Algorithmic developments using twin-width

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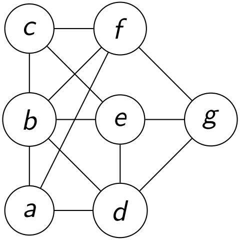
Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.



Maximum red degree $= 0$

overall maximum red degree = 0

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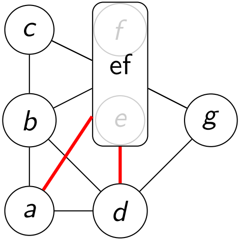

Maximum red degree $= 2$

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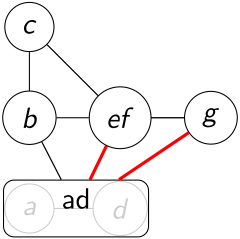

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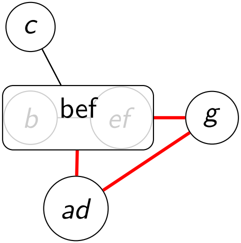

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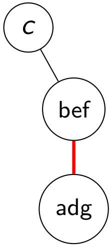

Maximum red degree $= 1$

overall maximum red degree = 2

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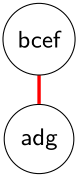

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Partition viewpoint: $\mathcal{S} = \mathcal{P}_n, \mathcal{P}_{n-1}, \ldots, \mathcal{P}_1$ with $\mathcal{P}_i$ a partition of $V(G)$ in i parts

Theorem (B., Geniet, Kim, Thomassé, Watrigant '20 & '21)

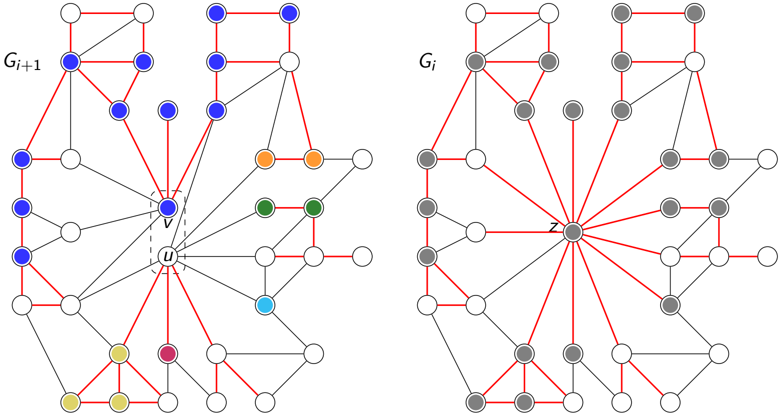
The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width or clique-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size,*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- *$\Omega(\log n)$-subdivisions of all the n-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from K_4,*
- *strong products of two bounded twin-width classes, one with bounded degree,*
- *first-order (FO) transductions of the above.*

Given a d-sequence, one can solve

any problem definable with a FO sentence φ in time $f(d, \varphi)|V(G)|$.

special cases like k-Independent Set in time $2^{O_d(k)}|V(G)|$



$O_d(1)$-approximate Min Dominating Set

How hard is computing twin-width?

Theorem (Bergé, B., Déprés '22)

It is NP-complete to decide if the twin-width is at most 4.

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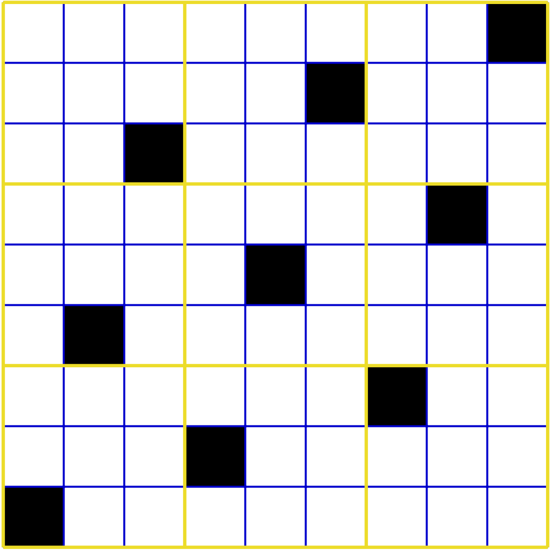
Question

Given a graph G and an integer d, is it possible to either provide an $f(d)$-sequence of G or correctly conclude that tww$(G) > d$, in time $g(d)|V(G)|^{O(1)}$ or $|V(G)|^{g(d)}$?

