPT $\neq$ AW[$*$]
$\overline{(viii)}$ intractable
$\overline{(iv)}$ independent

Equivalences in the ordered graph language

Theorem

Let $\mathscr{C}$ be a hereditary class of ordered graphs. The following are equivalent.

(1) *$\mathscr{C}$ has bounded twin-width.*

(2) *$\mathscr{C}$ is monadically dependent.*

(3) *$\mathscr{C}$ is dependent.*

(4) *$\mathscr{C}$ is small.*

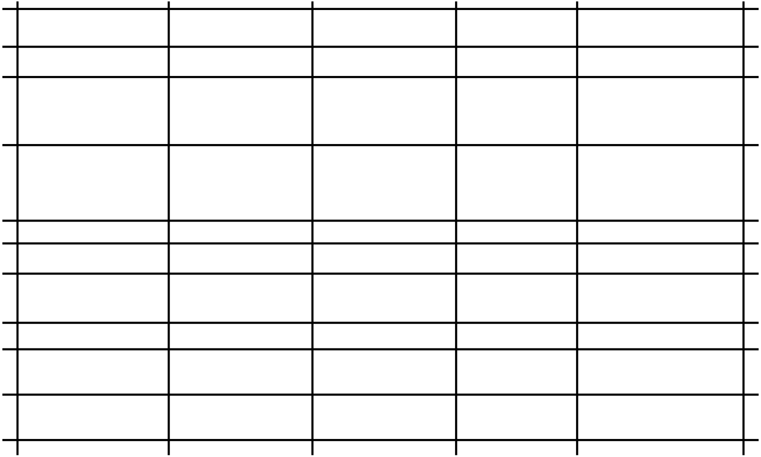
(5) *$\mathscr{C}$ contains $2^{O(n)}$ ordered n-vertex graphs.*

(6) *$\mathscr{C}$ contains less than $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k!$ ordered n-vertex graphs, for some n.*

(7) *$\mathscr{C}$ does not include one of 256 hereditary ordered graph classes $\mathscr{M}_{\eta,\lambda,\rho}$ with unbounded twin-width.*

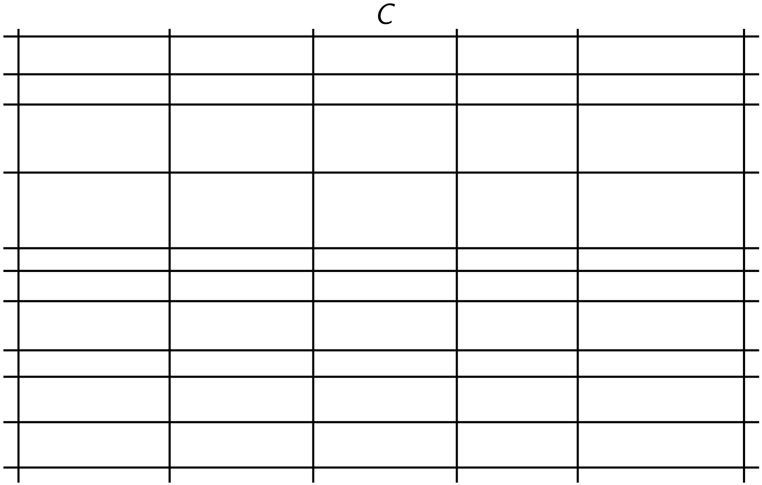
(8) *FO-model checking is fixed-parameter tractable on $\mathscr{C}$.*

k-Rich division



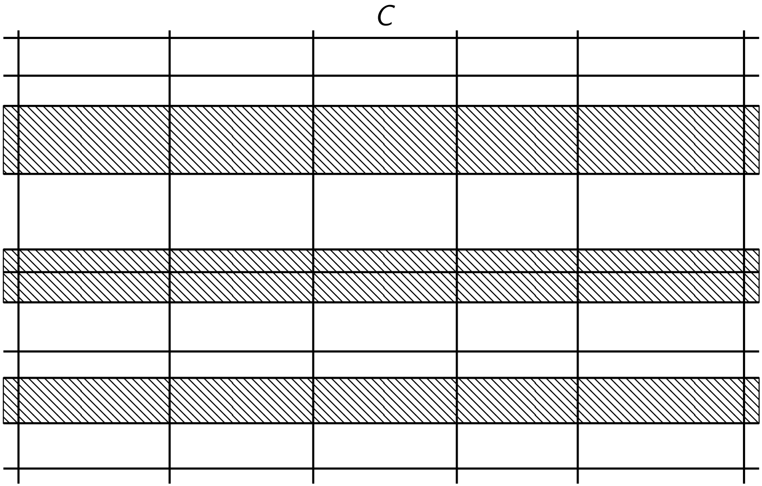
Division

k-Rich division



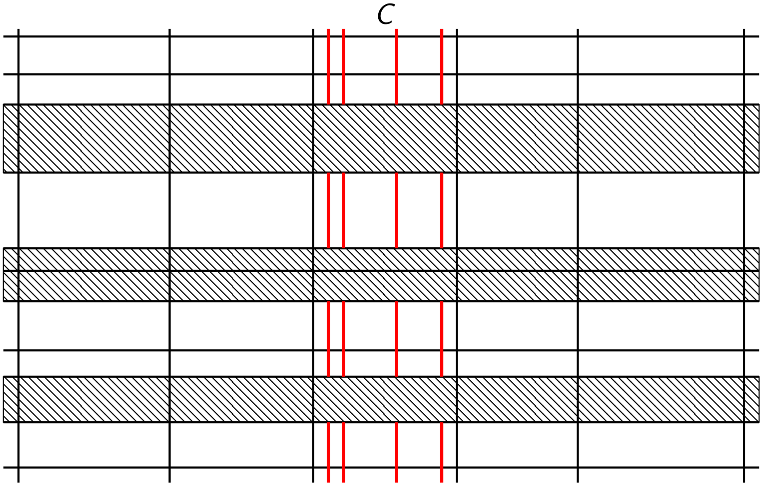
Division such that for each, say, column part C

