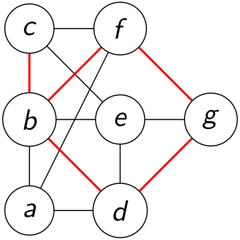
Twin-width

Édouard Bonnet
based on joint works with Colin Geniet, Ugo Giocanti, Eun Jung Kim, Patrice Ossona de Mendez, Stéphan Thomassé, and Rémi Watrigant

ENS Lyon, LIP

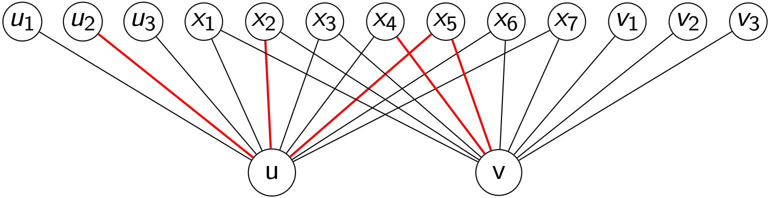
May 28th, 2021, CANADAM

Trigraphs



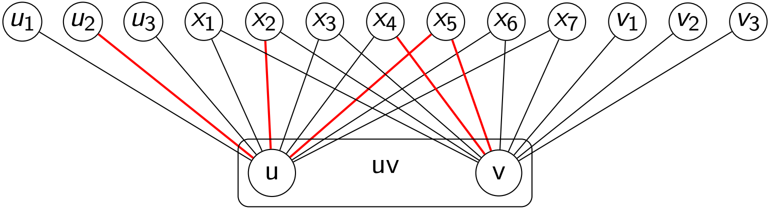
Three outcomes between a pair of vertices:
edge, or non-edge, or red edge (error edge)

Contractions in trigraphs



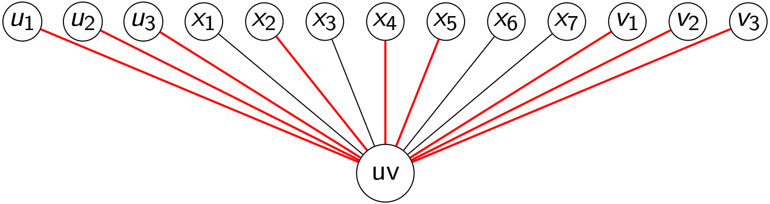
Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



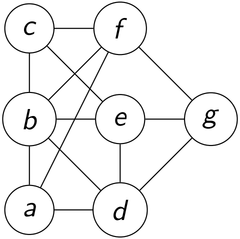
Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



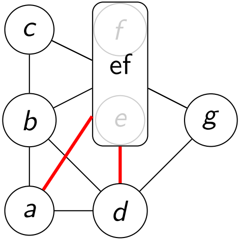
edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Contraction sequence



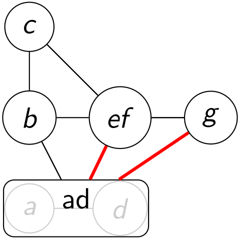
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



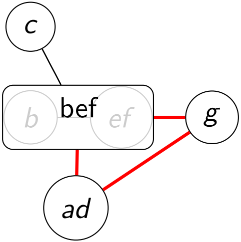
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



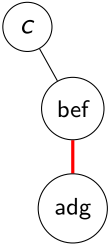
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



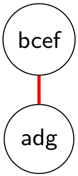
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that
G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



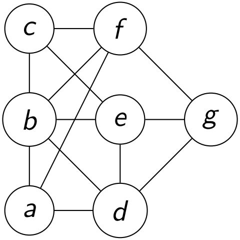
A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence

A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that
G_i is obtained by performing one contraction in G_{i+1}.

