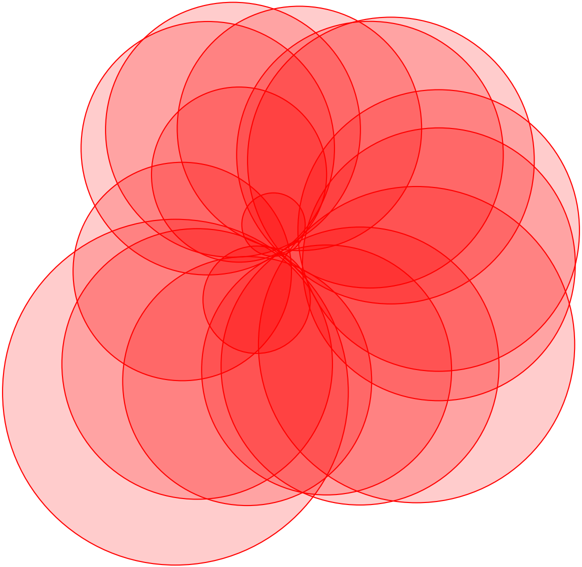
Maximum Clique on Disks

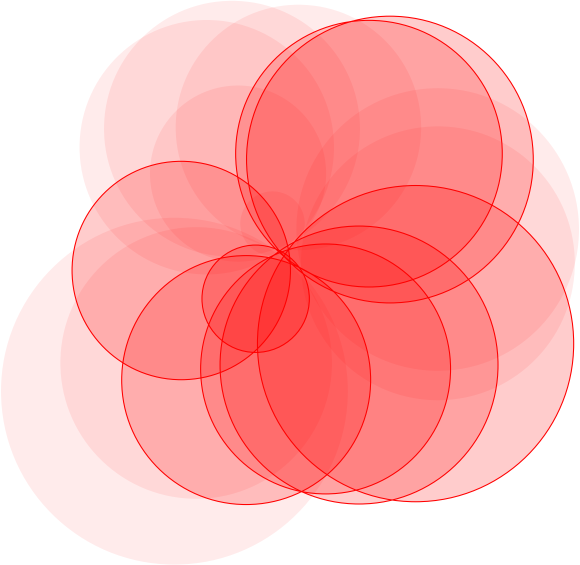
Édouard Bonnet joint work with Panos Giannopoulos, Eunjung Kim, Paweł Rzążewski, and Florian Sikora
and
Marthe Bonamy, Nicolas Bousquet, Pierre Chabit, and Stéphan Thomassé

LIP, ENS Lyon

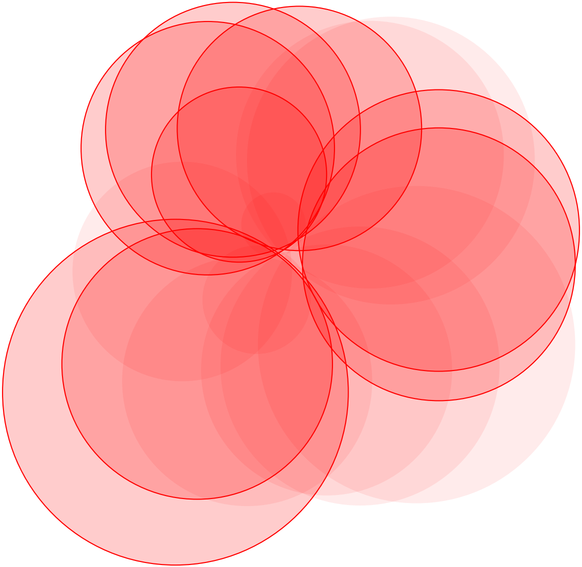
20 mars 2018, LIGM



Find a largest collection of disks that pairwise intersect

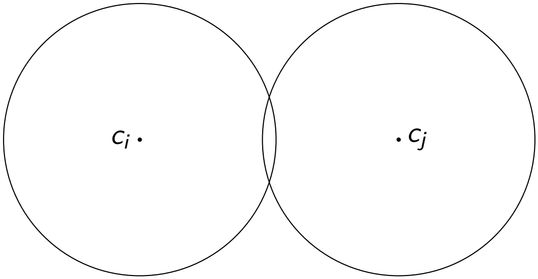


Like this



or that

Polynomial algorithm on unit disks by Clark, Colbourn, and Johnson, 1990.



Guess two farthest disks in an optimum solution S.

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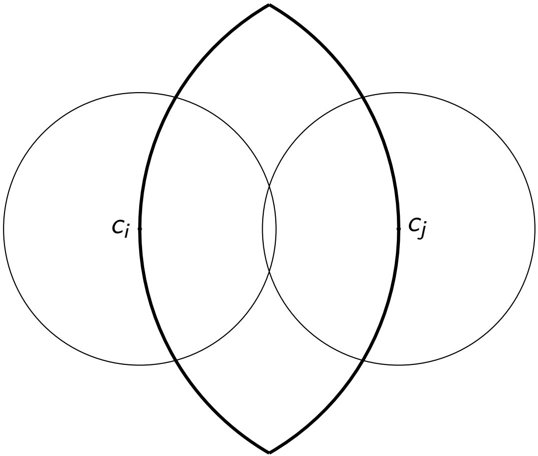

Hence, all the centers of S lie inside the bold digon.