k-Rich division



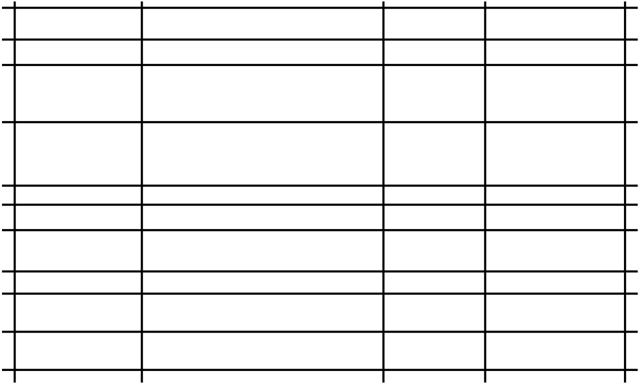
Division such that for each, say, column part C no removal of k row parts

k-Rich division



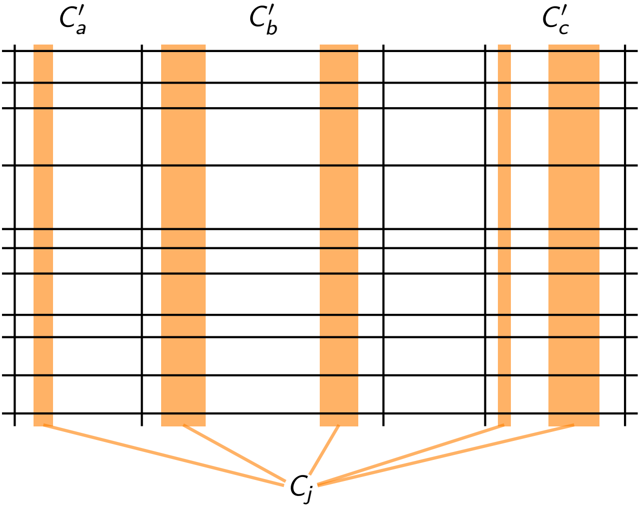
Division such that for each, say, column part C no removal of k row parts leaves C with less than k distinct column vectors

Large rich division $\Rightarrow$ unbounded twin-width



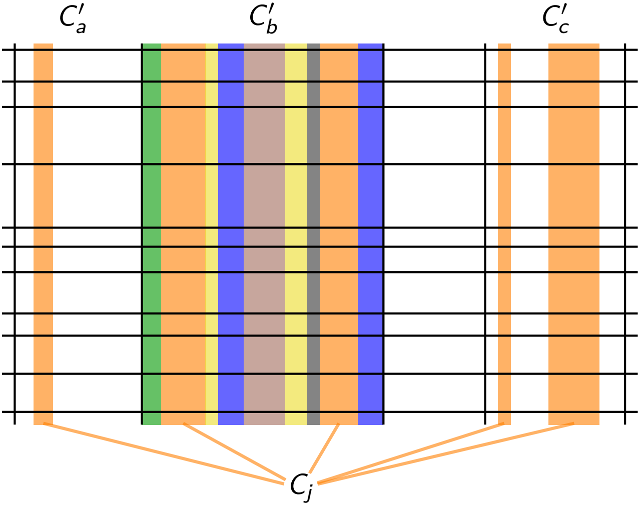
Fix an $2k(k+1)$-rich division $\mathcal{D}$, and assume there is a k-sequence $\mathcal{S}$

Large rich division ⇒ unbounded twin-width



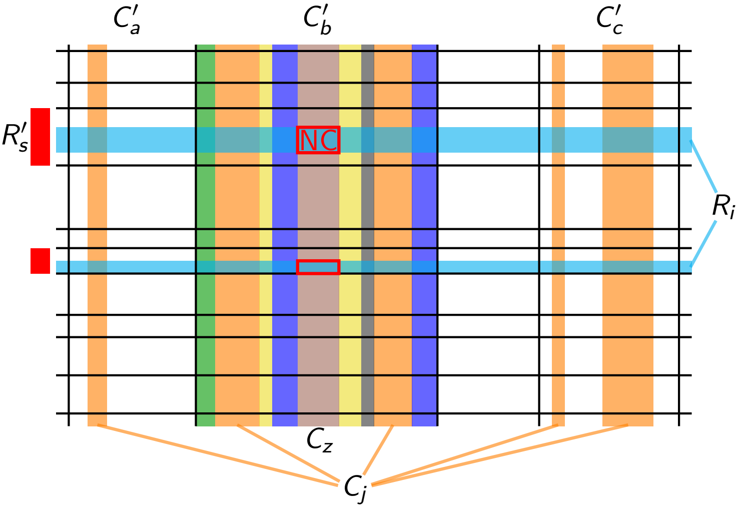
Consider the first time a part of $\mathcal{S}$ intersects 3 parts of $\mathcal{D}$