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

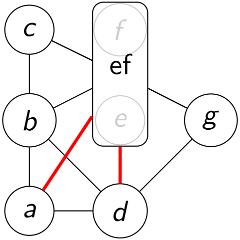


Maximum red degree = 0

overall maximum red degree = 0

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

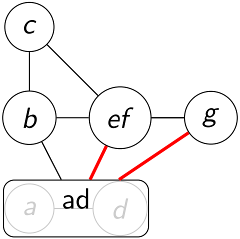


Maximum red degree $= 2$

overall maximum red degree = 2

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

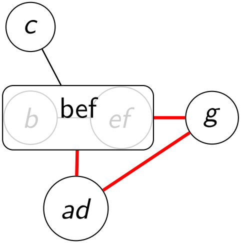


Maximum red degree = 2

overall maximum red degree = 2

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

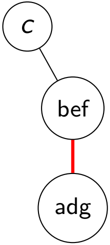


Maximum red degree = 2

overall maximum red degree = 2

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

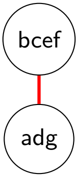


Maximum red degree $= 1$

overall maximum red degree = 2

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.



Maximum red degree $= 1$

overall maximum red degree = 2

Twin-width

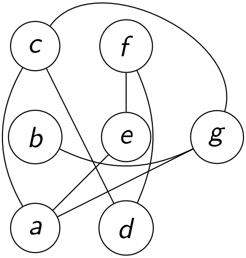
tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

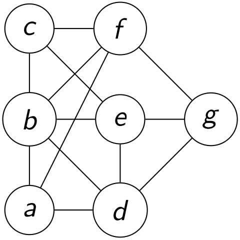
Maximum red degree $= 0$
overall maximum red degree = 2

Simple operations preserving small twin-width

- complementation: remains the same
- taking induced subgraphs: may only decrease
- adding one vertex linked arbitrarily: at most "doubles"

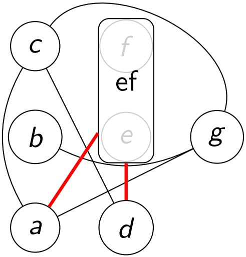
Complementation



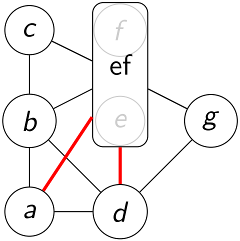


$$\text{tww}(\overline{G}) = \text{tww}(G)$$

Complementation



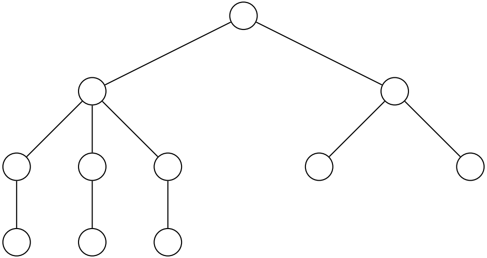
$\overline{G_6}$



G_6

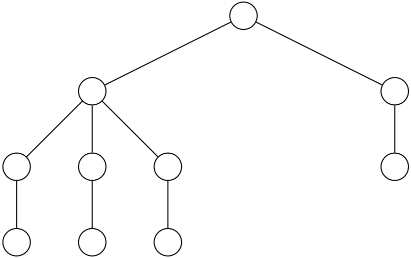
$$\text{tww}(\overline{G}) = \text{tww}(G)$$

Graphs with bounded twin-width – trees



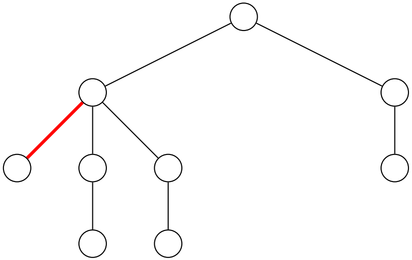
If possible, contract two twin leaves

Graphs with bounded twin-width – trees



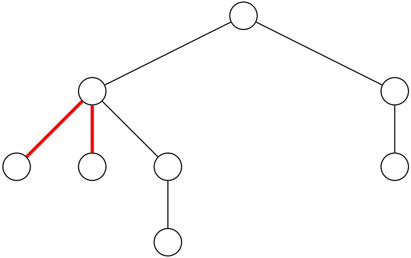
If not, contract a deepest leaf with its parent