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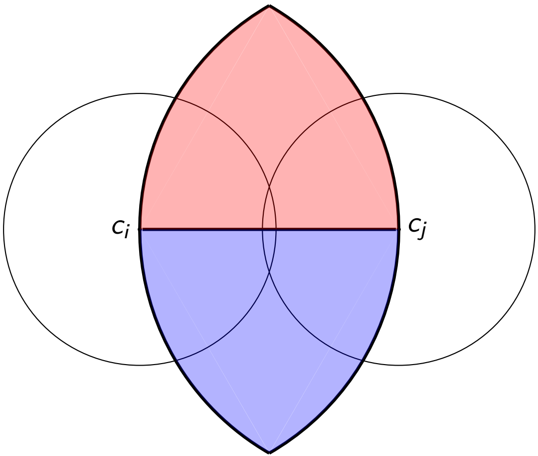

Two disks centered in the same-color region intersect.

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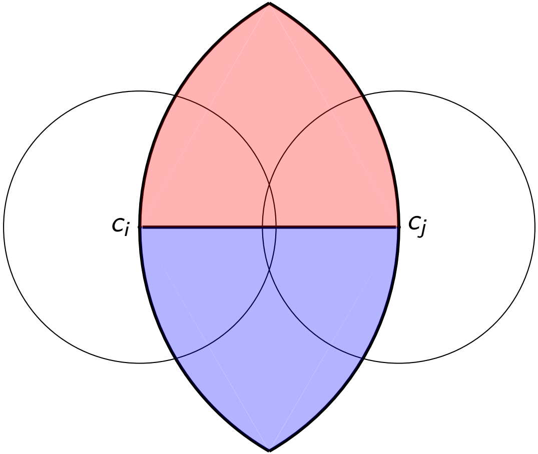

We solve MAX CLIQUE in a co-bipartite graph.

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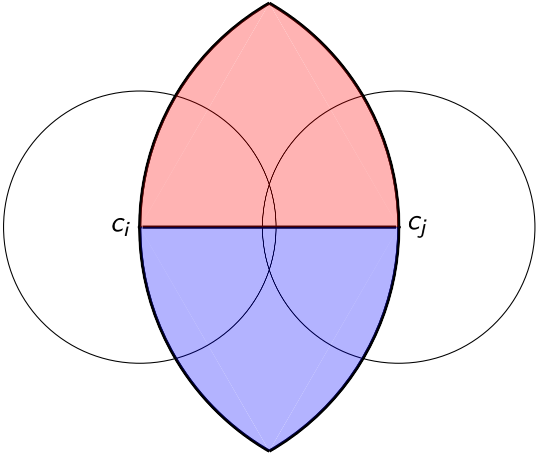


We solve Max Independent Set in a bipartite graph.

Disk graphs

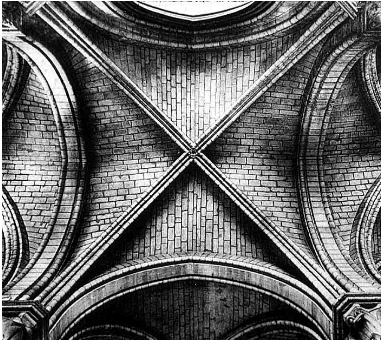
Unweighted problems

3-Colourability [?]	NP-complete	[+]Details
Clique [?]	Unknown to ISGCI	[+]Details
Clique cover [?]	NP-complete	[+]Details
Colourability [?]	NP-complete	[+]Details
Domination [?]	NP-complete	[+]Details
Feedback vertex set [?]	NP-complete	[+]Details
Graph isomorphism [?]	Unknown to ISGCI	[+]Details
Hamiltonian cycle [?]	NP-complete	[+]Details
Hamiltonian path [?]	NP-complete	[+]Details
Independent dominating set [?]	NP-complete	[+]Details
Independent set [?]	NP-complete	[+]Details
Maximum bisection [?]	NP-complete	[+]Details
Maximum cut [?]	NP-complete	[+]Details
Minimum bisection [?]	NP-complete	[+]Details
Monopolarity [?]	NP-complete	[+]Details
Polarity [?]	NP-complete	[+]Details
Recognition [?]	NP-hard	[+]Details

Inherits the NP-hardness of planar graphs.

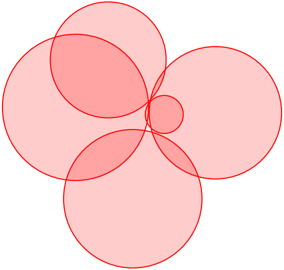
So what is known for Max Clique on disk graphs?

- Polynomial-time 2-approximation
  - For any clique there are 4 points hitting all the disks.
  - Guess those points and remove the non-hit disks.
  - The resulting graph is partitioned into 2 co-bipartite graphs.
  - Solve exactly on both co-bipartite graphs.
  - Output the best solution.
- No non-trivial exact algorithm known



And what is known about disk graphs?

- Every planar graph is a disk graph.
- Every triangle-free disk graph is planar (centers $\to$ vertices).
- So a triangle-free non-planar graph like $K_{3,3}$ is not disk.
- A subdivision of a non-planar graph is not a disk graph (more generally not a string graph).
- ...