Question

Is twin-width at most d polytime recognizable? (for $d \in \{2, 3\}$)

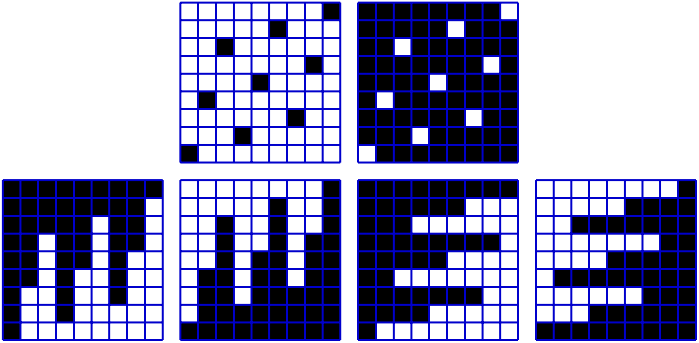
k-grid permutation



Here with $k = 3$, it has every 3-permutation as subpermutation

The 6 universal patterns of unbounded twin-width

Theorem (B., Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk '22)
$\exists f$ s.t. all the adjacency matrices of a graph of twin-width $\geqslant f(k)$ contains a k-grid permutation submatrix or one of its 5 encodings



Semi-induced matching, antimatching, and 4 half-graphs or ladders

Twin-width win-win

Goal: compute FO-definable parameter p in FPT time in $\mathcal{C}$.

Show that $\exists f$ non-decreasing, such that $\forall G \in \mathcal{C}$ an $f(p(G))$-sequence of G can be computed in FPT time

- Width $> f(k)$: report $p(G) > k$
- Width $\leqslant f(k)$: use FO model checking algorithm

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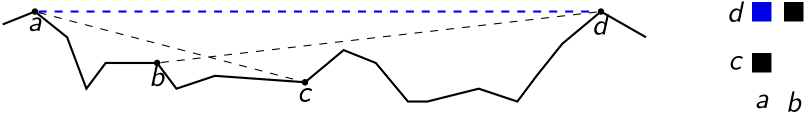
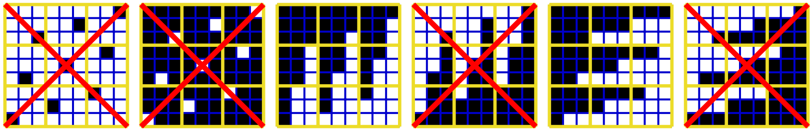
$\rightarrow$ k-Biclique in visibility graphs of 1.5D terrains
$\rightarrow$ k-Independent Set in visibility graphs of simple polygons

[B., Chakraborty, Kim, Köhler, Lopes, Thomassé '22]

Visibility graphs of 1.5D terrains

Order along x-coordinates





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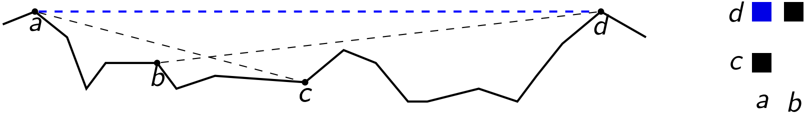
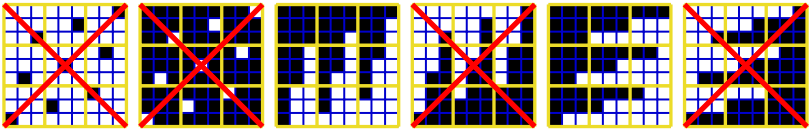
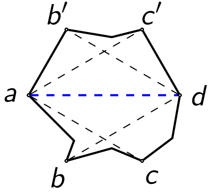
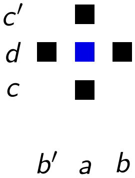




k-Biclique and k-Ladder are FPT in this class

Ordering along the boundary of the polygon





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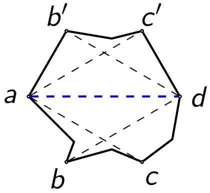
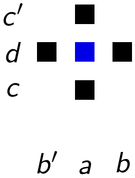
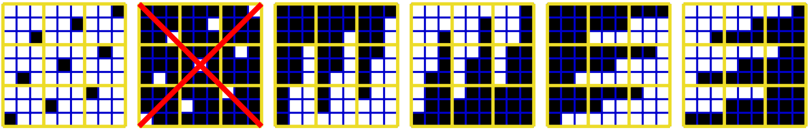
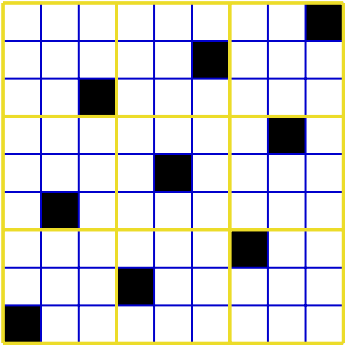