Large rich division ⇒ unbounded twin-width



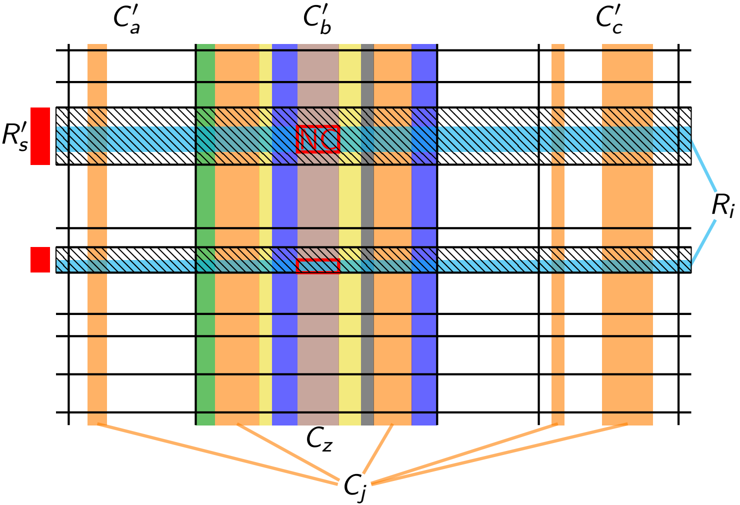
There are at most k other column parts intersecting C'_b (red degree of C_j)

Large rich division ⇒ unbounded twin-width



Each such part C_z is non-constant in at most $2k$ zones of $\mathcal{D}$

Large rich division ⇒ unbounded twin-width



Thus removing $2k(k+1)$ row parts of $\mathcal{D} \to \leqslant k+1$ distinct columns

No large rich division $\Rightarrow$ bounded twin-width

Build greedily a division where every part contradicts the richness

- can only be stopped by a large rich division
- turned into a contraction sequence as in Tww I

No large rich division $\Rightarrow$ bounded twin-width

Build greedily a division where every part contradicts the richness

- can only be stopped by a large rich division
- turned into a contraction sequence as in Tww I

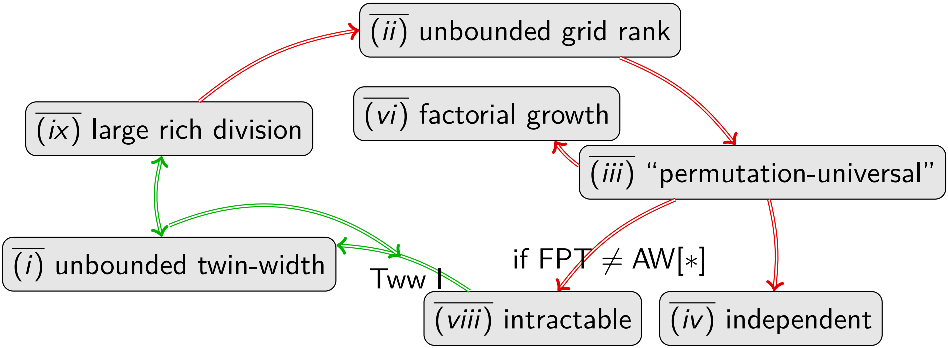
$\rightarrow$ approximation of twin-width for ordered binary structures

Theorem

There is a fixed-parameter algorithm, which, given an ordered binary structure G and a parameter k, either outputs

- *a $2^{O(k^4)}$-sequence of G, implying that $tww(G) = 2^{O(k^4)}$, or*
- *a $2k(k+1)$-rich division of $M(G)$, implying that $tww(G) > k$.*

Roadmap


$\overline{(ii)}$ unbounded grid rank
$\overline{(vi)}$ factorial growth
$\overline{(ix)}$ large rich division
$\overline{(iii)}$ "permutation-universal"
$\overline{(i)}$ unbounded twin-width
if FPT $\neq$ AW[$*$]
Tww I
$\overline{(viii)}$ intractable
$\overline{(iv)}$ independent

k-rank division

rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$
rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$
rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$
rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$

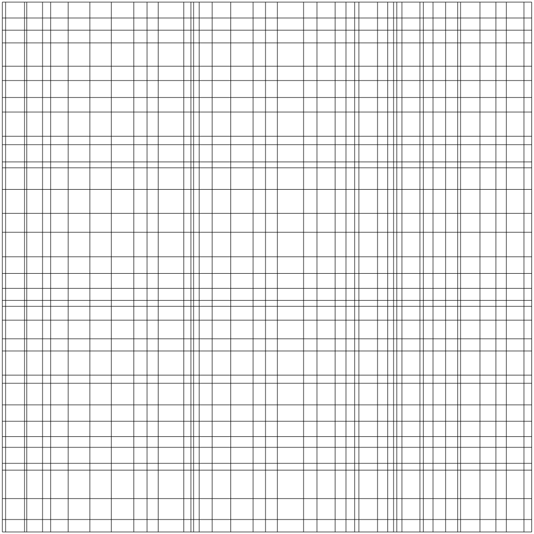
k-by-k division where every cell has rank at least k

k-rank division

rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$
rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$
rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$
rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$	rank $\geqslant k$

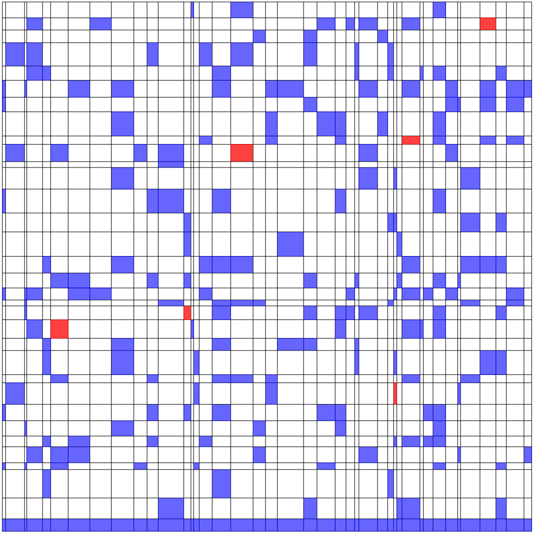
Grid rank of M = largest k such that M admits a k-rank division