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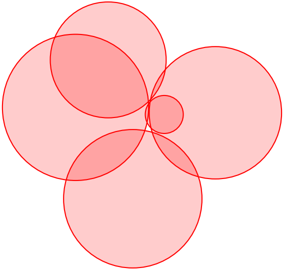
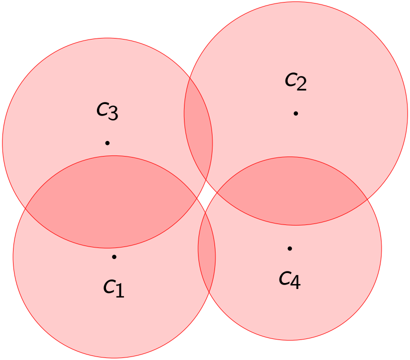


Other ways of showing that a graph is not disk?

Say the 4 centers encoding a $K_{2,2} = \overline{2K_2}$ are in convex position.



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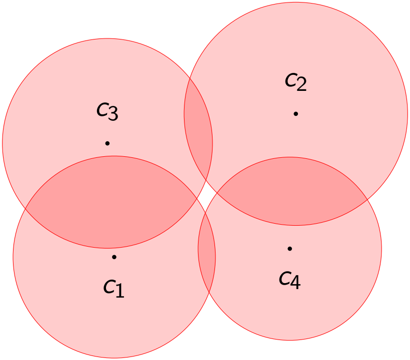

Then the two non-edges should be diagonal.

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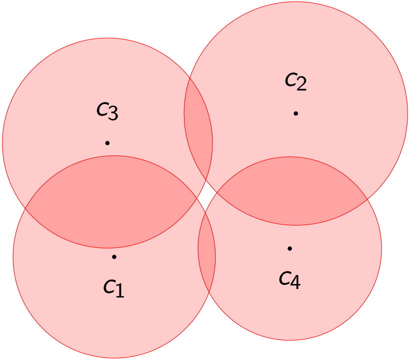
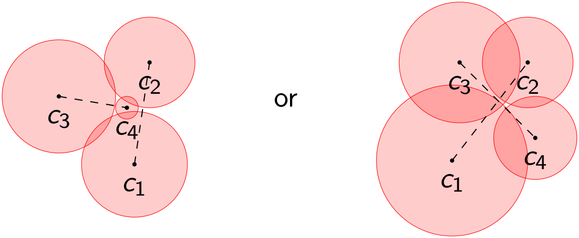


Then the two non-edges should be diagonal.

Suppose $d(c_1, c_3) > r_1 + r_3$ and $d(c_2, c_4) > r_2 + r_4$.
But $d(c_1, c_3) + d(c_2, c_4) \leqslant d(c_1, c_2) + d(c_3, c_4) \leqslant r_1 + r_2 + r_3 + r_4$,
a contradiction.

Conclusion: the 4 centers of an induced $\overline{2K_2}$ are either

- not in convex position or
- in convex position with the non-edges being *diagonal*.



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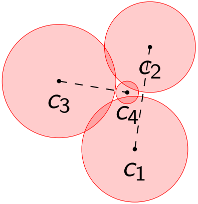

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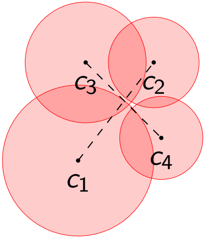
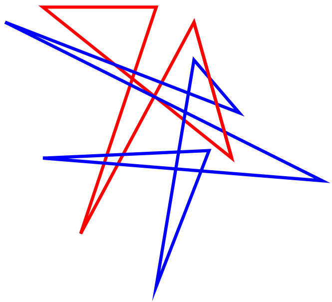


Reformulation: either

- the line $\ell(c_1, c_2)$ crosses the segment c_3c_4, or
- the line $\ell(c_3, c_4)$ crosses the segment c_1c_2, or
- both; equivalently, the segments c_1c_2 and c_3c_4 cross.

Assume $\overline{C_s + C_t}$ is a disk graph.

Link consecutive centers of the two disjoint cycles (non-edges).



For each red segment s_i, we denote by:

- a_i the number of blue segments crossed by $\ell(s_i)$.
- b_i the number of blue segments whose extension cross s_i.
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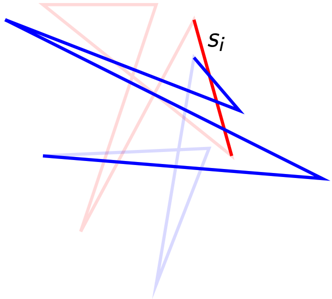

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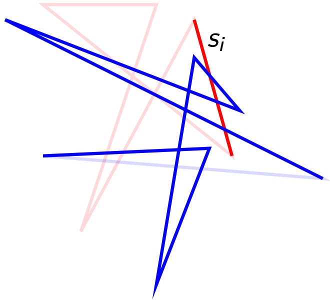

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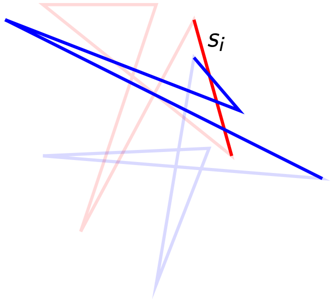

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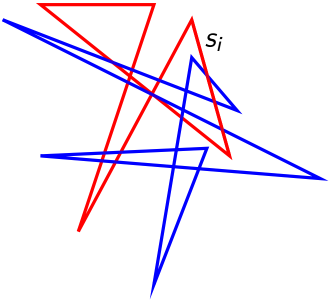