Graphs with bounded twin-width – trees



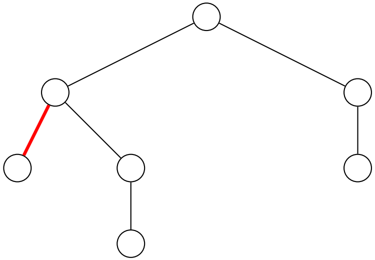
If not, contract a deepest leaf with its parent

Graphs with bounded twin-width – trees



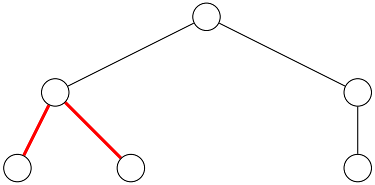
If possible, contract two twin leaves

Graphs with bounded twin-width – trees



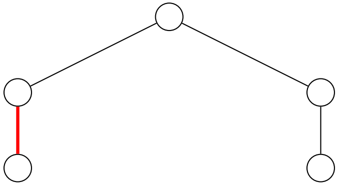
Cannot create a red degree-3 vertex

Graphs with bounded twin-width – trees



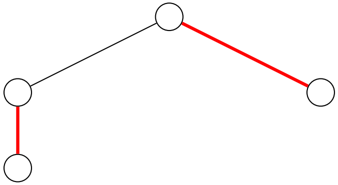
Cannot create a red degree-3 vertex

Graphs with bounded twin-width – trees



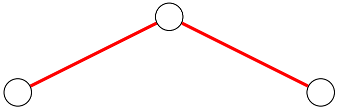
Cannot create a red degree-3 vertex

Graphs with bounded twin-width – trees



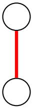
Cannot create a red degree-3 vertex

Graphs with bounded twin-width – trees



Cannot create a red degree-3 vertex

Graphs with bounded twin-width – trees

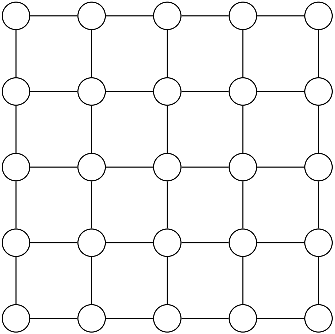


Cannot create a red degree-3 vertex

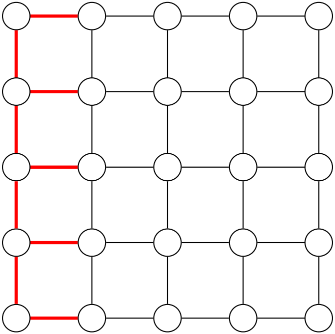
Graphs with bounded twin-width – trees

Generalization to bounded *treewidth* and even bounded *rank-width*

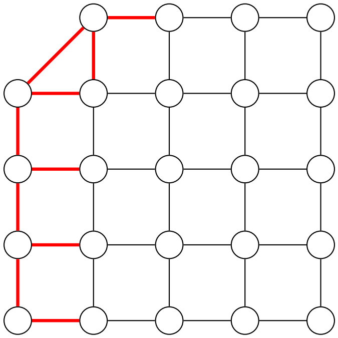
Graphs with bounded twin-width – grids



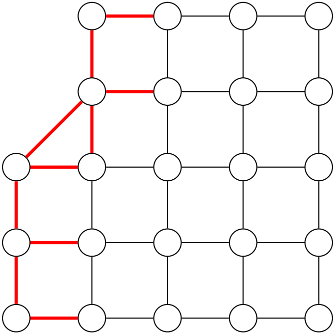
Graphs with bounded twin-width – grids



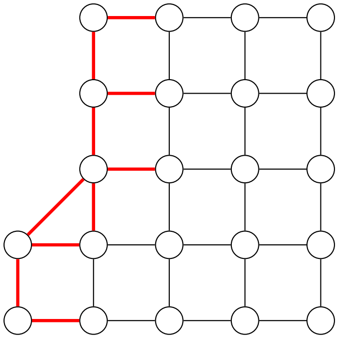
Graphs with bounded twin-width – grids



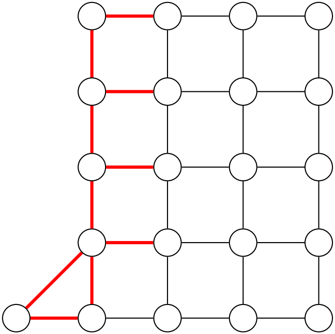
Graphs with bounded twin-width – grids



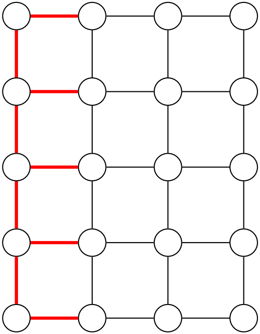
Graphs with bounded twin-width – grids



Graphs with bounded twin-width – grids



Graphs with bounded twin-width – grids



4-sequence for planar grids, $3d$-sequence for d-dimensional grids