Large rich division $\Rightarrow$ unbounded grid rank



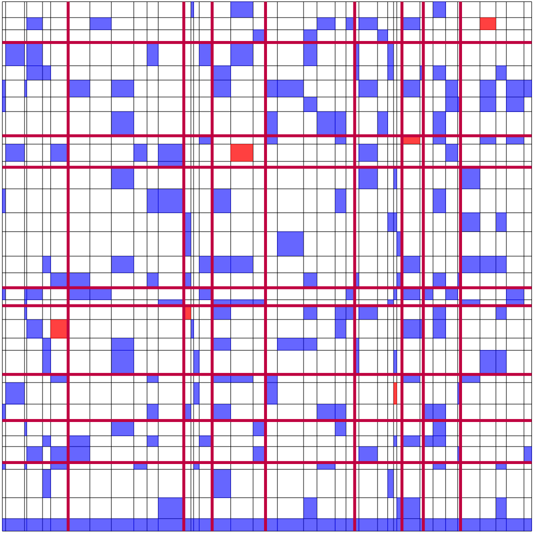
Fix a large rich division $\mathcal{D}$

Large rich division ⇒ unbounded grid rank



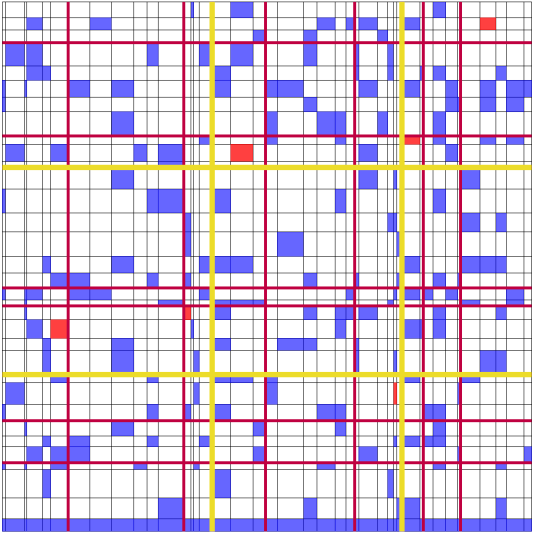
Red zones = large rank; Blue zones = first of its column to contain a particular row vector

Large rich division ⇒ unbounded grid rank



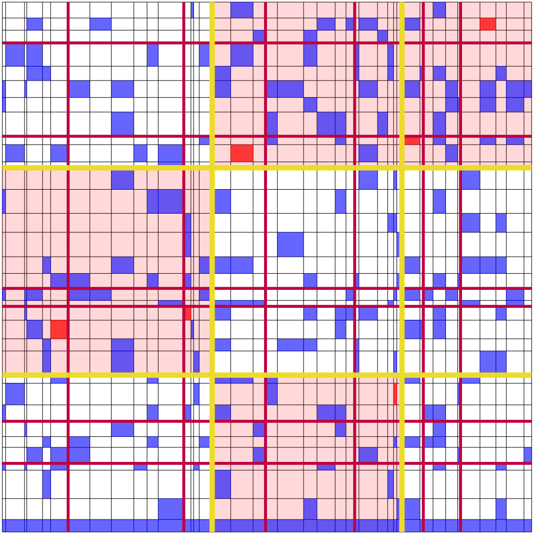
Marcus-Tardos theorem applied to the colored zones → division $\mathcal{D}'$

Large rich division ⇒ unbounded grid rank



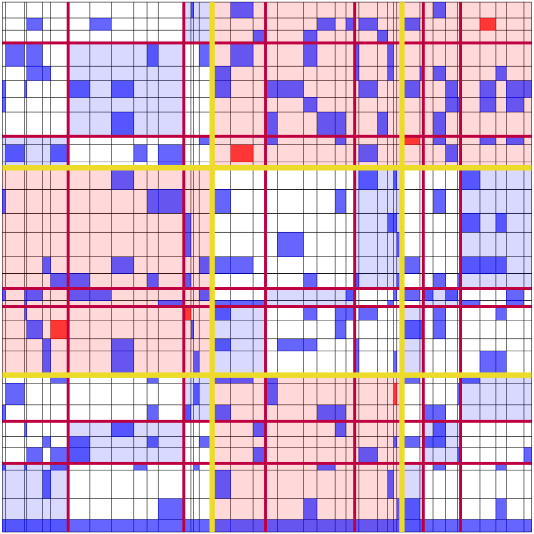
Coarser division $\mathcal{D}''$, 1 zone of $\mathcal{D}'' \equiv 2^k \times 2^k$ zones of $\mathcal{D}'$

Large rich division ⇒ unbounded grid rank



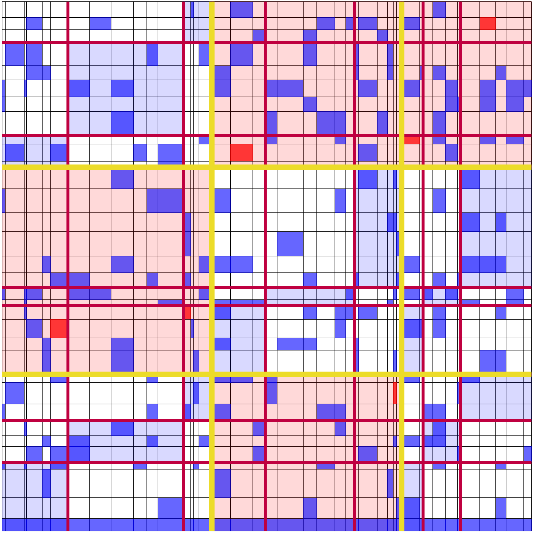
A zone of $\mathcal{D}''$ containing a red zone has large rank