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It should be that $a_i + b_i - c_i = t$.

$$\sum_{1 \leqslant i \leqslant s} a_i + b_i - c_i = st$$

1) a_i is even:

$$\sum_{1\leqslant i\leqslant s} a_i + b_i - c_i = st$$

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2) $\sum_{1 \leqslant i \leqslant s} b_i = \sum_{1 \leqslant i \leqslant t} a'_i$ is therefore even. (a'_j, b'_j, c'_j same for blue segments)

3) $\sum_{1 \leqslant i \leqslant s} c_i$ is even:

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$$\sum_{1\leqslant i\leqslant s} a_i + b_i - c_i = \sum_{1\leqslant i\leqslant s} a_i + \sum_{1\leqslant i\leqslant t} a'_i - \sum_{1\leqslant i\leqslant s} c_i \text{ is even.}$$

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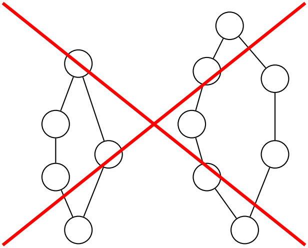
$\sum_{1 \leqslant i \leqslant s} a_i + b_i - c_i = \sum_{1 \leqslant i \leqslant s} a_i + \sum_{1 \leqslant i \leqslant t} a'_i - \sum_{1 \leqslant i \leqslant s} c_i$ is even.

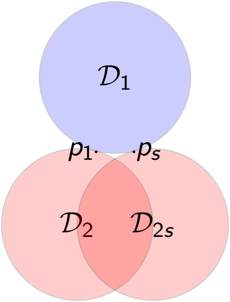
Hence s and t cannot be both odd.

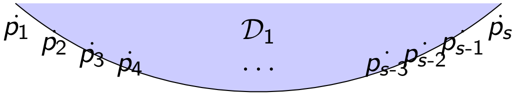
The complement of two odd cycles is not a disk graph.



Are there other graphs of co-degree 2 which are not disk?

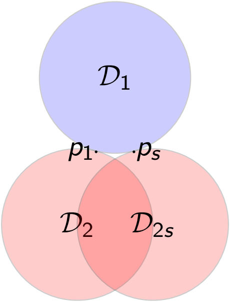
Complement of an even cycle $1, 2, \ldots, 2s$

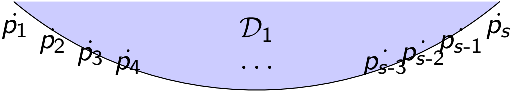




We start by positioning $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_{2s}$.
We draw a convex chain between p_1 and p_s.

Complement of an even cycle $1, 2, \ldots, 2s$





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D_{2i}: same radius and boundary crosses p_i with tangent $\ell(p_{i-1}p_{i+1})$

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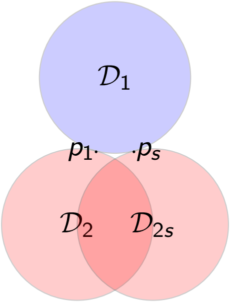

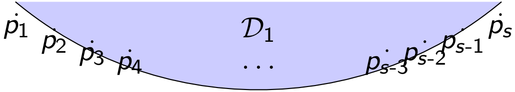

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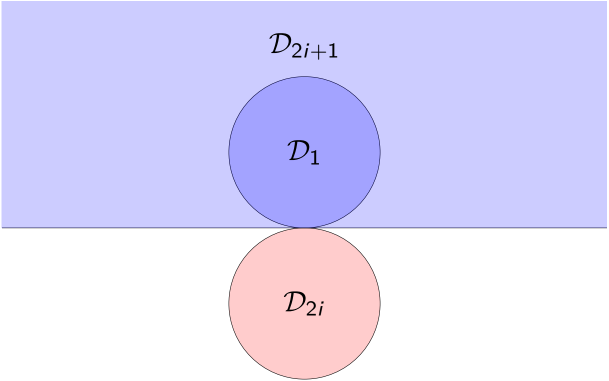
D_{2i}: same radius and boundary crosses p_i with tangent $\ell(p_{i-1}p_{i+1})$
D_{2i+1}: larger radius and "co-tangent" to D_{2i} and D_{2i+2}.

Stacking complements of even cycles



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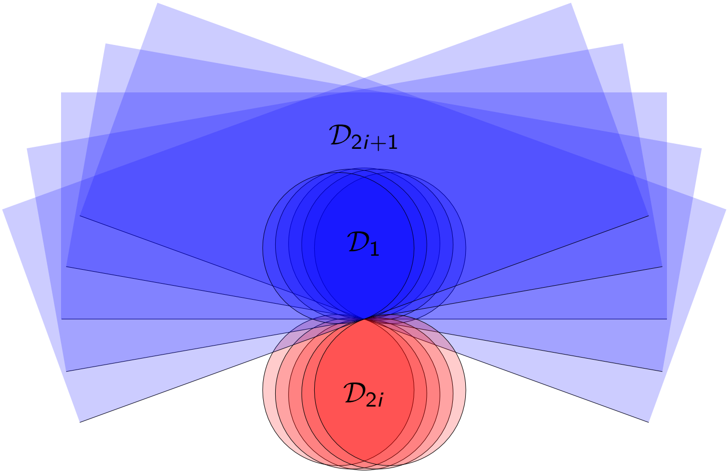