Theorem

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded tree-width, and even, rank-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size (seen as digraphs),*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- *$\Omega(\log n)$-subdivisions of all the n-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from K_4,*
- *strong products of two bounded twin-width classes, one with bounded degree, etc.*

χ-boundedness

$\mathcal{C}$ χ-bounded: $\exists f, \forall G \in \mathcal{C}$, $\chi(G) \leqslant f(\omega(G))$

Theorem

Every twin-width class is χ-bounded.
More precisely, every graph G of twin-width at most d admits a proper $(d+2)^{\omega(G)-1}$-coloring.

χ-boundedness

$\mathcal{C}$ χ-bounded: $\exists f, \forall G \in \mathcal{C}$, $\chi(G) \leqslant f(\omega(G))$

Theorem

Every twin-width class is χ-bounded.
More precisely, every graph G of twin-width at most d admits a proper $(d+2)^{\omega(G)-1}$-coloring.

Are they polynomially χ-bounded? i.e., $\chi(G) = O(\omega(G)^d)$

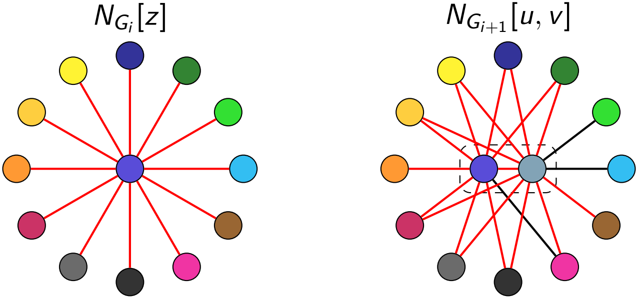
Bounded twin-width graphs do satisfy strong Erdős-Hajnal

$d + 2$-coloring in the triangle-free case

Algorithm: **Start from $G_1 = K_1$, color its unique vertex 1, and rewind the d-sequence. A contraction seen backward is a split and we shall find colors for the two new vertices.**

$d + 2$-coloring in the triangle-free case

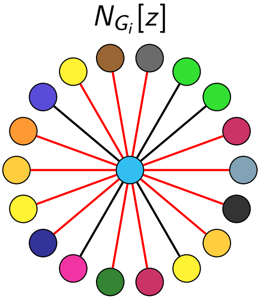
Algorithm: **Start from $G_1 = K_1$, color its unique vertex 1, and rewind the d-sequence. A contraction seen backward is a split and we shall find colors for the two new vertices.**