Large rich division ⇒ unbounded grid rank



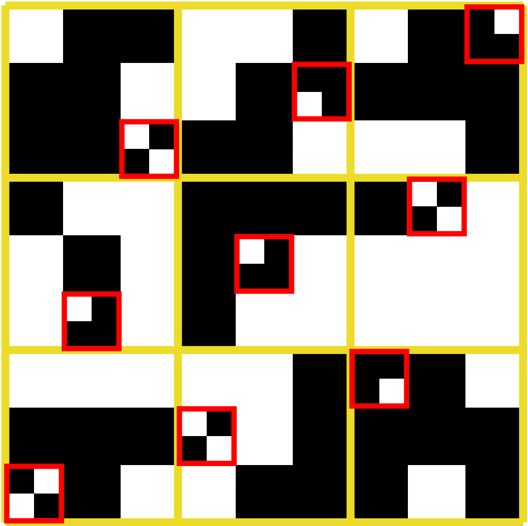
Other zones have diagonals of blue zones

Large rich division ⇒ unbounded grid rank



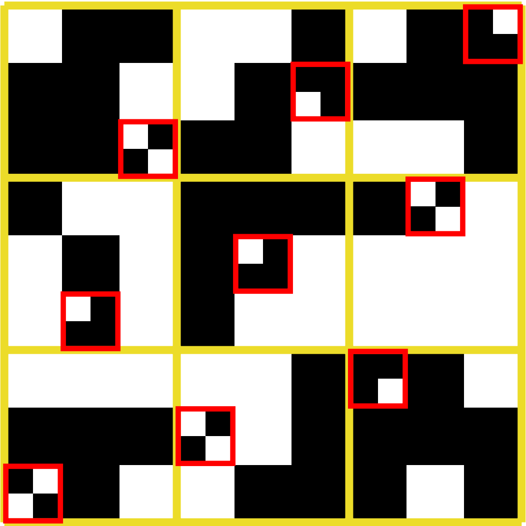
2^k distinct row vectors in each zone of $\mathcal{D}''$

Large rank division ⇒ large rank Latin division



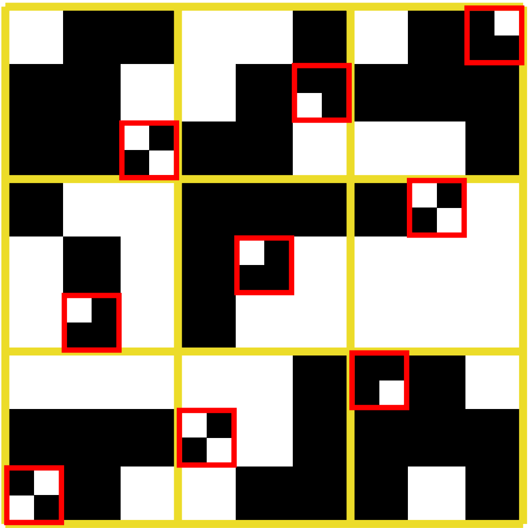
Latin rank division: high-rank zones are boxed (red) in a universal permutation pattern,

Large rank division ⇒ large rank Latin division



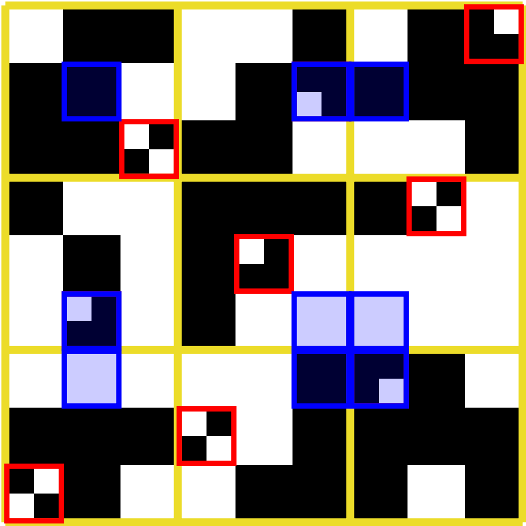
...they are the usual suspects: diagonal, anti-diagonal, upper triangular, upper anti-triangular, and their *complements*

Large rank division ⇒ large rank Latin division



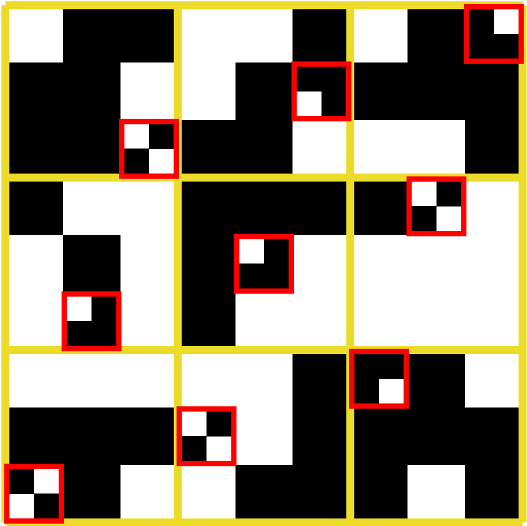
…while every other subzones are constant