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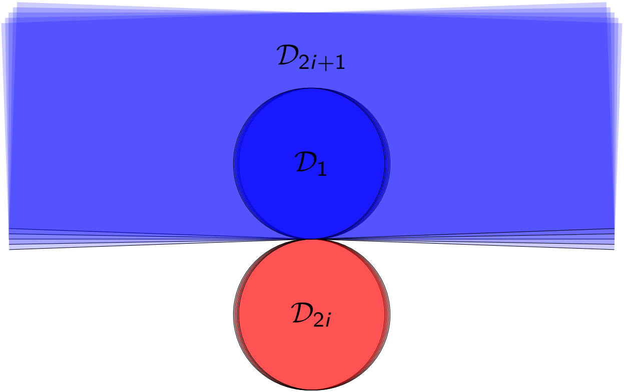
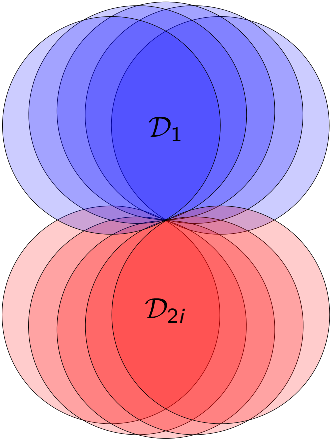
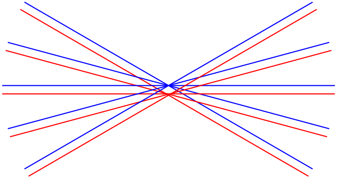
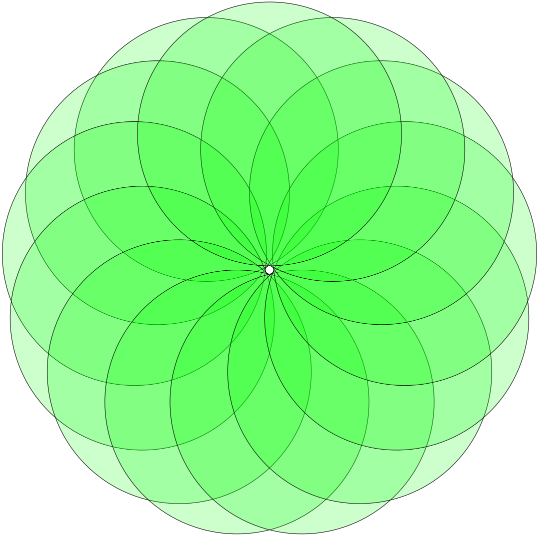


Disks of different cycle complements intersect





Complement of odd cycle by unit disks (Atminas & Zamaraev)



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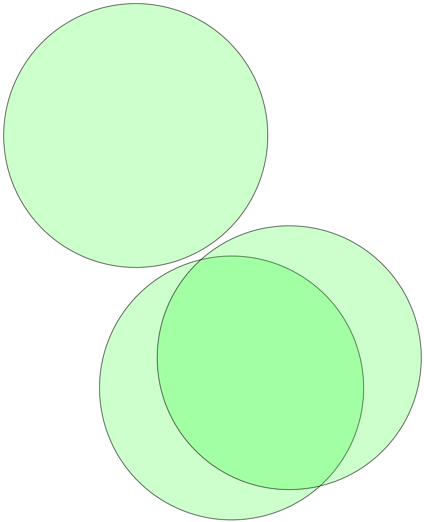

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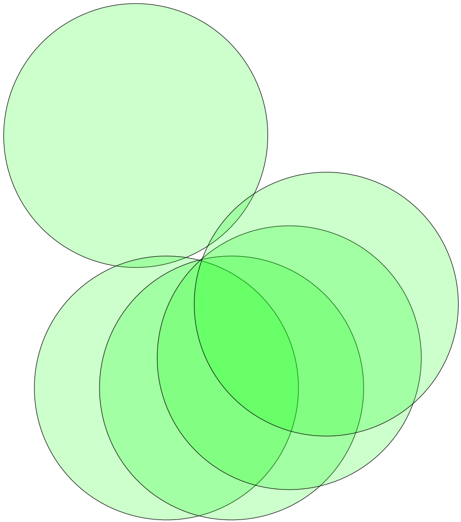
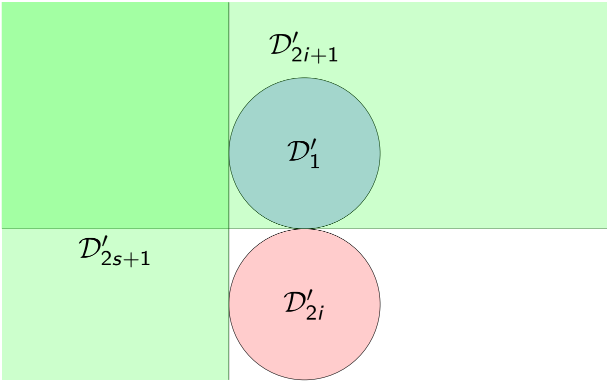
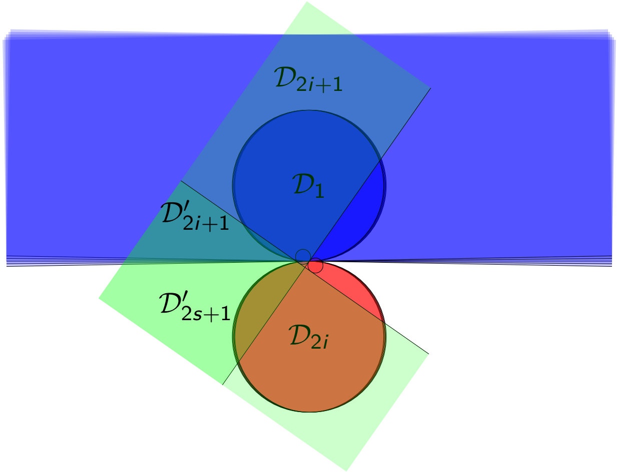