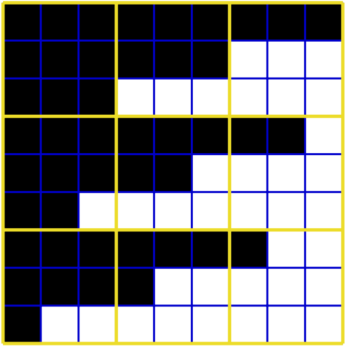




Extractions

Here we only need a decreasing pattern





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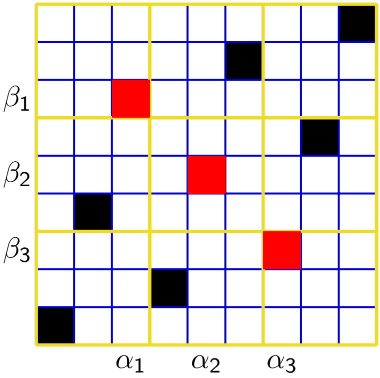

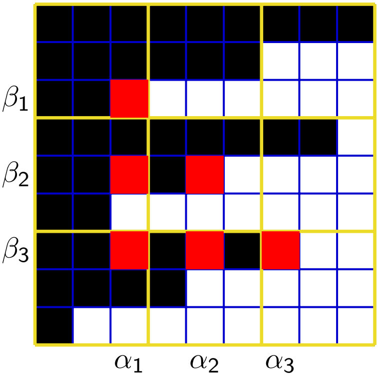

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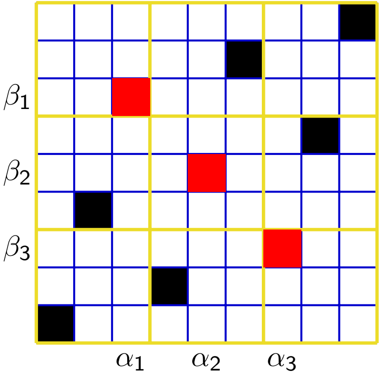
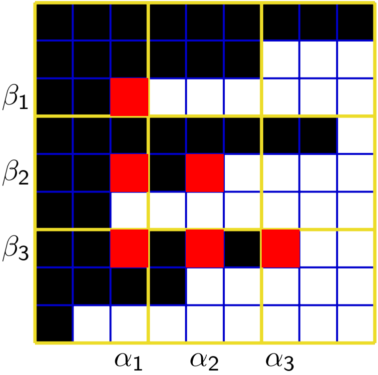
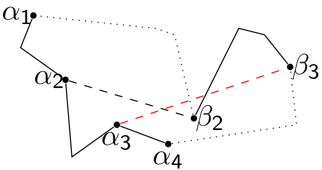




By Ramsey's theorem, we can assume that the α_is and the β_is both induce a clique.

Geometric arguments



Quadrangle $\alpha_2\alpha_3\beta_3\beta_2$ is not self-crossing

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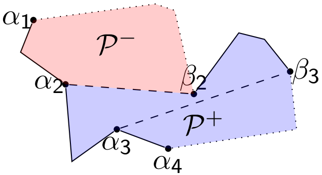

Quadrangle $\alpha_2\alpha_3\beta_3\beta_2$ has to be convex

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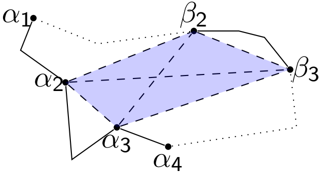

Then $\alpha_2, \alpha_3, \beta_3, \beta_2$ induce K_4, a contradiction

Approximation algorithms

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

For every d, there is a D such that every n-vertex graph with twin-width at most d iteratively admits $\frac{n}{D}$ disjoint pairs that can be contracted in a D-sequence.