Large rank division ⇒ large rank Latin division



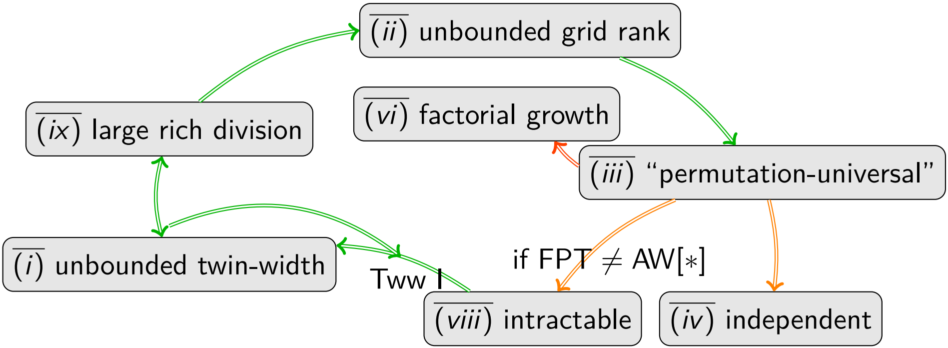
Reversible encoding of $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by a 6×6 matrix

Large rank division $\Rightarrow$ large rank Latin division



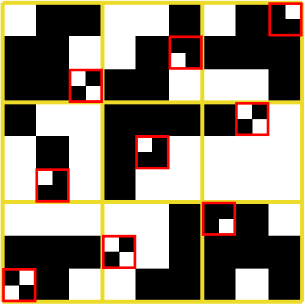
Injection from $\mathfrak{S}_n$ to $\mathcal{M}_{2n} \to |\mathcal{M}_n| \geqslant \lfloor \frac{n}{2} \rfloor !$

Roadmap



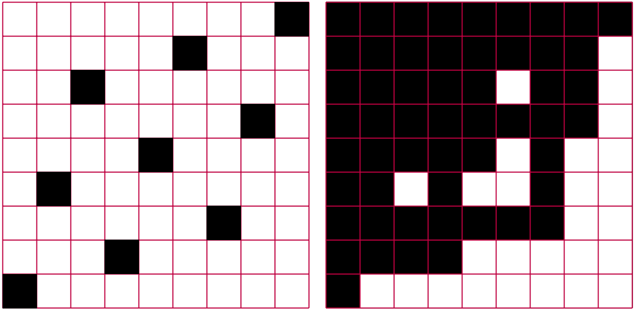
$\overline{(ii)}$ unbounded grid rank
$\overline{(vi)}$ factorial growth
$\overline{(ix)}$ large rich division
$\overline{(iii)}$ "permutation-universal"
$\overline{(i)}$ unbounded twin-width
if FPT $\neq$ AW[$*$]
Tww I
$\overline{(viii)}$ intractable
$\overline{(iv)}$ independent

Further extractions in the rank Latin division



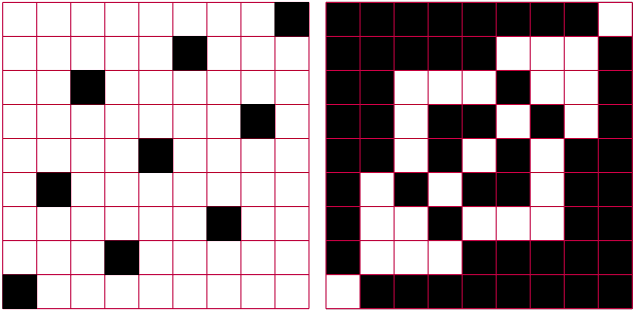
Submatrix agreeing on 1 of 16 patterns for the constant zones
$\eta : \{-1,1\}^2 \cup \{(0,0)\} \to \{0,1\}$ with $\eta(0,0) = 1 - \eta(1,1)$

Large rank Latin division $\Rightarrow$ permutation-universal



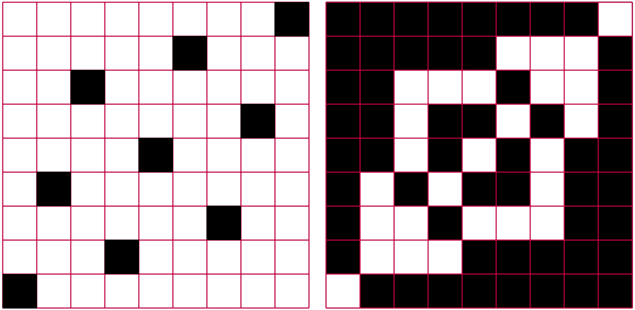
An example of a pattern with $\eta(x, y) = 0$ iff $x = y = 1$

Large rank Latin division $\Rightarrow$ permutation-universal



Another example

Large rank Latin division $\Rightarrow$ permutation-universal



Now injection from $\mathfrak{S}_n$ to $\mathcal{M}_n$, so $|\mathcal{M}_n| \geqslant n!$

Growth gap of hereditary ordered graph class

Conjecture (Balogh, Bollobás, Morris)

Every hereditary class of ordered graphs have growth $2^{O(n)}$ *or at least* $n^{n/2+o(n)}$.