Different representation with non-unit disks

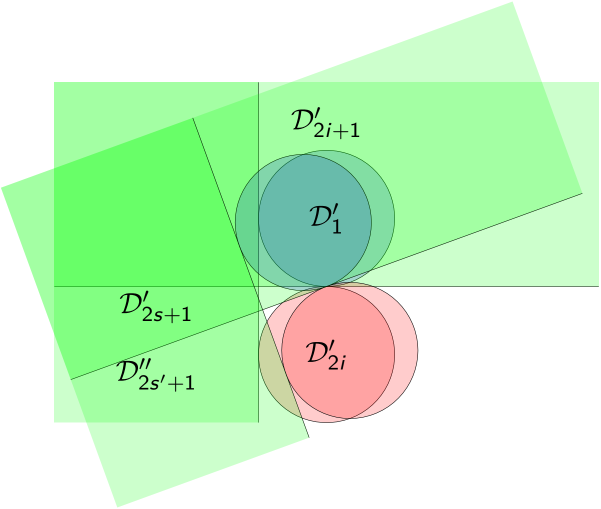


Same construction except $\mathcal{D}'_1$ intersects $\mathcal{D}'_{2s}$ and $\mathcal{D}'_{2s+1}$ is "co-tangent" to $\mathcal{D}'_1$ and $\mathcal{D}'_{2s}$.

Complement of many even cycles and one odd cycle



Sanity check: trying to stack complements of odd cycles



$\mathcal{D}''_{2s'+1}$ cannot possibly intersect $\mathcal{D}'_1$

Going back to algorithms.

Can we solve Max Independent Set more efficiently if there are no two vertex-disjoint odd cycles as an induced subgraph?

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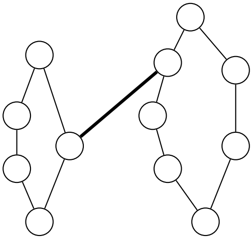
Another way to see it:
at least one edge between two vertex-disjoint odd cycles



Quasi-polynomial time approximation-scheme (QPTAS)

ocp(G): maximum size of an odd cycle packing.

Theorem (Bock et al. 2014)

PTAS for MAX INDEPENDENT SET *for* $ocp = o(n/\log n)$.

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Lemma

Let H *complement of a disk graph with* n *vertices.*
If $ocp(H) > n/\log^2 n$, *then vertex of degree at least* $n/\log^4 n$.

Proof.

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The shortest odd cycle has size at most $\log^2 n$.
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Branching factor $(1, n/\log^4 n)$ (in $2^{\log^5 n}$), and PTAS otherwise.

Subexponential algorithm

Theorem (Györi et al. 1997)

A graph of odd girth c has an odd cycle cover of size $\tilde{O}(n/c)$.

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Let H be a complement of disk graph, Δ its degree, c its odd girth. We can:

- branch in time $2^{\tilde{O}(n/\Delta)}$.
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$2^{\tilde{O}(\min(n/\Delta, n/c, c\Delta))} \leqslant 2^{\tilde{O}(n^{2/3})}$ for $\Delta = c = n^{1/3}$.

Filled ellipses and triangles

2-subdivisions: graphs where each edge is subdivided exactly twice
co-2-subdivisions: complements of 2-subdivisions

Lemma (technical)

For some $\alpha > 1$, MAX INDEPENDENT SET on 2-subdivisions is not α-approximable algorithm in $2^{n^{0.99}}$, unless the ETH fails.

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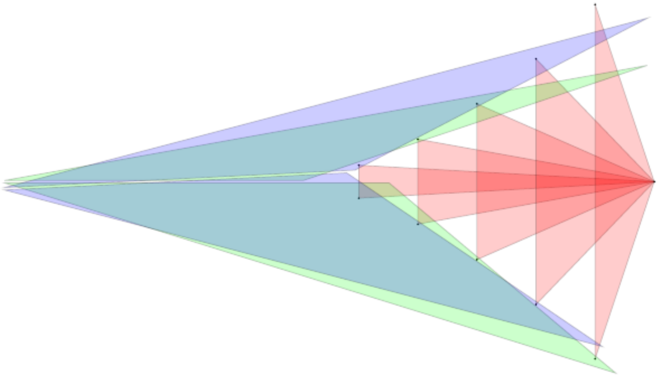
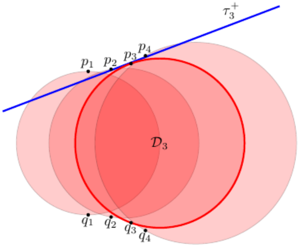
For some $\alpha > 1$, $\textsc{Max Independent Set}$ on 2-subdivisions is not α-approximable algorithm in $2^{n^{0.99}}$, unless the ETH fails.