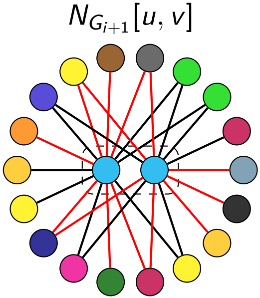


z has only red incident edges $\rightarrow$ $d + 2$-nd color available to v

$d + 2$-coloring in the triangle-free case

Algorithm: **Start from $G_1 = K_1$, color its unique vertex 1, and rewind the d-sequence. A contraction seen backward is a split and we shall find colors for the two new vertices.**





z incident to at least one black edge $\rightarrow$ non-edge between u and v

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem

Bounded twin-width classes are small.

Unifies and extends the same result for:
σ-free permutations [Marcus, Tardos '04]
K_t-minor free graphs [Norine, Seymour, Thomas, Wollan '06]

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem

Bounded twin-width classes are small.

Subcubic graphs, interval graphs, triangle-free unit segment graphs have *unbounded* twin-width

Construction of subcubic graphs with large twin-width?

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem

Bounded twin-width classes are small.

Is the converse true for hereditary classes?

Conjecture (small conjecture)

A hereditary class has bounded twin-width if and only if it is small.

Growth gap of hereditary ordered graph class

Conjecture (Balogh, Bollobás, Morris)

Every hereditary class of ordered graphs have growth $2^{O(n)}$ *or at least* $(n/2)!$.

Solved:

- Bounded twin-width: growth is $2^{O(n)}$
- Unbounded twin-width: witness progressively turned into a canonical family of $\geqslant n!$ ordered (n, n)-bipartite graphs

Growth gap of hereditary ordered graph class

Conjecture (Balogh, Bollobás, Morris)

Every hereditary class of ordered graphs have growth $2^{O(n)}$ *or at least* $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k! = n^{n/2+o(n)}$

Solved:

- Bounded twin-width: growth is $2^{O(n)}$
- Unbounded twin-width: witness progressively turned into a canonical family of $\geqslant n!$ ordered (n, n)-bipartite graphs

A bit more work to get the fine-grained bound

Sparse classes with bounded twin-width

Theorem

Let $\mathcal{C}$ a hereditary class of bounded twin-width. TFAE:

- *graphs in $\mathcal{C}$ are $K_{t,t}$-free;*
- *graphs in $\mathcal{C}$ have linearly many edges;*
- *$\mathcal{C}$ has bounded expansion;*
- *The subgraph-closure of $\mathcal{C}$ has bounded twin-width.*

Sparse classes with bounded twin-width

Theorem

Let $\mathcal{C}$ a hereditary class of bounded twin-width. TFAE:

- *graphs in $\mathcal{C}$ are $K_{t,t}$-free;*
- *graphs in $\mathcal{C}$ have linearly many edges;*
- *$\mathcal{C}$ has bounded expansion;*
- *The subgraph-closure of $\mathcal{C}$ has bounded twin-width.*

Still an interesting family of classes including bounded queue/stack number, K_t-minor free, and some expander classes

Does polynomial expansion imply bounded twin-width?

Open questions

- Polynomial χ-boundedness
- Explicit construction of cubic graphs with large twin-width
- ~~Small conjecture~~ refuted
- Does polynomial expansion imply bounded twin-width?
- Algorithm to compute/approximate twin-width in general

Open questions

- Polynomial χ-boundedness
- Explicit construction of cubic graphs with large twin-width
- ~~Small conjecture~~ refuted
- Does polynomial expansion imply bounded twin-width?
- Algorithm to compute/approximate twin-width in general

Thank you for your attention!

On arxiv

Twin-width I: tractable FO model checking [BKTW '20]

Twin-width II: small classes [BGKTW '20]

Twin-width III: Max Independent Set, Min Dominating Set, and Coloring [BGKTW '21]

Twin-width IV: low complexity matrices [BGOdMT '21]

Twin-width and permutations [BNOdMST '21]

Stanley-Wilf conjecture / Marcus-Tardos theorem

Question

For every k, is there a c_k such that every $n \times m$ $0,1$-matrix with at least c_k 1 per row and column admits a k-grid minor?