Solved:

- Bounded twin-width: growth is $2^{O(n)}$ (Tww II)
- Unbounded twin-width: $\geqslant n!$ ordered (n, n)-bipartite graphs

Growth gap of hereditary ordered graph class

Conjecture (Balogh, Bollobás, Morris)

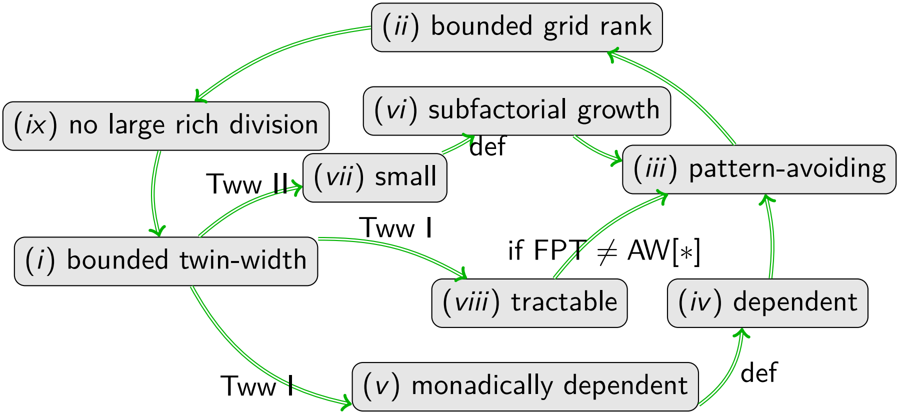
Every hereditary class of ordered graphs have growth $2^{O(n)}$
or at least $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k! = n^{n/2+o(n)}$

Solved:

- Bounded twin-width: growth is $2^{O(n)}$ (Tww II)
- Unbounded twin-width: $\geqslant n!$ ordered (n, n)-bipartite graphs

A bit more work to get the fine-grained bound

Roadmap



(ii) bounded grid rank
(ix) no large rich division
(vi) subfactorial growth
def
(vii) small
Tww II
(iii) pattern-avoiding
Tww I
(i) bounded twin-width
if FPT ≠ AW[∗]
(viii) tractable
(iv) dependent
def
Tww I
(v) monadically dependent

The interplay between twin-width and permutations

Twin-width originates from a permutation width defined by Guillemot and Marx to show Permutation Pattern $\in$ FPT

The interplay between twin-width and permutations

Twin-width originates from a permutation width defined by Guillemot and Marx to show Permutation Pattern $\in$ FPT

Likewise, twin-width heavily relies on Marcus-Tardos theorem

The interplay between twin-width and permutations

Twin-width originates from a permutation width defined by Guillemot and Marx to show $\textsc{Permutation Pattern} \in \mathsf{FPT}$

Likewise, twin-width heavily relies on Marcus-Tardos theorem

"Permutation-universal" matrices were here key

The interplay between twin-width and permutations

Twin-width originates from a permutation width defined by Guillemot and Marx to show Permutation Pattern $\in$ FPT

Likewise, twin-width heavily relies on Marcus-Tardos theorem

"Permutation-universal" matrices were here key

Theorem (B., Nešetřil, Ossona de Mendez, Siebertz, Thomassé '21+)

A class of binary structures has bounded twin-width if and only if it is an FO transduction of a proper permutation class.

The interplay between twin-width and permutations

Twin-width originates from a permutation width defined by Guillemot and Marx to show PERMUTATION PATTERN $\in$ FPT

Likewise, twin-width heavily relies on Marcus-Tardos theorem

"Permutation-universal" matrices were here key

Theorem (B., Nešetřil, Ossona de Mendez, Siebertz, Thomassé '21+)
A class of binary structures has bounded twin-width if and only if it is an FO transduction of a proper permutation class.

Theorem (Tww I + Tww IV)
A class of binary structures has bounded twin-width if and only if it the reduct of a monadically dependent class of totally ordered binary structures.

Future directions

Main questions:
Algorithm to compute/approximate twin-width in general
Fully classify classes with tractable FO model checking
Small conjecture

Future directions

Main questions:
Algorithm to compute/approximate twin-width in general
Fully classify classes with tractable FO model checking
Small conjecture

Thank you for your attention!

On arxiv
Twin-width I: tractable FO model checking [BKTW '20]
Twin-width II: small classes [BGKTW '20]
Twin-width III: Max Independent Set, Min Dominating Set, and Coloring [BGKTW '21]
Twin-width IV: low complexity matrices [BGOdMT '21]
Twin-width and permutations [BNOdMST '21]

Stanley-Wilf conjecture / Marcus-Tardos theorem

Question

For every k, is there a c_k such that every $n \times m$ $0,1$-matrix with at least c_k 1 per row and column admits a k-grid minor?

Stanley-Wilf conjecture / Marcus-Tardos theorem

Conjecture (reformulation of Füredi-Hajnal conjecture '92)

For every k, there is a c_k such that every $n \times m$ 0,1-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Stanley-Wilf conjecture / Marcus-Tardos theorem

Conjecture (reformulation of Füredi-Hajnal conjecture '92)

For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Conjecture (Stanley-Wilf conjecture '80s)

Any proper permutation class contains only $2^{O(n)}$ n-permutations.