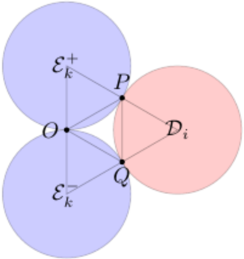
Graphs of filled ellipses or filled triangles contain all the co-2-subdivisions.

Filled triangles



Filled ellipses





The QPTAS and subexponential algorithm only work for intersection graphs of 2D-objects that are *exactly* disks.

- Approximation scheme with better running time?
- Have higher-dimensional disks the same obstruction?

EPTAS

VC dim of $\mathcal{S}$ = maximum size of a set with all intersections with $\mathcal{S}$.
VCdim(G) = VC dimension of the neighborhood set-system.
$\alpha(G)$ = size of a maximum independent set in G.
iocp(G) = same as ocp but induced.

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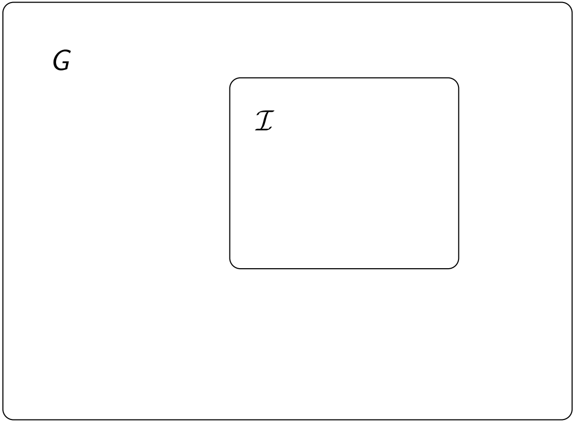
Theorem

MAX INDEPENDENT SET *can be* $1+\varepsilon$*-approximated in time* $2^{\tilde{O}(1/\varepsilon^3)} n^{O(1)}$ *on graphs G with*

- *VCdim*$(G) = O(1)$,
- $\alpha(G) = \Omega(|V(G)|)$, *and*
- *iocp*$(G) = 1$.

First step of the algorithm: sampling

$\mathcal{I}$ is a fixed maximum solution.



Select randomly a subset S of $\tilde{O}(1/\varepsilon^3)$ vertices.

First step of the algorithm: sampling

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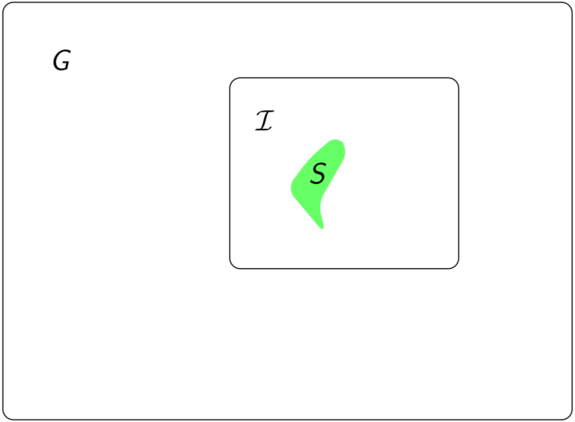

With probability $1/2^{\tilde{O}(1/\varepsilon^3)}$, $S \subseteq \mathcal{I}$.

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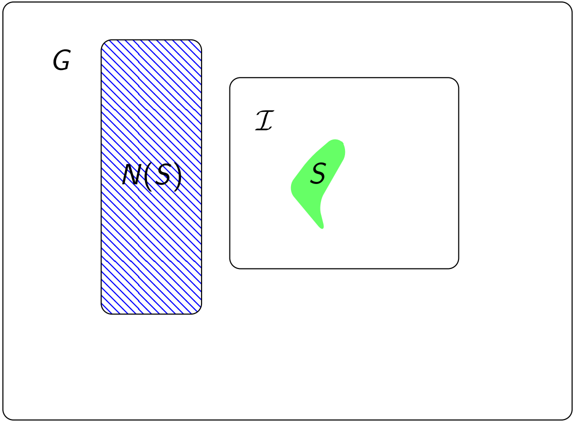

Remove the neighborhood of S.

Classic result of Haussler and Welzl in VC dimension theory

Theorem

A set-system $(\mathcal{S}, U)$ with VC dimension d and only sets of size at least $\varepsilon|U|$ has a hitting set of size $O(\frac{d}{\varepsilon}\log\frac{1}{\varepsilon})$.

Furthermore, any sample of size $\frac{10d}{\varepsilon}\log\frac{1}{\varepsilon}$ is a hitting set w.h.p.

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We apply that result to the set-system
$(\{N(u) \cap \mathcal{I} \mid u \in V(G), |N(u) \cap \mathcal{I}| \geqslant \varepsilon^3|\mathcal{I}|\}, \mathcal{I})$.
In words, the large neighborhoods over I.

Second step of the algorithm: win-win on odd girth

Now we removed $N(S)$, all the vertices have degree $< \varepsilon^3|\mathcal{I}|$ in $\mathcal{I}$.