Stanley-Wilf conjecture / Marcus-Tardos theorem

Conjecture (reformulation of Füredi-Hajnal conjecture '92)

For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Stanley-Wilf conjecture / Marcus-Tardos theorem

Conjecture (reformulation of Füredi-Hajnal conjecture '92)

For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Conjecture (Stanley-Wilf conjecture '80s)

Any proper permutation class contains only $2^{O(n)}$ n-permutations.

Klazar showed Füredi-Hajnal $\Rightarrow$ Stanley-Wilf in 2000
Marcus and Tardos showed Füredi-Hajnal in 2004

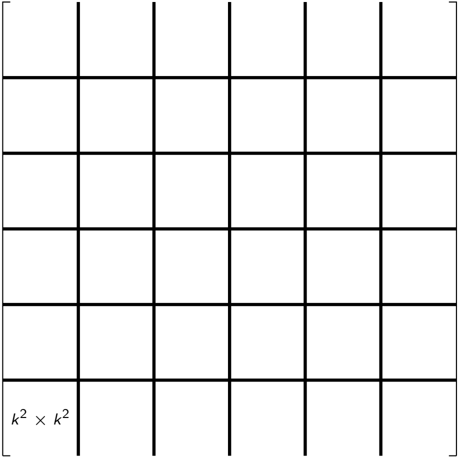
Marcus-Tardos one-page inductive proof

$$M = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Let M be an $n \times n$ $0,1$-matrix without k-grid minor

Marcus-Tardos one-page inductive proof

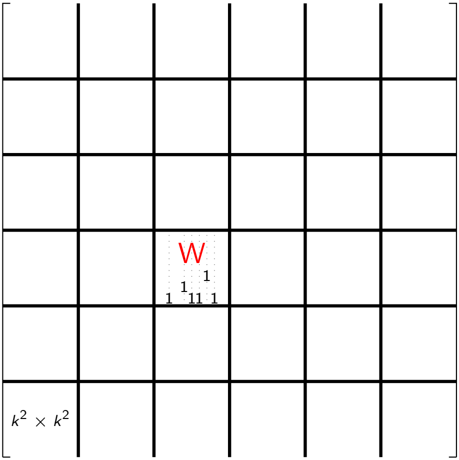
$M =$



Draw a regular $\frac{n}{k^2} \times \frac{n}{k^2}$ division on top of M

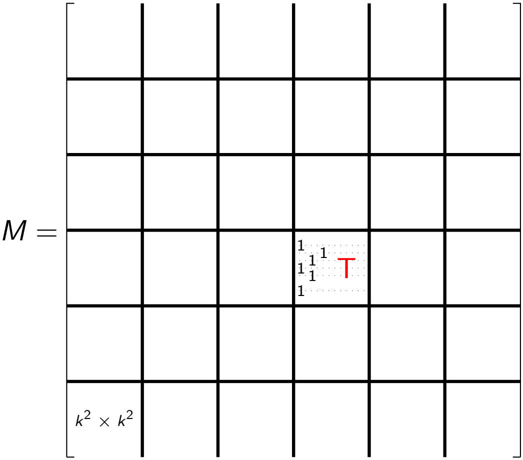
Marcus-Tardos one-page inductive proof

$M =$



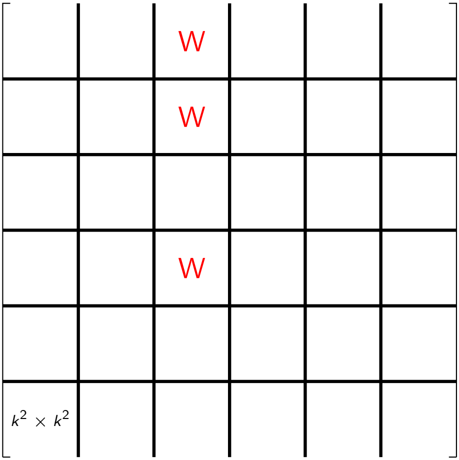
A cell is *wide* if it has at least k columns with a 1