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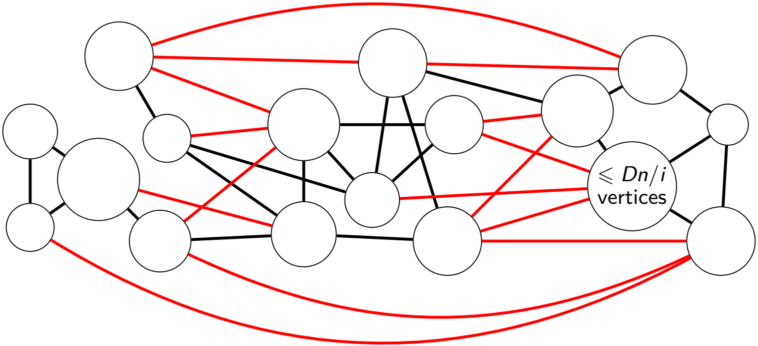
Consequence: We can turn a d-sequence into a *balanced* D-sequence $\mathcal{S}$, i.e., such that $\forall \mathcal{P}_i \in \mathcal{S}, \forall P \in \mathcal{P}_i$, $|P| \leqslant D\frac{n}{i}$



Approximating Max Independent Set

In general graphs: an $n^{1-\varepsilon}$-approximation or $r(n)$-approximation in time $\exp(\frac{n^{1-\varepsilon}}{r(n)^{1+\varepsilon}})$ are unlikely

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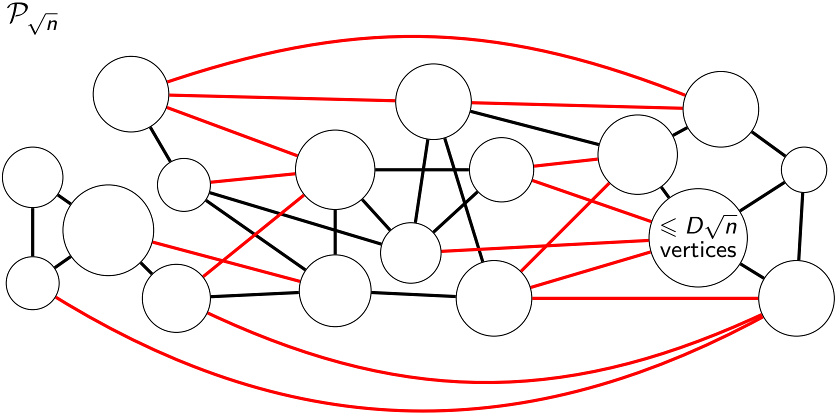

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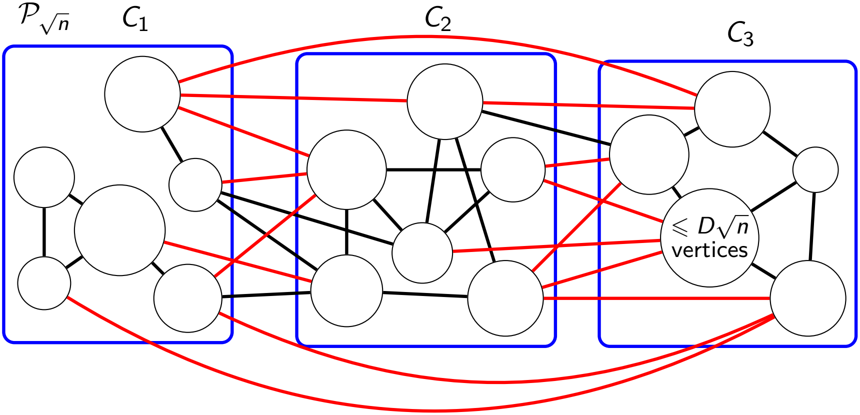

$D+1$-color the red graph of $G/\mathcal{P}_{\sqrt{n}}$ in polynomial time

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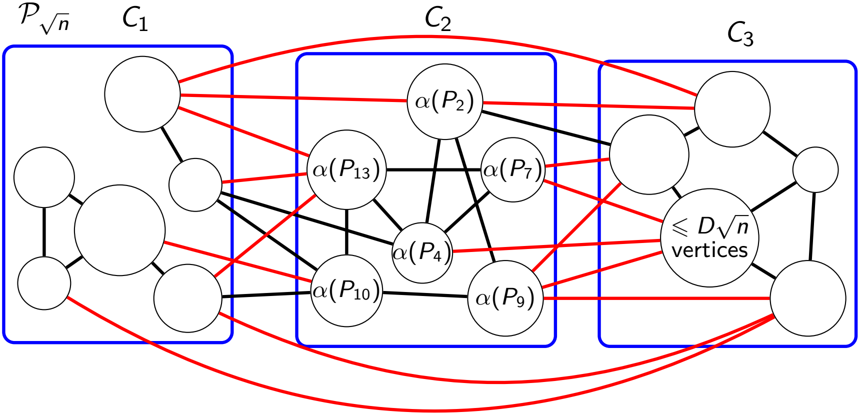