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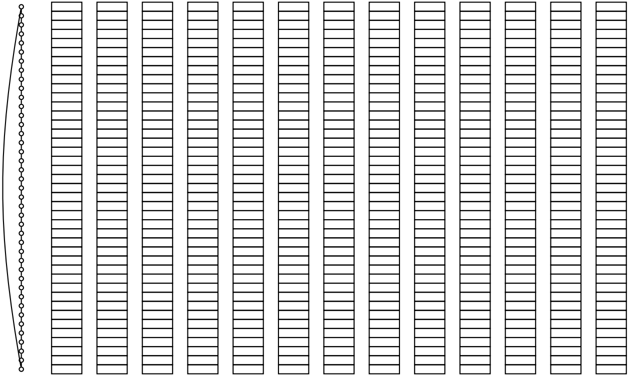
We compute a shortest odd cycle C.

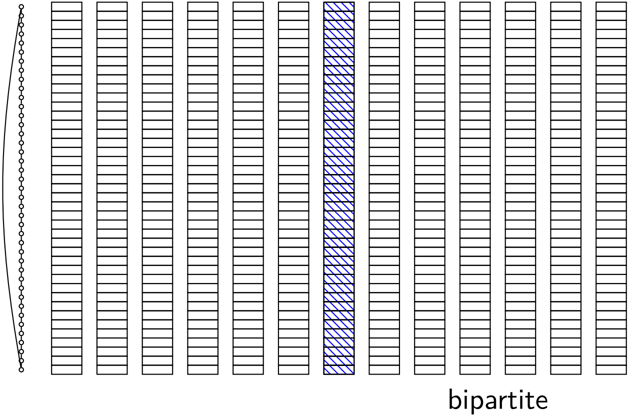
- If $|C| \leqslant 1/\varepsilon^2$, we remove $N[C]$ from the graph. The resulting graph is bipartite and $|N(C)| < \frac{\varepsilon^3|\mathcal{I}|}{\varepsilon^2} = \varepsilon|\mathcal{I}|$.
- If $|C| > 1/\varepsilon^2$, we need a fresh slide.

Finding a small odd cycle transversal



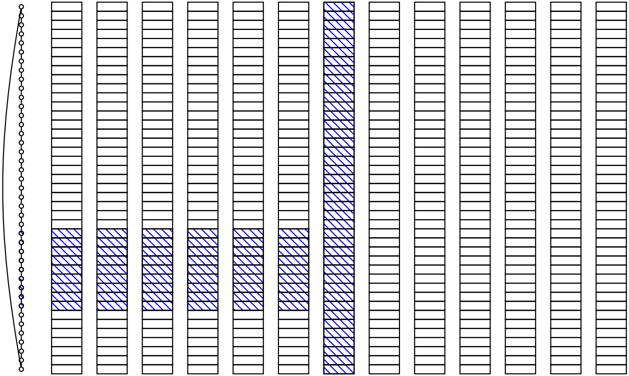
Columns are the successive neighborhoods of C, *layers*.
Rows indicate the closest vertex on C, *strata*.

Finding a small odd cycle transversal



We delete the lightest among the first $\approx 1/\varepsilon$ layers.

Finding a small odd cycle transversal

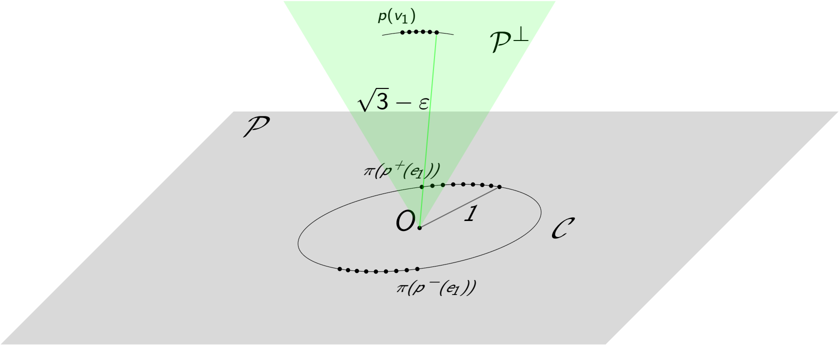


$\approx 1/\varepsilon$ consecutive strata are an odd cycle transversal.

Regarding the higher-dimensional question:

- unit 4D-disk graphs
- ball (3D-disk) graphs with radii arbitrary close to 1

contain all the co-2-subdivisions
(so, no approximation scheme and no subexponential algorithm)



What about unit ball graph?

Let $x_1, \dots, x_s$ be the consecutive centers in $\mathbb{R}^3$ of a co-odd-cycle. Consider the trace on the 2-sphere of the following vector walk.

- Start at vector $\overrightarrow{ab}$ with $a = x_1$ and $b = x_2$.
- move continuously a from x_1 to x_3 following the segment x_1x_3.
- move continuously b from x_2 to x_4 following the segment x_2x_4.
- and so on, until back to $\overrightarrow{x_1x_2}$.

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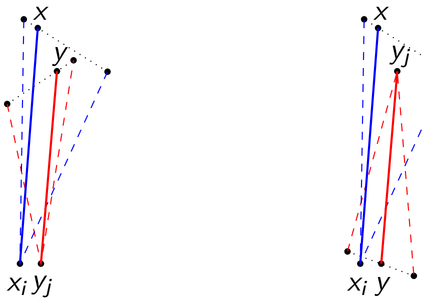
As s is odd, half-way through we reach $\overrightarrow{x_2x_1}$.
Hence the curve drawn on the 2-sphere is antipodal.

As two antipodal curves intersect, we have one of the following configurations:



Open questions

- Is Max Clique NP-hard on disk and unit ball graphs?
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Thank you for your attention!