Klazar showed Füredi-Hajnal $\Rightarrow$ Stanley-Wilf in 2000
Marcus and Tardos showed Füredi-Hajnal in 2004

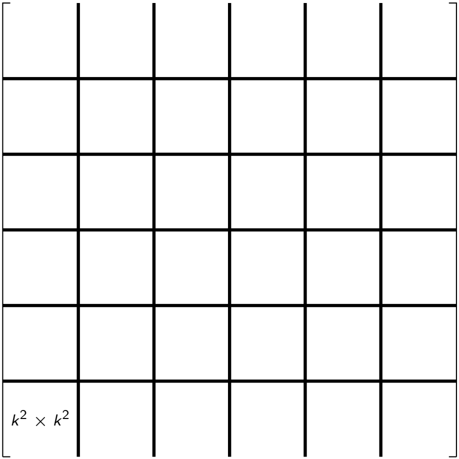
Marcus-Tardos one-page inductive proof

$$M = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

Let M be an $n \times n$ $0,1$-matrix without k-grid minor

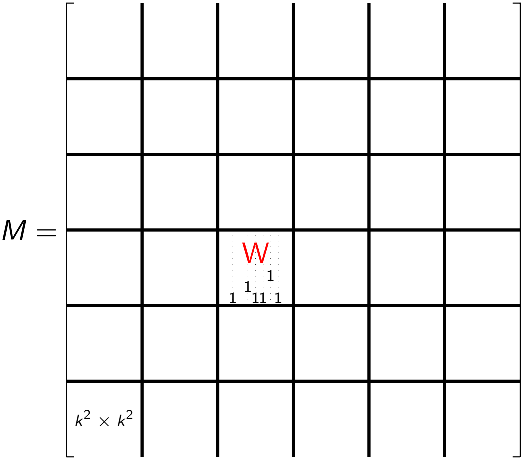
Marcus-Tardos one-page inductive proof

$M =$



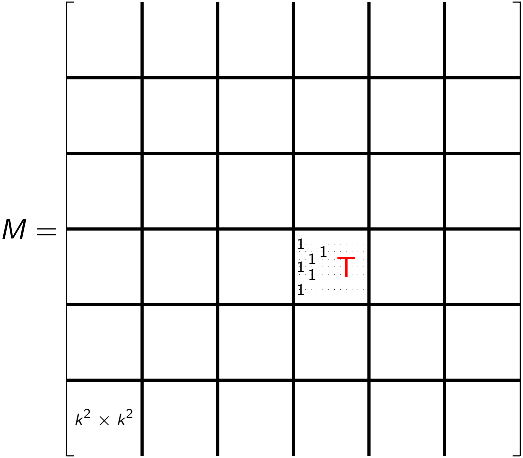
Draw a regular $\frac{n}{k^2} \times \frac{n}{k^2}$ division on top of M

Marcus-Tardos one-page inductive proof



A cell is *wide* if it has at least k columns with a 1

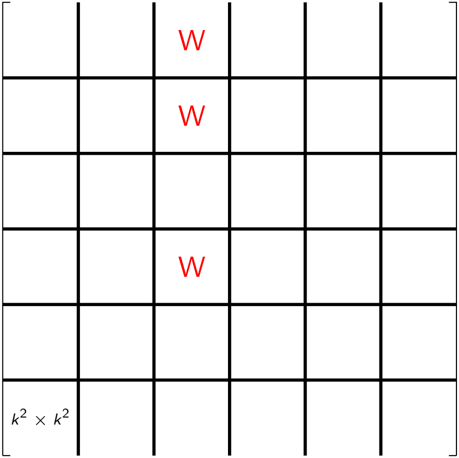
Marcus-Tardos one-page inductive proof



A cell is *tall* if it has at least k rows with a 1

Marcus-Tardos one-page inductive proof

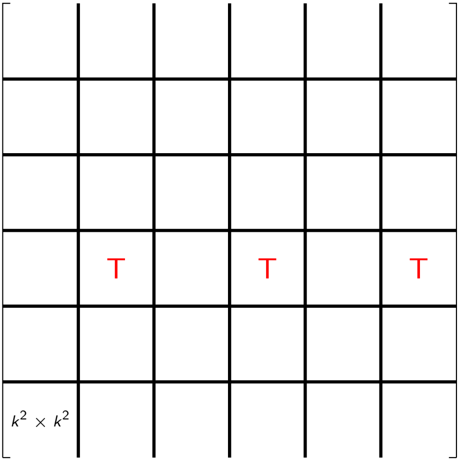
$M =$



There are less than $k\binom{k^2}{k}$ wide cells per column part. Why?

Marcus-Tardos one-page inductive proof

$M =$



There are less than $k\binom{k^2}{k}$ tall cells per row part

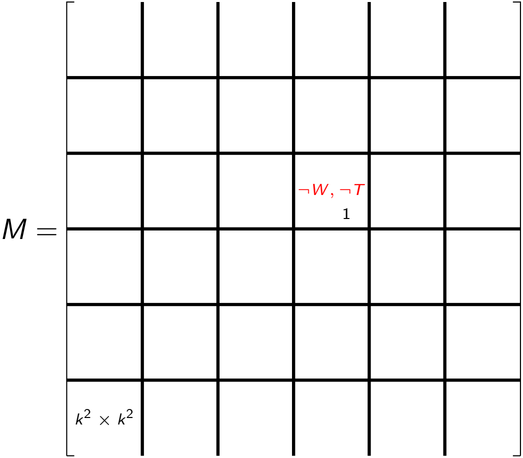
Marcus-Tardos one-page inductive proof

$M =$

		W			
	W	W			T
	T	W	T		T
		T			
$k^2 \times k^2$					W

In W and T, at most $2 \cdot \frac{n}{k^2} \cdot k\binom{k^2}{k} \cdot k^4 = 2k^3\binom{k^2}{k}n$ entries 1

Marcus-Tardos one-page inductive proof



There are at most $(k-1)^2 c_k \frac{n}{k^2}$ remaining 1. Why?

Marcus-Tardos one-page inductive proof

$M =$

		W			
	W	W			T
			$\neg W, \neg T$ 1		
	T	W	T		T
		T			
$k^2 \times k^2$					W

Choose $c_k = 2k^4\binom{k^2}{k}$ so that $(k-1)^2 c_k \frac{n}{k^2} + 2k^3\binom{k^2}{k} n \leqslant c_k n$