Solve MIS in $G[P_j]$ for every $P_j \in \mathcal{P}_{\sqrt{n}}$ in $2^{O(D\sqrt{n})}$ time

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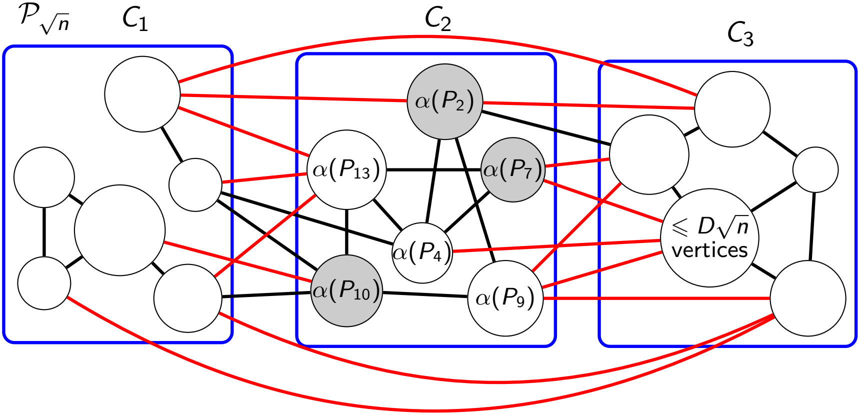

Solve weighted MIS in $G/\mathcal{P}_{\sqrt{n}}[C_i]$, $\forall i \in [D+1]$ in $2^{O(\sqrt{n})}$ time

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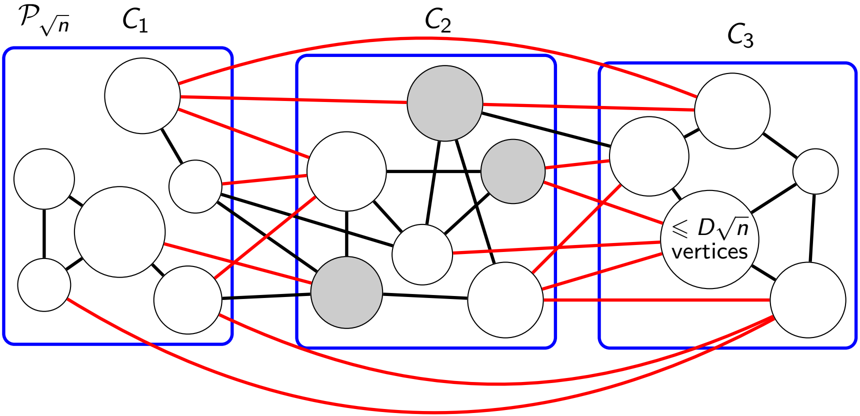

A heaviest such solution is a $(D+1)$-approximation

Approximating MIS given a d-sequence

Theorem (Bergé, B., Déprés, Watrigant '22+)

MIS *can be* $O_d(1)$*-approximated in time* $2^{O_d(\sqrt{n})}$.

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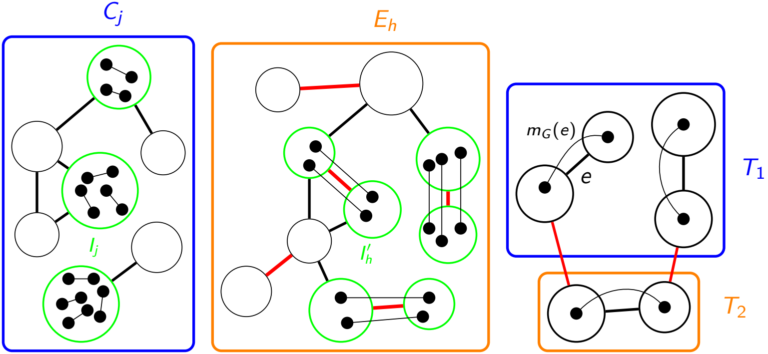
Setting $q := \log \frac{\varepsilon \log n}{O_d(1)}$

Theorem (Bergé, B., Déprés, Watrigant '22+)

MIS *can be* n^ε*-approximated in polynomial time.*

Coloring, Max Induced Matching

Similar results for these problems



Open questions

FPT/XP approximation of twin-width

Practical FPT algorithms for the problems on polygons/terrains

Better than n^{ε}-approximation for MIS given $O(1)$-sequences?
(every such approximation would then self-improve)

unexpected use of the FO model checking algorithm on bounded tww (recent example with 3-terminal pairs Directed Multicut)

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Thank you for your attention!