Marcus-Tardos one-page inductive proof



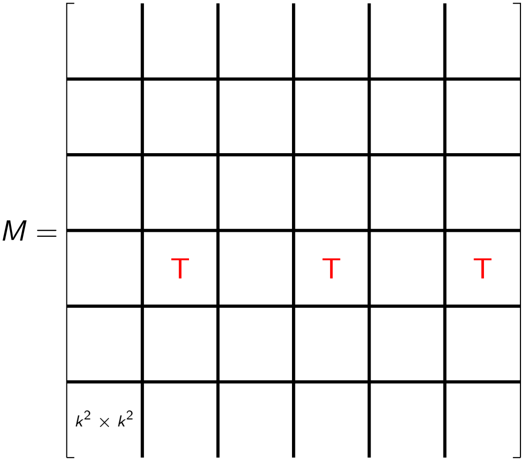
A cell is *tall* if it has at least k rows with a 1

Marcus-Tardos one-page inductive proof

$M =$ 

There are less than $k\binom{k^2}{k}$ wide cells per column part. Why?

Marcus-Tardos one-page inductive proof



There are less than $k\binom{k^2}{k}$ tall cells per row part

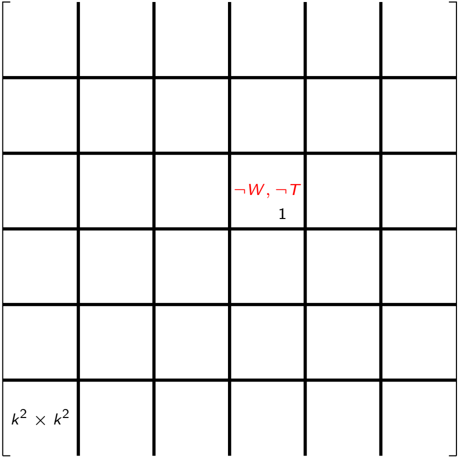
Marcus-Tardos one-page inductive proof

$$M = \begin{bmatrix} & & W & & & \\ & W & W & & & T \\ & & & & & \\ & T & W & T & & T \\ & & T & & & \\ k^2 \times k^2 & & & & & W \end{bmatrix}$$

In W and T, at most $2 \cdot \frac{n}{k^2} \cdot k\binom{k^2}{k} \cdot k^4 = 2k^3\binom{k^2}{k} n$ entries 1

Marcus-Tardos one-page inductive proof

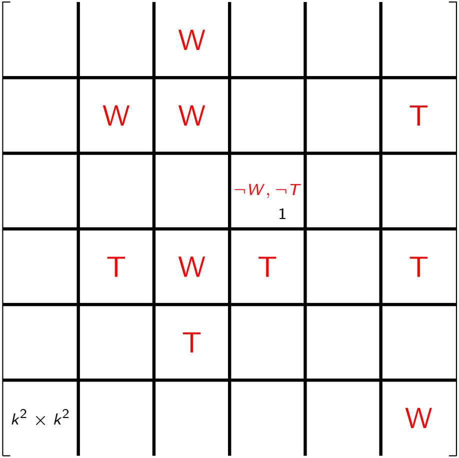
$M =$



There are at most $(k-1)^2 c_k \frac{n}{k^2}$ remaining 1. Why?

Marcus-Tardos one-page inductive proof

$M =$



Choose $c_k = 2k^4\binom{k^2}{k}$ so that $(k-1)^2 c_k \frac{n}{k^2} + 2k^3\binom{k^2}{k} n \leqslant c_k n$